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FIXING THE QUORUM:
REPRESENTATION VERSUS ABSTENTION

SANNE ZWART

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Abstract. The majority of the participating voters in referenda does not necessarily reflect the majority of the whole population since voters can abstain. This paper shows that a quorum exists for which the outcome of the referendum coincides with the population preference. However, a second equilibrium can exist in which the proposal is always rejected. When insufficient information makes the optimal quorum unknown, it is in general more harmful to set the quorum too high than too low. Robustness of the results is analyzed by allowing pressure groups to encourage or discourage participation after the quorum is set.

Keywords. Electoral engineering, quorum, referendum, voting/not-voting decision, voting rules.

JEL Classification. D72.

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1. Introduction

In June 2005 a referendum was held in Italy to block a fertility law. The law banned research using stem cells from embryos and imposed stringent requirements on test-tube pregnancies. Adversaries of the law initiated the referendum to aim for its abrogation. To succeed, a quorum of 50% was to be met and a majority of the participating voters had to support the abrogation. This gave advocates of the law two different possibilities to avoid abrogation: i) encouraging no-voters to take the effort to vote so that they would form the majority; ii) discouraging no-voters from voting so that the quorum would not be met and the referendum would be invalid. In Italy, the advocates of the law chose for the second option, for example the speakers of the Senate and the Chamber of Deputies as well as the Roman Church discouraged people from voting. The New York Times (2005) writes that "Italian prelates have told parishioners to head to the beach instead of the polling places on Sunday and Monday, so that the quorum will not be met." The strategy succeeded, the turnout was too low and the referendum was invalid. Since 26% of the population voted, while almost 90% of the participating voters were in favor, 23.4% of the population was in favor and voted. When a handful of people in favor was discouraged by the forecasts of an invalid referendum, encouraging no-voters to cast their ballot could indeed have led to a valid referendum in which the abrogation would have been approved.

Referenda are becoming increasingly widespread in democratic countries (Waters (2003) and Matsusaka (2005) discuss recent trends, see also the web sites of The Initiative & Referendum Institute). One of the main reasons is the wish to give voters a direct say in the issues at stake. An additional reason might be that direct democracy would contribute to voters' involvement with and trust in the political system. However, referenda are known to be imperfect decision making tools in the sense that a counter-intuitive relationship between the voters' preferences and the outcome can occur. Nurmi (1998) lists various voting paradoxes, including problems stemming from multiple proposals or multiple alternatives and the possibility of conflicting opinions between the majorities of the voters and their representatives. As the referendum in Italy shows, a quorum gives rise to an additional potential problem by giving opponents of change an additional tool to reach their aim. Fishburn and Brahms (1983) call this the "no-show" paradox.

The objective of this paper is twofold. In the first part we address the question whether there is any theoretical support for imposing a quorum in a referendum. The focus of the second part is on the robustness of the results. More specifically, we first look at the magnitude of the distortion when the quorum is set either too low or too high and then at the impact of pressure groups which can affect the voter turnout after the quorum is set.

The role of the quorum is analyzed in a stylized referendum model with heterogenous voters. The existence of a quorum makes the turnout a decisive variable for determining the outcome. But even for referenda without a quorum, the voting/not-voting decision is an important aspect of explaining the outcome. Analyzing this decision usually leads to the conclusion that people who vote do not form a representative subset of the population. For example, Fort and Bunn (1998) find for referenda concerning nuclear power that actual participation has more explanatory power for the yes/no decision than both economic and preference variables. Successfully navigating the hurdles of registering, going to the booth etc. made a no-vote more likely. In the model, this asymmetry between opponents and proponents is reflected by their possibly different probabilities of voting.

In the first part of the paper we show that with the appropriate choice of the quorum and the default outcome that occurs if the quorum is not met, the population majority outcome can be attained. To see how the referendum should be designed, suppose that proponents are more likely to cast their ballots then opponents. In order to offset the bias towards accepting, the default outcome needs to be rejection. A higher quorum needs more participating voters. To be precise, it needs a higher fraction of yes-voters in the population since they are more likely to vote. A higher quorum thus reduces the cases where the majority of participating voters is in favor while the majority of the population is not. The population majority outcome is attained for the quorum for which they equal.

Interestingly, when voters care more about the outcome when they are participating, the optimal quorum does not necessarily lead to the population majority outcome. A second equilibrium can exist in which the default outcome always occurs. In this case, the referendum clearly is an imperfect tool for decision making.

The second part of the paper analyzes the robustness of the results in two ways. When the social planner has insufficient knowledge about the population parameters or insufficient political power to set the quorum at its optimal level, a non-optimal quorum can arise. When the default outcome is set correctly, we show that setting the quorum too low is less harmful than setting it too high. The reason is that the default outcome will always occur when the quorum is too high, while when the quorum is too low both outcomes might still occur. Since in most real-life applications there is not much flexibility in setting the quorum, this finding implicates that only topics for which *both* sides have a high expected turnout should be subjected to referenda. A non-optimal quorum can also arise when pressure groups have the possibility to affect the turnout after the quorum is set, like in the Italian referendum discussed above. When the default outcome is rejecting the proposal, yes-pressure groups should always encourage people to vote. For no-pressure groups it is optimal to encourage voters to participate only if it is likely that there are relatively many no-voters, otherwise they should be discouraged from voting.

Since the basis of democracy is that all people are equally important, we consider the preference of the population majority as the benchmark outcome. We thus abstain from social welfare considerations that balance an "optimal outcome" with the cost of representation. The model can easily be adapted to address different intensities of voters' preferences. In case a quorum is exogenously imposed to guarantee a certain level of representativeness, the second part of the paper can be read as analyzing the difference between the referendum outcome and the population majority outcome. We assume that participation is voluntary, as compulsory voting would trivially result in the population majority outcome (however, Franklin (1999) and Jakee and Sun (2006) raise arguments against compulsory voting).

Theoretical support for the importance of the population majority outcome follows from the axiomatization of May (1952) as the only voting rule that is decisive, anonymous, not-favoring any of the outcomes and positively responding (i.e. when one voter changes opinion then the group decision becomes more favorable towards that opinion). However, when voters can abstain from participating, Côrte-Real and Pereira (2004) find that in general no voting rule that is independent of the abstainers' preferences can achieve the population majority outcome. They show that this outcome can be achieved if in the case of a turnout below the quorum, the underlying reasons determine the outcome. In the equilibrium setting of this paper's model, this interpretation of an insufficient turnout is done *ex ante* when the referendum is designed.

The model is based on the decision-theoretic approach initiated by Downs (1957). Voters participate in the referendum when they receive a positive net utility from voting. Following Riker and Ordeshook (1968) and in line with empirical evidence discussed extensively by Blais (2000), the net utility of voting depends on the outcome of the referendum, the cost of casting the ballot and a "consumption benefit" that represents the fulfillment of a voter's "civic duty". The main difference between their and our model is how a voter derives utility from the outcome of the referendum and from participating. In their model, they consider the benchmark of a utility function that is linear in the outcome of the referendum. However, there might be nonlinearities involved with respect to the outcome and participation. More specifically, the utility of the referendum outcome might depend on whether a voter has participated or not. On top of this, when there are many potential voters, the probability that a particular voter's action is decisive is almost zero. Myerson (2000) derives estimates of the order 10^{-9}. Hence, unless the utility difference between the outcomes is extremely large relative to the cost of voting, the nonlinear effect might be far more important. It is not clear what the direction of this nonlinear effect should be: there are convincing arguments for all possibilities. When it is zero the outcome of the referendum does not affect a voter's participation decision. When it is negative, a voter exhibits an underdog-mentality: the less likely her preferred outcome, the more likely she will vote. When the nonlinear effect is positive, a voter likes to be part of the winning side. In this paper we consider all types. Moreover, we show that if all types can occur simultaneously, the average type drives the results.

Although the literature on voting is vast, there are few papers on referenda. Herrera and Mattozzi (2007) discuss a group-based referendum model. As in this paper, the turnout of each group is endogenous. However, instead of having the referendum outcome directly affecting the voters' utility, their groups weigh the cost of increasing the turnout with its effect on the referendum outcome. They find a "quorum paradox": the equilibrium turnout might only exceed the quorum if the quorum is not imposed. Myatt (2007) discusses a model in which a finite number of privately informed voters have to chose between two alternatives that are preferred to the status quo. In contrast with the model of this paper, strategic voting can occur when a voter fears that her most preferred alternative will not receive sufficient support. Marquette and Hinckley (1988) and Kanazawa (1998) suggest that a voter's recall of previous elections is also relevant for current turnout. Closely related to the model of this paper, Kanazawa (1998) proposes to substitute the Riker-Ordeshook probability regarding the current election with the probability that the voter's preferred outcome occurred when she participated in past elections. Hence, instead of computing the probability that her preferred candidate wins as in this paper's model, a voter uses an estimation based on past experience.

The outline of the paper is as follows: Section 2 presents the model, Section 3 shows that there is a quorum for which the population majority outcome can occur and analyzes its properties, Section 4 addresses the robustness of the results by considering a not-optimally set quorum and allowing for pressures groups. Appendices A and B contain precise formulations of claims made in the main text. Proofs are deferred to Appendix C.

2. The Referendum Model

2.1. **The Referendum.** A referendum is held in order to decide whether a proposal should be accepted or rejected. Each voter has three options: i) to vote in favor of the proposal; ii) to vote against it; iii) not to vote. Voters who do indeed vote are called *participating voters*. The referendum is only valid if a quorum is met, that is if more than a certain fraction of the voters is indeed voting. The proposal is accepted if the referendum is valid and if the majority of the participating voters is in favor.[1] When the quorum is met but a majority of the participating voters is against, the proposal is rejected. In case the referendum is invalid, a preset default outcome determines whether the proposal is accepted or not. Although in some real-life referenda the default outcome is not explicitly set, in most cases it is rather clear what will happen when the referendum is not valid. For example, in the referendum about the European Constitution in the Netherlands there was no formal default outcome. Though, all major political parties were in favor and it was clear that the European Constitution would be accepted in case the quorum would

[1]When the intensities of the voters' preferences differ, a qualified majority can be used to protect a minority from the majority, see Appendix A for details.

not be met. In this paper, designing a referendum is thus choosing the quorum and default option.

There is a continuum of voters with measure one. Each voter knows whether she is in favor of the proposal or against it, but there is uncertainty about the overall fraction of voters in favor of the topic.[2] The assumption that the preferences of voters are endogenously determined is rather standard. However, Rosema (2004) discusses the psychology of voting and finds that possible election outcomes are used in the decision what to vote. Making voters' preferences endogenous though, justifies research on its own and is outside the scope of this paper. Hence, denote by y the proportion of voters in favor of the proposal. The very reason that a referendum is needed, is that the value of y is unknown. Hence, y is a random variable which takes its values in an interval $[\underline{y}, \overline{y}] \subset [0, 1]$. The distribution of y is common knowledge. This can be the case if for example forecasting agencies provide correct projections when not everyone has made up her mind yet. The proportion of voters in favor has full support on $[\underline{y}, \overline{y}]$. The model is not relevant when the majority is either always in favor or always against, so it is assumed that $\underline{y} < \frac{1}{2} < \overline{y}$.

When the proportion of yes-voters y were observable, no referendum is needed to have the proposal accepted or rejected according to the majority of the voters. This benchmark case is referred to as the *population majority outcome.* To be precise, denote by A the event that the proposal is accepted and by $R = A^c$ the event that it is rejected. The population majority outcome is then defined as A when $y > \frac{1}{2}$ and R when $y < \frac{1}{2}$. When $y = \frac{1}{2}$, the population majority outcome prescribes both A and R with probability $\frac{1}{2}$. However, for notational convenience A is prescribed but we assume that this case does not occur, i.e. $\mathbb{P}[y = \frac{1}{2}] = 0$.

Since voters have the possibility to abstain from voting, the proportion of yes-voters y is not directly observable. This paper analyzes whether a referendum can be designed in such a way that the population majority outcome always occurs.

2.2. **The Voters.** A voter who is in favor of the proposal is referred to as a yes-voter, a voter who is against the proposal as a no-voter. The typical yes-voter will be indicated by index i and the typical no-voter by index j. Whether a voter will indeed participate depends on her net benefit of doing so. A voter participates in the referendum if her net utility of doing so is positive. In our model, this net utility of voting has the form proposed by Riker and Ordeshook (1968). As in their model, the net utility consists of three terms: i) a cost of voting; ii) a "consumption benefit from the act of voting" and iii) a utility from the outcome of the referendum depending on its probability of occurrence. The main

[2]It is possible to allow for voters who are indifferent with respect to the proposal by assuming that this group has a fixed size and that due to a lack of motivation these voters do never participate in the referendum.

difference between their model and mine is how the utility depends on the outcome of the referendum.

A voter who decides to indeed cast her vote, incurs a cost $c > 0$ representing the effort to go to the ballot box. Since there is a continuum of voters, the impact of a single voter on the outcome is nil. If voters were only concerned about the strategic benefit of voting and its cost, this would lead to the well-known paradox that none of the voters would take the effort to cast a ballot. Cultural theories of voting argue that the incorporation of "civic engagement" eliminates the paradox. In an empirical study, Blais, Young and Lapp (2000) find support for this hypothesis. In explaining voter turnout, the cost of voting and a return depending on the outcome of the referendum matter, but only among the voters with a relatively weak civic engagement.

In the model this civic engagement is a moral pressure to vote that differs across voters. Let m_i be the moral pressure of yes-voter i. The moral pressure of a yes-voter has a uniform distribution on the interval $[\bar{m}^y - \frac{\alpha}{2}, \bar{m}^y + \frac{\alpha}{2}]$ so that the average moral pressure of yes-voters is given by $\bar{m}^y$. Similarly, assume that the moral pressure of no-voters has a uniform distribution on the interval $[\bar{m}^n - \frac{\alpha}{2}, \bar{m}^n + \frac{\alpha}{2}]$. The moral pressure is felt as a disutility when a voter is not voting. Since there are no strong arguments why yes- and no-voters should have differently shaped moral pressure distributions, they are taken as identical. Hence, the scaling parameter α that determines the within-group heterogeneity is the same for both sides. The average moral pressures though can be different. This allows for the proposal to unequally affect the yes- and no-voters, so that one side might be more inclined to vote. Different average moral pressures can thus cause a bias towards accepting or rejecting the proposal.

The dependence on the outcome is modeled in the following way. A yes-voter wants the proposal to be accepted and derives utility in this case. The utility a yes-voter derives from acceptance of the proposal can depend on whether the voter indeed participates in the referendum or not. Let the utility of an accepted proposal for a participating yes-voter be γ^v, while it is γ^{nv} for a non-participating yes-voter.[3] Similarly, when the proposal is rejected, a participating no-voter derives utility γ^v while a non-participating no-voter derives utility γ^{nv}. For $\gamma^v > \gamma^{nv}$, voters derive more utility from their preferred outcome when they have participated. When the reversed inequality holds, a voter likes her preferred outcome best when it occurs without costing her any effort. If $\gamma^v \neq \gamma^{nv}$, the additional bias towards accepting or rejecting the proposal might either offset or strengthen the bias stemming from different average moral pressures.

[3]This is equivalent to the more elaborate modelling where disutility is derived from rejection of the proposal. For example, when participating yes-voters derive utility $\beta^{vA}\mathbb{P}[A]$ in case of acceptance and $\beta^{vR}\mathbb{P}[R]$ in case of rejection, the total utility is $(\beta^{vA} - \beta^{vR})\mathbb{P}[A] - \beta^{vR}$. Defining γ^v as $\beta^{vA} - \beta^{vR}$ and noting that the constant can be absorbed by rescaling of $\bar{m}^y$, as will be made clear below, gives the result.

	utility of yes-voter i	
voting	$\gamma^v \mathbb{P}^v[A] - c$	
not voting	$\gamma^{nv} \mathbb{P}^{nv}[A] - m_i$	
net utility of voting	$\gamma^v(\mathbb{P}^v[A] - \mathbb{P}^{nv}[A]) + (\gamma^v - \gamma^{nv})\mathbb{P}^{nv}[A]$	$-c + m_i$
Riker-Ordeshook	$\gamma^v(\mathbb{P}^v[A] - \mathbb{P}^{nv}[A])$	$-c + m_i$
this model	$(\gamma^v - \gamma^{nv})\mathbb{P}[A]$	$-c + m_i$

TABLE 1. Riker and Ordeshook (1968) assume that the utility of the outcome does not depend on participation, so $\gamma^v = \gamma^{nv}$. However, when the impact of a single voter is nihil, the probability of acceptance $\mathbb{P}[A]$ is independent of voter i's participation and the outcome only affects the participation decision through differences between γ^v and γ^{nv}.

The outcome of the referendum is unknown when the voters have to make their decisions. The *ex ante* expected utility thus depends on the probability of acceptance or rejection. Theoretically these probabilities can depend on whether a voter participates or not, so denote the probability of acceptance by $\mathbb{P}^v[A]$ when a voter participates and by $\mathbb{P}^{nv}[A]$ when she does not. For a yes-voter, the expected utility derived from the outcome of the referendum is thus $\gamma^v\mathbb{P}^v[A]$ or $\gamma^{nv}\mathbb{P}^{nv}[A]$ depending on whether she is participating or not.

The utilities of a yes-voter are summarized in Table 1, for a no-voter identical expressions hold when the probability of acceptance is replaced by the probability of rejection.[4] The net utility of voting is shown in the third line. The first term is a utility difference caused by voter i's impact on the outcome, the second term is a utility difference due to different valuations of the outcome when a voter participates or not. Econometricians would call the latter an interaction effect. It captures nonlinearities that arise from the participation and the outcome. Riker and Ordeshook (1968) assume that the utility of the outcome does not depend on the voter's decision, so $\gamma^v = \gamma^{nv}$. The outcome thus only affects voters' decisions through different probabilities of acceptance. However, the probability that a particular voter is pivotal is extremely small when the population is large. For example, consider a population of 5 million voters of which 50.1% is expected to be in favor. Feddersen (2004) uses a formula derived by Myerson (2000) to find estimates for the probability of a pivotal vote of the order 10^{-9}. This shows that even when γ^v and γ^{nv} are close, different valuations of the outcome may be far more important than the utility difference caused by the voter's impact. Although voter's tend to overestimate

[4]We implicitly assume that whenever a voter cast her ballot, she votes according to whether she is in favor or against. In other words, all voters are sincere. It is necessary to assume this since each voter is atomistic and her decision is not affecting the outcome. However, sincere voting is guaranteed when the voter's morality leads to a large negative utility when she votes for the non-preferred outcome.

their impact, as for example found be Blais et al. (2000), their biases should be of a very high order to outweigh the effects of different valuations.

To focus on how different valuations affect the referendum outcome, we abstain from the small impact of a single voter by assuming a continuum of voters. Hence, no strategic concerns are incorporated in the decision making process at the individual level.[5] The probability of acceptance does not depend on the voter's action and is denoted by $\mathbb{P}[A]$; the probability of rejection is then $\mathbb{P}[R] = 1 - \mathbb{P}[A]$. The expression of the net utility shows that the levels of the utilities derived from acceptance or rejection are not relevant for the behavior of the voters, only their difference matters. Define $\gamma = \gamma^v - \gamma^{nv}$ as the excess utility of the preferred outcome of voting relative to not-voting.

It is not clear what the sign of γ should be, or even whether it should be non-zero. We hence do not make any assumptions and discuss the model for all possible values of γ. When $\gamma = 0$, the outcome of the referendum is not relevant for the decision of a voter whether to vote or not. For this reason we refer to these voters as *simple-hearted voters*.

When $\gamma < 0$, the outcome of the referendum will give a higher utility when the voter does not cast her vote. This captures the feeling of a voter who likes her preferred outcome best if she does not have to do anything for it to occur. A higher probability of her preferred outcome makes a voter less willing to vote. This resembles the "underdog effect" reported by Levine and Palfrey (2007) in a laboratorial experiment: voters supporting the less popular alternative have higher participation rates. Another way of interpreting this behavior is suggested by Haan and Kooreman (2003). For a finite number of voters they show that the side with the highest number of supporters can still be the most likely to lose due to free-riding behavior. When $\gamma < 0$ voters balance their moral pressures with the outcome of the referendum, and we therefore refer to them as *calculating voters*.

When $\gamma > 0$, the more likely it is that the preferred outcome will occur, the more likely a voter will participate. This represents a voter who wants to be part of the winning team: the higher the probability of winning, the more likely she wants to take action to support it. This is in line with the expressive voting model of Schuessler (2000) in which benefits from attachment to a collective lead to a preference for the winning party. For example, Ashworth, Geys and Heyndels (2006) find evidence that although in Belgian municipal elections turnout is highest when the largest party obtains a small majority, turnout is again stimulated when there is a clear winner with at least two thirds of the votes. Further support that some voters want to be a winner is given by Bartels (1988) who shows that the public opinion before US presidential elections tends towards the winner of the most recent primary election. Remarkably, Clausen (1968) finds that in post-election recall surveys the winning candidate's support is overestimated and concludes that apparently

[5]In Section 4 we will give interest groups the possibility to coordinate the individuals. This allows individuals to indirectly strategically affect the outcome.

too many people "remember" to have contributed to the victory. Since voters cluster together when $\gamma > 0$, we refer to them as *affectionate voters*.

The above expressions show that the cost c of casting the ballot can be absorbed in the mean moral pressures $\bar{m}^y$ and $\bar{m}^n$. Without loss of generality, the exposition of the model can thus focus on the case $c = 0$. It also shows that there is an alternative interpretation of the model in which all voters have the same moral pressures, but differ in their cost of voting.

2.3. **Equilibrium.** Since all voters have the same information, they make the same inference about $\mathbb{P}[A]$ and $\mathbb{P}[R]$. For notational convenience we assume that when a voter is indifferent between voting or abstaining will vote. An equilibrium can then be characterized by two switching points $-\gamma p$ and $-\gamma r$ such that yes-voter i only votes if $m_i \geq -\gamma p$, no-voter j only votes if $m_j \geq -\gamma r$, $\mathbb{P}[A] = p$ and $\mathbb{P}[R] = r$. Since $p + r = \mathbb{P}[A] + \mathbb{P}[R] = 1$, an equilibrium is fully characterized by p. To find the equilibria, it thus suffices to analyze for all $p \in [0, 1]$, whether $p - \mathbb{P}[A] = 0$ when the yes- and no-voter switching points are $-\gamma p$ and $-\gamma(1 - p)$ respectively.

Let $Y = \mathbb{P}[m_i \geq -\gamma p]$ denote the probability that yes-voter i will vote. Invoking the law of large numbers, see Judd (1985), Y also denotes the proportion of yes-voters who are voting. Hence, Y will be referred to as the propensity to vote of yes-voters. Similarly, define the propensity to vote of no-voters $N = \mathbb{P}[m_j \geq -\gamma(1 - p)]$. Then

$$Y = \min\left\{\max\left\{\frac{\bar{m}^y + \frac{\alpha}{2} + \gamma p}{\alpha}, 0\right\}, 1\right\} = \frac{1}{2} + \min\left\{\max\left\{\frac{\bar{m}^y + \gamma p}{\alpha}, -\frac{1}{2}\right\}, \frac{1}{2}\right\}. \quad (1)$$

A similar expression holds for N. Note that Y and N are both functions of p.

When the proportion of yes-voters equals y, the measure of participating yes-voters is given by yY and the measure of participating no-voters by $(1 - y)N$. The *participation rate* is thus given by $yY + (1 - y)N$. When $q \in [0, 1]$ denotes the quorum, the referendum is valid if $yY + (1 - y)N \geq q$. This is the *quorum condition*. When the referendum is valid, the proposal is accepted if the majority of the participating voters is in favor, so if $yY \geq (1 - y)N$ (for notational convenience the proposal is accepted when exactly half of the voters is in favor). This is the *majority condition*. In case the referendum is not valid, the preset default outcome $D \in \{A, R\}$ determines the outcome.

Table 2 relates the probabilities of accepting the proposal with the propensities to vote and the quorum. Suppose that the default outcome is rejecting the proposal, $D = R$ (the case $D = A$ follows from symmetric arguments). First suppose that yes-voters are more likely to participate than no-voters, so $Y > N$. A higher proportion y of yes-voters makes a valid referendum more likely since more voters will actually vote (a yes-voter is more likely to vote than a no-voter), and it makes it more likely that the proposal is accepted (there are more participating yes-voters). When the quorum is below $2NY/(Y + N)$, the quorum is relatively easily met and the majority condition determines the probability

condition	constraint	$\mathbb{P}[A]$
$Y > N$ and $q \leq \frac{2NY}{Y+N}$	majority	$\mathbb{P}[y \geq \frac{N}{Y+N}]$
$Y > N$ and $q \geq \frac{2NY}{Y+N}$	quorum	$\mathbb{P}[y \geq \frac{q-N}{Y-N}]$
$Y = N$	both	$\mathbb{P}[y \geq \frac{1}{2}]\mathbb{1}_{\{Y \geq q\}}$
$Y < N$	both	$\mathbb{P}[\frac{q-N}{Y-N} \geq y \geq \frac{N}{Y+N}]$

TABLE 2. Binding constraints and the probability of accepting the proposal when the default outcome is rejection.

of acceptance (note that for $q = 2NY/(Y+N)$ the majority and quorum constraint coincide). For a higher quorum instead it is determined by the quorum constraint. Now suppose that $Y < N$. A higher fraction of yes-voters y makes a valid referendum less likely since less voters will actually vote (a yes-voter is less likely to vote than a no-voter), but if the referendum is valid it is more likely that the proposal is accepted (there are more participating yes-voters). Both constraints are binding, the quorum constraint from above, the majority constraint from below. Note that when $Y = N$, the quorum can only be met if $q \leq Y = N$. In this case the probability of accepting is determined by the majority condition.

An equilibrium in case $D = R$ is thus a solution of $p - \mathbb{P}[A] = 0$, where $\mathbb{P}[A]$, Y and N are as discussed above. This equilibrium characterization is at the core of the analysis.

3. The Quorum and the Population Majority Outcome

3.1. **Simple-Hearted Voters.** Suppose that the voters are simple-hearted, so $\gamma = 0$. The expectations about the outcome of the referendum do not affect the voter's decision whether to vote or not. This implies that the choice of the quorum does not affect the propensities to vote. Any bias that stems from different average moral pressures can thus be directly addressed by a quorum. The following proposition states that with the right choice of the quorum and the default option, the population majority outcome occurs.

Proposition 1. (Simple-Hearted Voters and the Population Majority Outcome)
Assume that $\gamma = 0$ *and* $\bar{m}^y, \bar{m}^n \in (-\frac{\alpha}{2}, \frac{\alpha}{2})$.
i) When $\bar{m}^y = \bar{m}^n$*, the population majority outcome is only achieved in the unique equilibrium of the referendum with a quorum of at most* $q^* = \frac{1}{2} + \frac{\bar{m}^y + \bar{m}^n}{2\alpha}$ *and default outcome* $D \in \{A, R\}$.
ii) When $\bar{m}^y \neq \bar{m}^n$*, the population majority outcome is only achieved in the unique equilibrium of the referendum with quorum* $q^* = \frac{1}{2} + \frac{\bar{m}^y + \bar{m}^n}{2\alpha}$ *and default outcome* $D = R$ *if* $\bar{m}^y > \bar{m}^n$ *and* $D = A$ *otherwise.*

In order to discuss the implications of the proposition, it is insightful to look first at the propensities to vote. The condition that $\bar{m}^y$ and $\bar{m}^n$ are contained in $(-\frac{\alpha}{2}, \frac{\alpha}{2})$ implies that they are given by $Y^* = \frac{1}{2} + \bar{m}^y/\alpha$ and $N^* = \frac{1}{2} + \bar{m}^n/\alpha$ and that they are contained in $(0, 1)$, see Equation (1). This assures that on each side some voters do abstain from voting while others cast their votes. It hence excludes the less relevant cases where all voters of a side vote or all of them do not vote. The first statement of the proposition assumes that the propensities to vote are equal for yes- and no-voters. Obviously, a majority of yes-voters in the whole population, $y \geq \frac{1}{2}$, will then lead to a majority of yes-voters among the participating voters. The participation rate is constant and equal to $yY^* + (1 - y)N^* = Y^* = N^*$. In this case, any quorum below or equal to the propensity Y^* or N^* is automatically met and the default outcome is free to choose (in the proposition the average propensity $\frac{1}{2}(Y^* + N^*)$ is used to stress the similarity with the optimal quorum in the second statement). Since the majority of the participating voters perfectly reflects the majority among the population, the population majority outcome is achieved. Note especially that the quorum $q = 0$ is allowed, which is identical to the case of not having a quorum. Intuitively, when the propensities to vote are equal, there is no bias towards accepting or rejecting the proposal and no quorum is needed. However, since the participation rate is constant, any sufficiently low quorum does no harm.

The second statement assumes that the propensities to vote are different. With the found expressions for Y^* and N^*, the optimal quorum can be expressed as the average propensity to vote $\frac{1}{2}(Y^* + N^*)$. To see why this is the case, assume that $\bar{m}^y > \bar{m}^n$ (symmetric arguments hold for the opposite case). This assumption implies that $Y^* > N^*$. Yes-voters are more likely to vote and without a quorum there is a bias towards accepting the proposal. When a quorum is introduced, it can only offset this bias if the default outcome is rejecting the proposal, $D = R$. The participation rate $yY^* + (1 - y)N^*$ is strictly increasing in y. This shows that a majority of the population is in favor of the proposal, $y \geq \frac{1}{2}$, if and only if the participation rate is higher than $\frac{1}{2}(Y^* + N^*)$. The population majority outcome can thus be achieved by the quorum $q^* = \frac{1}{2}(Y^* + N^*)$. Note that the majority constraint is redundant: whenever the referendum is valid, a majority of the participating voters is in favor of the proposal. Instead of the fraction of participating voters in favor, the participation rate is the decisive variable. The model thus has a strong prediction: for a correctly set quorum the default outcome will never occur as the outcome of a valid referendum.

At first sight it might seem counterintuitive that the optimal quorum is increasing in the propensity to vote of both yes- and no-voters: the bias towards accepting is increased when yes-voters become more likely to vote, but it is decreased when no-voters become more likely to vote. An increased bias might need a higher quorum and a decreased bias a lower quorum. This reasoning correctly assesses the effect on the bias *in the absence of a* *quorum*. However, when the optimal quorum is imposed, the previous paragraph showed that the majority constraint is redundant. An increase in the propensity to vote of yes-voters has an identical effect on the quorum constraint as an increase in the propensity to vote of no-voters. More voters will indeed vote, so the quorum is more likely to be met and the probability of accepting the proposal is increased. To achieve the population majority outcome, an increase in the quorum is needed.

3.2. **Calculating Voters.** Now suppose that the voters are calculating, so $\gamma < 0$. The potential disutility of an unnecessary vote makes that less voters indeed take the effort to cast their ballots compared to the simple-hearted voters. *Ceteris paribus*, this leads to a lower optimal quorum. To construct a referendum that achieves the population majority outcome, the probability of a majority of yes-voters among the whole population is needed. Let ξ denote this probability, so $\xi = \mathbb{P}[y \geq \frac{1}{2}]$. From the assumptions on the distribution of y it follows that $\xi \in (0,1)$. The following proposition states that with the right design of the referendum, the population majority outcome occurs.

Proposition 2. (Calculating Voters and the Population Majority Outcome)
Assume that $\gamma < 0$ and $\bar{m}^y, \bar{m}^n \in (-\frac{\alpha}{2} - \gamma, \frac{\alpha}{2})$.
i) When $\bar{m}^y = \bar{m}^n + \gamma(1-2\xi)$, the population majority outcome is only achieved in the unique equilibrium of the referendum with a quorum of at most $q^* = \frac{1}{2} + \frac{\bar{m}^y + \bar{m}^n + \gamma}{2\alpha}$ and default outcome $D \in \{A, R\}$.
ii) When $\bar{m}^y \neq \bar{m}^n + \gamma(1-2\xi)$, the population majority outcome is only achieved in the unique equilibrium of the referendum with quorum $q^* = \frac{1}{2} + \frac{\bar{m}^y + \bar{m}^n + \gamma}{2\alpha}$ and the default outcome $D = R$ if $\bar{m}^y > \bar{m}^n + \gamma(1-2\xi)$ and $D = A$ otherwise.

The intuition for the proposition follows again from first looking to the propensities to vote. In the population majority outcome the probability that the proposal is accepted is given by ξ. The probability that the proposal is rejected is then given by $1-\xi$. This means that the propensities to vote of yes-voters and no-voters are given by $Y^* = \frac{1}{2} + (\bar{m}^y + \gamma\xi)/\alpha$ and $N^* = \frac{1}{2} + (\bar{m}^n + \gamma - \gamma\xi)/\alpha$ respectively. The condition that $\bar{m}^y$ and $\bar{m}^n$ are contained in $(-\frac{\alpha}{2} - \gamma, \frac{\alpha}{2})$ implies that for all $\xi \in (0,1)$ the propensities to vote Y^* and N^* are between 0 and 1. In other words, the condition ensures that for a fraction γ/α of the voters indeed their voting decisions depend on their expectations (that $\gamma < \alpha$ follows from the same condition). The first statement of the proposition now claims that when the propensities to vote are equal for yes- and no-voters, the referendum with a quota below or equal to $\frac{1}{2}(Y^* + N^*)$ achieves the population majority outcome. The reason is the same as for the simple-hearted voters: with equal propensities to vote the fractions of yes- and no-voters among the participating voters are identical to the population fractions. No quorum is needed, but a sufficiently small quorum does not affect the outcome of the referendum since the participation rate is constant.

When the propensities are not equal, according to the second statement a quorum is needed to achieve the population majority outcome. In fact, the optimal quorum is again the average of the propensities to vote, but now evaluated at the equilibrium, $q^* = \frac{1}{2}(Y^* + N^*)$. To get more intuition, assume that $\bar{m}^y > \bar{m}^n + \gamma(1 - 2\xi)$ (symmetric arguments hold for the opposite case). This implies that $Y^* > N^*$. Similar to the model with simple-hearted voters, a quorum with rejecting as default outcome, $D = R$, is needed to offset the bias towards accepting. The participation rate $yY^* + (1 - y)N^*$ is strictly increasing in y. The majority of the population is in favor if and only if the participation rate is higher than $\frac{1}{2}(Y^* + N^*)$. Since in this case the yes-voters constitute a majority, the quorum $q^* = \frac{1}{2}(Y^* + N^*)$ achieves the population majority outcome.

Compared to the model with simple-hearted voters, there are two important differences. Firstly, *ceteris paribus* the optimal quorum is lower in case of calculating voters. Comparing the expressions for q^* in the second statements of Propositions 1 and 2 shows that in the model with calculating voters the quorum is $-\gamma/\alpha$ lower. Some of the voters who would have cast their ballot when they would have been simple-hearted, prefer not to do so when they are calculating. A lower quorum is needed to offset a lower participation rate. This shows that when the referendum is designed for a population of simple-hearted voters while instead the voters are calculating, the quorum is set too high. In case $\bar{m}^y > \bar{m}^n + \gamma(1 - 2\xi)$, the quorum will only be met when the true proportion of yes-voters is at least y^* for $y^* > \frac{1}{2}$. The proposal is thus rejected for $y \in [\frac{1}{2}, y^*)$. When $\mathbb{P}[y \in [\frac{1}{2}, y^*)] > 0$, the referendum with the incorrectly set quorum will not achieve the population majority outcome and there is a tendency towards the default outcome R.

A second difference compared to the model with simple-hearted voters is that the design of the optimal referendum requires knowledge of $\xi = \mathbb{P}[y \geq \frac{1}{2}]$. Somewhat surprisingly, this knowledge is not needed for setting the optimal quorum. Instead, the knowledge of ξ is needed for setting the default outcome optimally. Intuitively, for the optimal quorum only the sum of the reductions in voters matters, while for the optimal default outcome the difference matters. When $\gamma = 0$ the propensity to vote is independent of the expectations. However, when $\gamma < 0$ the propensities to vote will in general depend on γ. Only when a population majority of yes- and no-voters is equally likely, so $\xi = \frac{1}{2}$, the default outcomes coincide with those in case of simple-hearted voters. When $\xi \neq \frac{1}{2}$, there will be fewer participating yes- and no-voters in equilibrium than in case of simple-hearted voters. When $\xi > \frac{1}{2}$, the decrease in yes-voters is larger than the decrease in no-voters. The choice of the default outcome needs to take account of this effect. The term $\gamma(1-2\xi)$ in the conditions accomplishes this. This effect is increasing in the extent to which voters calculate, γ. Note that the model with simple-hearted voters can be seen as the limiting case of the model with calculating voters and $\gamma \to 0$.

3.3. **Affectionate Voters.** Now consider the model with affectionate voters, so $\gamma > 0$. The expectations about the outcome of the referendum again matter. But now the higher the probability that the preferred outcome occurs, the more likely that a voter indeed casts her ballot. *Ceteris paribus*, this leads to more participating voters and hence to a higher optimal quorum than in case of simple-hearted voters. Compared to those voters, the affectionate voters have a tendency to behave in a coordinated way. This gives raise to the possibility of multiple equilibria. The following proposition states that although the referendum can be designed such that the population majority outcome occurs, under a certain condition there is indeed another equilibrium.

Proposition 3. (Affectionate Voters and the Population Majority Outcome)
Assume that $\gamma > 0$ *and* $\bar{m}^y, \bar{m}^n \in (-\frac{\alpha}{2}, \frac{\alpha}{2} - \gamma)$.
i) The population majority outcome is achieved in an equilibrium of the referendum designed as specified in Proposition 2.
ii) For the quorum q^**, the equilibrium mentioned in i) is the unique equilibrium when* $|\bar{m}^y - \bar{m}^n| \geq \gamma$*, otherwise there is a single alternative equilibrium which is characterized by* $\mathbb{P}[D] = 1$.

The first statement shows that the expressions for the optimal quorum in case of calculating voters also hold for affectionate voters. Compared to the model with simple-hearted voters, the optimal quorum is higher with affectionate voters since voters are more likely to participate. Comparing the expressions for the optimal quorum of the three models shows that the quorum is increasing in the extent of affection γ (or decreasing in the extent voters calculate $-\gamma$).

The proposition states that multiple equilibria can indeed arise. The second statement claims that when $\bar{m}^y$ and $\bar{m}^n$ are sufficiently close to each other, the optimal quorum does not necessarily lead to the population majority outcome.[6] In fact, this quorum can discourage the opponents of the default outcome from voting, an effect that is aggravated by the tendency to coordinate. This might give raise to an equilibrium where none of the voters expects the quorum to be met and because the voters adapt their behavior to this expectation, the quorum will indeed never be met. When $|\bar{m}^y - \bar{m}^n| < \gamma$ the fact that voters base their decisions to vote on expectations together with their tendency to coordinate gives rise to self-fulfilling equilibria. When instead the difference between $\bar{m}^y$ and $\bar{m}^n$ is sufficiently big, the equilibrium with $\mathbb{P}[D] = 1$ is not feasible anymore. To see why, suppose $\bar{m}^y \geq \bar{m}^n + \gamma$. Even when $\mathbb{P}[R] = 1$ the propensity to vote of yes-voters is (weakly) higher as that of no-voters. There will be a positive probability of accepting the proposal, which is a contradiction.

[6]In case $\bar{m}^y = \bar{m}^n + \gamma(1 - 2\xi)$ and $q < q^*$ the equilibrium can be unique, but there can also be two other equilibria, see Appendix B for details.

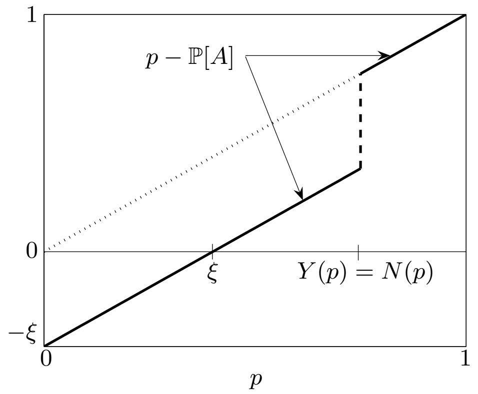

FIGURE 1. In case of calculating voters the optimal quorum leads to a unique equilibrium with the population majority outcome ($\mathbb{P}[A] = \xi$).

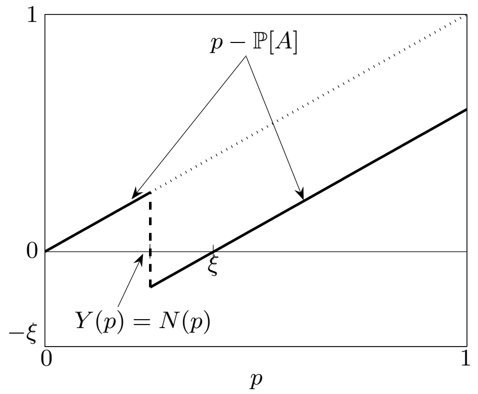

FIGURE 2. In case of affectionate voters the optimal quorum can also lead to a second equilibrium in which the proposal is never accepted ($\mathbb{P}[A] = 0$).

A graphical representation provides additional insight in why the equilibrium is necessarily unique for the calculating voters but not for the affectionate voter. In Figures 1 and 2, $p - \mathbb{P}[A]$ is shown as function of p for calculating and affectionate voters respectively. Recall that in equilibrium $p - \mathbb{P}[A] = 0$. In case of calculating voters, $\gamma < 0$, the propensity to vote $Y = \frac{1}{2} + (\bar{m}^y + \gamma p)/\alpha$ is decreasing in p. The propensity to vote $N = \frac{1}{2} + (\bar{m}^n + \gamma - \gamma p)/\alpha$ is increasing in p at the same rate. The participation rate for $y = \frac{1}{2}$ is thus independent of p. But as discussed above, for the optimal quorum only the quorum constraint is binding. This implies that for all p the quorum constraint is also satisfied if and only if $y \geq \frac{1}{2}$. For small p the probability of accepting the proposal is then ξ until the no-voters are more likely to participate than yes-voters. In this case the quorum constraint and the majority constraint cannot be simultaneously met and $\mathbb{P}[A] = 0$. The function $p - \mathbb{P}[A]$ is thus strictly increasing and has a un upwards jump. Since it is increasing, is crosses the x-axis at most once. The choice of the default outcome implies that the jump is after ξ, so that indeed an equilibrium exists.

In case of affectionate voters, $\gamma > 0$, Y is increasing in p and N decreasing. Arguments opposite to the ones above show that $\mathbb{P}[A]$ is zero for small p, while it jumps to ξ for larger p. This implies that $p - \mathbb{P}[A]$ is not strictly increasing in p. There can be two equilibria: one with $\mathbb{P}[A] = 0$ and one with $\mathbb{P}[A] = \xi$. The choice of the default outcome guarantees that the latter equilibrium exists. When $|\bar{m}^y - \bar{m}^n| < \gamma$, yes-voters have a lower propensity to vote than no-voters for $p = 0$. This implies that the quorum constraint and the majority constraint cannot be simultaneously. Since then $\mathbb{P}[A] = 0$, there is a second equilibrium in which the default outcome always occurs.

3.4. **Heterogenous Voter Types.** We now allow for heterogenous voters. To be more specific, the population can consist of simple-hearted, calculating and affectionate voters. Moreover, the parameters α and γ can differ across voters. This means that a voter k is defined by her preference, i.e. in favor or against the proposal, and the parameters $(\bar{m}_k, \alpha_k, \gamma_k)$. Define the parameter set $\mathcal{P} = \mathbb{R} \times (0, \infty) \times \mathbb{R}$. Now define the subset $\hat{\mathcal{P}}$ of $\mathcal{P}$ as follows

$$\hat{\mathcal{P}} = \left\{ (\bar{m}, \alpha, \gamma) \in \mathcal{P} \;\middle|\; \bar{m} \in \Big(-\frac{\alpha}{2} + \max\{0, -\gamma\}, \frac{\alpha}{2} + \min\{0, -\gamma\} \Big) \right\}.$$

Note that this restriction resembles the assumptions on $\bar{m}^y$ and $\bar{m}^n$ in Proposition 1-3. In fact, for any parameters $(\bar{m}_k, \alpha_k, \gamma_k) \in \hat{\mathcal{P}}$ the assumption in the proposition indicated by γ_k is satisfied for $\bar{m}_k$, α_k and γ_k. Denote the distribution function of the parameters of yes-voter i by Φ^y and of no-voter j by Φ^n. By the law of large numbers, Φ_y and Φ^n are also the population distributions. Denote the density functions by ϕ^y and ϕ^n respectively. The first condition on the density functions is that $\phi^y(\bar{m}_k, \alpha_k, \gamma_k) = \phi^n(\bar{m}_k, \alpha_k, \gamma_k) = 0$ if $(\bar{m}_k, \alpha_k, \gamma_k) \notin \hat{\mathcal{P}}$. This assures that of all the yes- or no-voters with a type $(\bar{m}_k, \alpha_k, \gamma_k)$ that can occur, some will indeed vote while others will not. Now define the following average parameters of the yes-voters

$$\bar{m}^y = \mathbb{E}_y\left[\frac{\bar{m}_i}{\alpha_i}\right] = \int_{\hat{\mathcal{P}}} \frac{\bar{m}_i}{\alpha_i} \mathrm{d}\Phi^y(\bar{m}_i, \alpha_i, \gamma_i),$$
$$\gamma^y = \mathbb{E}_y\left[\frac{\gamma_i}{\alpha_i}\right] = \int_{\hat{\mathcal{P}}} \frac{\gamma_i}{\alpha_i} \mathrm{d}\Phi^y(\bar{m}_i, \alpha_i, \gamma_i).$$

Denote the counterparts for the no-voters by $\bar{m}^n$ and γ^n. The second condition on the density function is that $\gamma^y = \gamma^n$. Since this is equivalent to $\mathbb{E}_y[\gamma_i/\alpha_i] = \mathbb{E}_n[\gamma_j/\alpha_j]$, this condition is satisfied if for example $\bar{m}_k$ and (α_k, γ_k) are independently distributed and the density function for (α_k, γ_k) is independent of being in favor or against the proposal. The common density function is the analogue of the assumption made in the previous section that γ is a population parameter and that the scaling parameter α of the moral pressure distribution is equal for both voter groups. Although this assumption is mainly made to keep the model tractable, there are no reasons to assume that γ^y and γ^n are very different. When they are close to each other, the outcomes will be similar to when they are identical. Define $\gamma = \gamma^y = \gamma^n$. The second condition implies that both the average type, i.e. simple-hearted, calculating or affectionate, and the extent of the affection (or the extent to which voters are calculating) scaled by α are equal among yes- and no-voters. The following proposition claims that knowledge of these average parameters together with $\xi = \mathbb{P}[y \geq \frac{1}{2}]$ is sufficient to design a referendum that achieves the population majority outcome.

Proposition 4. (Heterogenous Voters and the Population Majority Outcome)
Assume that the supports of Φ^y *and* Φ^n *are contained in* $\hat{\mathcal{P}}$ *and that* $\mathbb{E}_y[\frac{\gamma_i}{\alpha_i}] = \mathbb{E}_n[\frac{\gamma_j}{\alpha_j}]$. *Then, the quorum, default outcome and uniqueness of the population majority outcome are as in the model with only the representative voter types defined by* $(\bar{m}^y, 1, \gamma)$ *and* $(\bar{m}^n, 1, \gamma)$.

The proposition states that when the population consists of simple-hearted, calculating and affectionate voters and when the other parameters are allowed to vary across the voters, the quorum and default options should be set as for the population that only consists of the representative voter types $(\bar{m}^y, 1, \gamma)$ and $(\bar{m}^n, 1, \gamma)$. Hence, the analysis in the first three subsections is not a simplification but instead describes models with heterogenous voter types as well. When the signs and sizes of individual γ_k's can be different, an increase in p has different effects on voters with different γ_k's. In case of different signs, it makes some voters more willing to vote and others less. Only the average effect counts for setting the optimal quorum. Note that the representative voter types also determine whether the optimal quorum necessarily results in the population majority outcome or that the equilibrium with $\mathbb{P}[D] = 1$ can occur as well.

4. A Non-Optimal Quorum

In this section we analyze the consequences of a non-optimal quorum. There are two reasons why a non-optimal quorum can arise. Firstly, the quorum could have been set non-optimally due to insufficient knowledge about the relevant parameters or for political reasons. Secondly, after the quorum is set, whether optimally or not, pressure groups have incentives to affect the behavior of voters in order to make their preferred outcome more likely.

Throughout it is assumed that the proportion of yes-voters y has a uniform distribution on $[\underline{y}, \overline{y}]$ with $\underline{y} < \frac{1}{2} < \overline{y}$. Let ϕ denote the density, so $\phi = (\overline{y} - \underline{y})^{-1}$. The probability of accepting the proposal according to the population majority is then given by $\xi = \phi(\overline{y} - \frac{1}{2})$.

The analyses for the default outcomes A and R are symmetric. We assume $D = R$ so the proposal can only be accepted when the referendum is valid and when a majority of the participating voters is in favor.

4.1. A Not-Optimally Set Quorum.

First consider the simple-hearted voters with $\gamma = 0$. The outcome of the referendum does not affect the behavior of the voters so the propensities to vote Y and N are fixed. When it is known which constraints are binding, the probability of accepting the proposal can be computed in a straightforward manner using the three cases considered in Subsection 2.3. Denote this probability by p_m when only the majority constraint is binding, by p_q when only the quorum constraint is binding and by p_b when both constraints are binding. Let s denote the sum of the propensities to vote, so $s = Y + N$. These probabilities of accepting the proposal given the binding constraints are then

$$p_m = \phi\left(\overline{y} - \frac{N}{s}\right),$$

$$p_q = \phi\frac{\overline{y}Y + (1-\overline{y})N - q}{Y - N},$$

$$p_b = \phi\frac{q - \frac{2YN}{s}}{Y - N}.$$

To analyze the effect of the quorum on the probability of accepting the proposal, these equilibrium probabilities are related to the quorum in the following proposition. Instead of framing the proposition in terms of the deep parameters $\bar{m}^y$, $\bar{m}^n$, α and γ, it is easier to use Y and N.

Proposition 5. (Simple-Hearted Voters and a Not-Optimally Set Quorum)
Suppose $\gamma = 0$.
i) Suppose $\bar{m}^y \geq \bar{m}^n$ *and* $\frac{N}{s} > \underline{y}$. *Then*

$$\mathbb{P}[A] = \begin{cases} p_m & \text{if } q \leq \frac{2YN}{s}, \\ p_q & \text{if } \frac{2YN}{s} < q \leq \overline{y}Y + (1-\overline{y})N, \\ 0 & \text{if } \overline{y}Y + (1-\overline{y})N < q. \end{cases}$$

ii) Suppose $\bar{m}^y < \bar{m}^n$ *and* $\frac{N}{s} < \overline{y}$. *Then*

$$\mathbb{P}[A] = \begin{cases} p_m & \text{if } q \leq \overline{y}Y + (1-\overline{y})N, \\ p_b & \text{if } \overline{y}Y + (1-\overline{y})N \leq q \leq \frac{2YN}{s}, \\ 0 & \text{if } \frac{2YN}{s} \leq q. \end{cases}$$

A first observation is that for every quorum an equilibrium exists. To see why this is the case, the function $p - \mathbb{P}[A]$ is key. Although for the optimal quorum q^* this function is discontinuous in p, it is continuous for a non-optimal quorum. Together with the fact that $\mathbb{P}[A] \in [0, 1]$ this shows that there is at least one $p \in [0, 1]$ for which $p - \mathbb{P}[A] = 0$. There thus exists an equilibrium.

When the propensity to vote is higher for yes-voters than for no-voters, $Y > N$, the default outcome is correctly set. This case is discussed in the first statement of the proposition and depicted in Figure 3. The probability of acceptance is constant for a low quorum. The quorum will always be met and the majority constraint is binding. The definition of p_m shows that in this case $p_m > \xi$. Intuitively, for a quorum below the optimal quorum q^*, the referendum will be too often valid and the probability of acceptance is above $\mathbb{P}[y \geq \frac{1}{2}]$. Note that the condition $N/s > \underline{y}$ implies that $p_m < 1$. When q increases, more participating voters are needed to meet the quorum. Since $Y > N$, the required proportion of yes-voters increases. When q increases further, the quorum constraint takes over from the majority constraint. The probability of acceptance decreases and crosses ξ. For higher q it can reach a level such that even with the highest participation rate

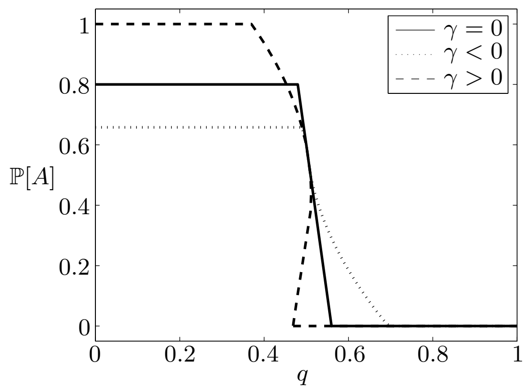

FIGURE 3. The effect of the quorum on the probability of acceptance when the default outcome is correctly set.[7]

$\overline{y}Y + (1-\overline{y})N$ the quorum can not be met. From here on, the probability of acceptance equals zero.

When the propensities to vote for yes- and no-voters are equal, the participation rate is constant. The quorum constraint is either always satisfied or never. According to the first statement, p_q does not occur since the two borders are equal. The probability of acceptance suddenly drops from $p_m = \xi$ to 0 if q raises above $\frac{1}{2}s$.

The second statement assumes that the default outcome is incorrectly set. The definition of p_m shows that even when the quorum is so low that it is not affecting the referendum, the probability of acceptance is below the population majority outcome ξ. The condition $N/s < \underline{y}$ implies that the probability of acceptance is positive. When the quorum constraint becomes binding, it imposes an upper bound on the proportion of yes-voters. Since the propensity to vote is lower for yes-voters than for no-voters, the quorum will not be met when there are too many yes-voters. When the quorum is higher than $2YN/s$ more than half of the participating voters should be no-voters, but then the majority constraint cannot be satisfied and the probability of accepting the proposal is zero.

The proposition shows that when the quorum is lower than the optimal quorum q^*, the probability of accepting the proposal is at most p_m. It also shows that when the quorum is set higher than the optimal quorum, it can be 0. Especially when the difference between the average moral pressures $\bar{m}^y$ and $\bar{m}^n$ is small, so that Y and N are similar and p_m is close to ξ, it is less harmful when the quorum is set too low than too high. Moreover,

[7] This figure uses $\alpha = 2$, $\gamma \in \{-0.9, 0, 0.9\}$, $\underline{y} = 0.3$, $\overline{y} = 0.8$ and thus $\xi = 0.6$. Since the range of admissible values of $\bar{m}^y$ and $\bar{m}^n$ is determined by γ, the average moral pressures need to be adjusted for different values of γ. Using $\bar{m}^y = 0.2$ and $\bar{m}^n = -0.2$ when $\gamma = 0$, the adjustment $\bar{m}^y = 0.2 - \frac{1}{2}\gamma$ and $\bar{m}^n = -0.2 - \frac{1}{2}\gamma$ achieves that the optimal quorum is the same for all γ and equal to $\frac{1}{2}$.

suppose that the exact values of $\bar{m}^y$ and $\bar{m}^n$ are not known. When the quorum is based on their expected values, it will be as often too low as too high. But to assess the effect on the outcome, it is important that a too high quorum is more harmful. Hence, the uncertainty about the average moral pressures causes the proposal to be rejected too often.

When $\gamma \neq 0$, the propensities to vote depend on the probability that the proposal is accepted, which in turn depends on the propensities to vote. As in the case for the simple-hearted voters, the equilibrium probabilities can be computed if it is known which constraints are binding. We here discuss the results using Figure 3; Appendix B contains the precise statements.

For the model with calculating voters, so $\gamma < 0$, an equilibrium exists for every quorum when γ is not too negative. This ensures that changes in the probability of accepting the proposal do not have too big impacts. Note that the interpretation of γ as the average across heterogenous voters suggest that the value of γ is not that extreme. Since the calculating voters show some "balancing" behavior, changes in q effect the equilibrium probability more gradually than for the simple-hearted voters. The effects of a not-optimally set quorum are thus similar though less severe.

The model with affectionate voters, $\gamma > 0$, is more complicated. Here, an upper bound on γ is needed to limit the effect of the equilibrium probability on the voters. As was shown in the previous section, even for the optimal quorum two equilibria can exist. When the quorum is not optimally set there can be up to three equilibria.[8] As before, multiple equilibria can arise since the model resembles a coordination game. Voters act according to what they expect and thereby make their expectations happen, in other words, there are self-fulfilling prophecies. Changes in q thus have a larger impact than for the simple-hearted voters. Note especially that when the quorum is set already slightly too high (in the figure the optimal quorum is 0.5), a sure rejection will result. Again, setting the quorum a bit too low is less harmful than setting it a bit too high.

In case of three equilibria, the middle one only serves to separate the others. This equilibrium is unstable in the sense that when a small fraction of voters changes behavior, this would trigger changes in the behavior of other voters that would ultimately lead to one of the other equilibria. Although their instability makes them less appealing, they cannot be completely ignored in the analysis. Clearly, the properties of the stable and unstable equilibria are opposites. So, a higher quorum decreases the probability of acceptance in the stable equilibria with a positive probability, but increases it in the unstable equilibria.

4.2. **Pressure Groups.** After the quorum is set, pressure groups might want to affect the turnout of the voters. For example, in the Italian referendum no-voters were urged

[8]Although it cannot be seen from the figure, there is a hole in the graph when $\gamma > 0$: for the optimal quorum $q^* = \frac{1}{2}(Y(\xi) + N(\xi))$ the equilibrium in the middle does not exist conform Proposition 3.

to go to the beach instead of the ballot box.[9] In our model, we assume that pressure groups cannot directly affect the behavior of voters of the other side: a yes-pressure group can only affect the average moral pressure of the yes-voters $\bar{m}^y$ and a no-pressure group only the average moral pressure of no-voters $\bar{m}^n$. In essence, the model has become a group-based voting model of mobilization.

We still assume that the preferences of the voters are given. Although before this was already a simplification, in the face of pressure groups, it needs even more justification. Apart from affecting the participation rate of their side, these pressure groups have of course incentives to try to convert voters. For example, Neijens and van Praag (2006) discuss the dynamics of opinion formation and show that a large fraction of the voters changes their opinion in the period before the election. The assumption that voters' preferences are given thus implies that the model deals with the short period directly preceding the referendum day. Since affecting the participation rate is just a part of the pressure group strategy, we will only analyze its marginal effect. Its sign already indicates in which direction a pressure group should affect the voters. Herrera and Mattozzi (2007) discuss a referendum model where pressure groups setting the participation rates play against each other.

4.2.1. *Yes-Pressure Groups.* The equilibrium probabilities of accepting the proposal follow from rewriting the conditions stated in Proposition 5. The analysis of the not-optimally set quorum dealt separately with a correctly and an incorrectly set default outcome. When the effect of the average moral pressures is analyzed, it matters wether the moral propensity to vote of the other side is above or below the quorum. Remember that the propensities to vote Y and N should be between 0 and 1.

Proposition 6. (Simple-Hearted Voters and Yes-Pressure Groups)
Suppose $\gamma = 0$ and $N > \frac{\underline{y}}{1-\underline{y}}$.
i) Suppose $N \leq q$. Then

$$\mathbb{P}[A] = \begin{cases} 0 & \text{if } Y < \frac{q-(1-\overline{y})N}{\overline{y}}, \\ p_q & \text{if } N \leq \frac{1}{2}q \quad \text{and} \quad \frac{q-(1-\overline{y})N}{\overline{y}} \leq Y, \\ & \text{or if } N > \frac{1}{2}q \quad \text{and} \quad \frac{q-(1-\overline{y})N}{\overline{y}} \leq Y \leq \frac{qN}{2N-q}, \\ p_m & \text{if } N > \frac{1}{2}q \quad \text{and} \quad \frac{qN}{2N-q} \leq Y. \end{cases}$$

ii) Suppose $N > q$. Then

$$\mathbb{P}[A] = \begin{cases} 0 & \text{if } Y \leq \frac{qN}{2N-q}, \\ p_b & \text{if } \frac{qN}{2N-q} \leq Y \leq \frac{q-(1-\overline{y})N}{\overline{y}}, \\ p_m & \text{if } \frac{q-(1-\overline{y})N}{\overline{y}} \leq Y. \end{cases}$$

[9]Hanafin (2006) discusses in detail the strategic lobbying that preceded the enacting of the fertility law in 2004 and the failure of the referendum.

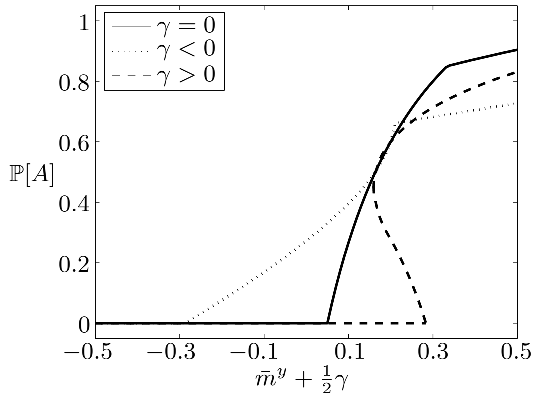

FIGURE 4. The effect of the average moral pressure of yes-voters on the probability of acceptance.[10]

The first statement assumes that the propensity to vote of no-voters is so low that the quorum is not met when everyone is against the proposal, when $N < q$, or exactly met when $N = q$. This case is depicted in Figure 4. Even for low values of the average moral pressure of yes-voters, the quorum will not be met. The propensity to vote needs to be higher than q before the quorum can be met to offset the low propensity of the no-voters. In this case the quorum constraint will be binding. Now suppose that N is not too low, so $N > \frac{1}{2}q$. When $\bar{m}^y$ is increased further, the quorum constraint is always met and it is the majority constraint that determines the equilibrium probability. When N is below $\frac{1}{2}q$, the majority constraint is always satisfied if the quorum constraint is satisfied. In this case the equilibrium probability remains p_q. The condition that $N > \underline{y}/(1-\underline{y})$ guarantees that $\mathbb{P}[A] < 1$. Comparison between this proposition and Proposition 5 shows that increasing Y is similar to decreasing q.

The second statement assumes that when all voters are no-voters, the quorum constraint is met. In this case, the quorum can already be met for $Y < q$. The quorum constraint is then binding from above, so the equilibrium probability is given by p_b. When Y is increased further, the quorum constraint is always satisfied. From here on p_m determines the equilibrium probability. Again, increasing Y is similar to decreasing q.

Figure 4 also shows the equilibria for calculating and affectionate voters. In both cases the equilibrium lines are similar to the mirrored images of those in Figure 3. This reflects that increasing the propensity to vote of yes-voters is comparable to decreasing the quorum. For the calculating voters there is again a unique equilibrium. The offsetting behavior leads to positive probabilities for lower values of Y and to smoother effects of $\bar{m}^y$ in

[10]Figure 4 uses $\bar{m}^n = -0.2 - \frac{1}{2}\gamma$, $\alpha = 2$, $\gamma \in \{-0.9, 0, 0.9\}$, $q = 0.5$, $\underline{y} = 0.3$ and $\overline{y} = 0.8$. Note that when $\gamma = 0$, Y ranges from 0.25 to 0.75.

general. Changes in the propensity to vote of yes-voters are partially undone by their own calculating attitude.

In case of affectionate voters multiple equilibria again exist for intermediate values of $\bar{m}^y$. The equilibria in the middle are unstable. Similar to the effect of a quorum slightly higher than the optimal quorum, a propensity to vote slightly below the value for which the quorum is optimal, which is $2q - N$, immediately leads to a sure rejection (in the figure the quorum is optimal for $\bar{m}^y + \frac{1}{2}\gamma = 0.2$).

For all voter types, an increase in Y leads *ceteris paribus* to more participating yes-voters. The majority constraint is met for lower values of y. Since there are more participating voters also the quorum constraint is met for lower y. This shows that apart from the unstable equilibria when $\gamma > 0$ and the equilibria with $\mathbb{P}[A] = 0$, an increase in Y raises the equilibrium probability of accepting the proposal. Loosely speaking, a yes-pressure group should always encourage voters to participate by increasing $\bar{m}^y$.

4.2.2. *No-Pressure Groups.* For no-pressures groups the recommendation is not that straightforward. On the one hand, an increase in N leads to more participating no-voters so that the participating no-voters are a majority for lower y. On the other hand, an increase in N leads to more participating voters so that the quorum is met for lower y. When the referendum is valid more often, this can lead to a higher probability of accepting the proposal. To analyze these opposite effects in more detail, the following proposition states the equilibrium probabilities as function of N.

Proposition 7. (Simple-Hearted Voters and No-Pressure Groups)
Suppose $\gamma = 0$ and $Y > \frac{1-\overline{y}}{\overline{y}}$.
i) Suppose $Y \geq q$ and $Y < \frac{q}{2\underline{y}}$. Then

$$\mathbb{P}[A] = \begin{cases} 0 & \text{if } Y < \frac{q}{\overline{y}} \quad \text{and} \quad N < \frac{q - \overline{y}Y}{1-\overline{y}}, \\ p_q & \text{if } Y < \frac{q}{\overline{y}} \quad \text{and} \quad \frac{q-\overline{y}Y}{1-\overline{y}} \leq N \leq \frac{qY}{2Y-q}, \\ & \text{or if } Y \geq \frac{q}{\overline{y}} \quad \text{and} \quad N \leq \frac{qY}{2Y-q}, \\ p_m & \text{if } \frac{qY}{2Y-q} \leq N. \end{cases}$$

ii) Suppose $Y < q$. Then

$$\mathbb{P}[A] = \begin{cases} 0 & \text{if } Y < \frac{1}{2}q, \\ & \text{or if } Y \geq \frac{1}{2}q \quad \text{and} \quad N < \frac{qY}{2Y-q}, \\ p_b & \text{if } Y \geq \frac{1}{2}q \quad \text{and} \quad \frac{qY}{2Y-q} \leq N \leq \frac{q-\overline{y}Y}{1-\overline{y}}, \\ p_m & \text{if } Y \geq \frac{1}{2}q \quad \text{and} \quad \frac{q-\overline{y}Y}{1-\overline{y}} \leq N. \end{cases}$$

The first statement assumes that $Y > q$. This case is depicted in Figure 5. When $Y < q/\overline{y}$ the quorum is not met when $N = 0$. The equilibrium probability equals zero until the quorum will be met when the proportion of yes-voters equals $\overline{y}$. When $Y \geq q/\overline{y}$ the quorum constraint is binding from the beginning onwards. When N is sufficiently

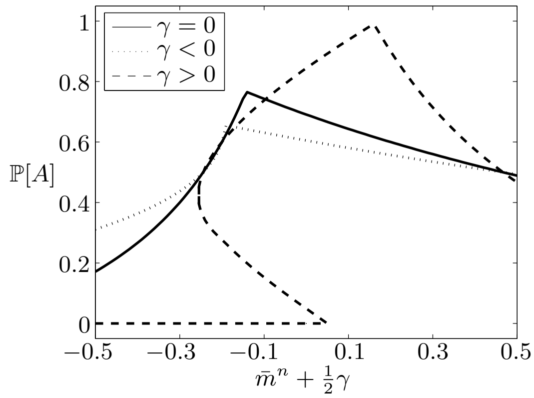

FIGURE 5. The effect of the average moral pressure of no-voters on the probability of acceptance.[11]

high, the quorum constraint is always met and the majority constraint determines the equilibrium probability. Since an increase in N makes a valid referendum more likely, p_b is increasing in N. On the other hand, an increase in N makes a majority of the participating no-voters more likely, so p_m is decreasing in N. It is clear that the maximum probability of accepting the proposal is attained for $N = qY/(2Y - q)$. The condition $Y < \frac{1}{2}q/\underline{y}$ implies that the maximum probability of accepting the proposal is below 1. The condition $Y > (1 - \overline{y})/\overline{y}$ implies that even when $N = 1$, the yes-voters can constitute the majority of the participating voters, so that $\mathbb{P}[A] > 0$.

The second statement assumes that the propensity to vote of the yes-voters is below the quorum. When the propensity is below $\frac{1}{2}q$, the quorum constraint and the majority constraint cannot be simultaneously met and the probability of accepting the proposal is 0. When $Y \geq \frac{1}{2}q$, the equilibrium probability is also zero for low N. Only for higher N it becomes positive. Note that in this case $N > q > Y$, so that both constraints are binding. The equilibrium probability is determined by p_b until N is so high that the quorum is always satisfied. From here on the majority constraint is binding.

For the calculating and the affectionate voters similar reasonings hold. It should not come as a surprise that the equilibrium for the calculating voters is unique and as function of $\bar{m}^n$ flatter than for the simple-hearted voters. For the affectionate voters there are multiple equilibria possible as before. Again, when N is slightly below the value implied by the quorum, which is $2q - Y$, the only equilibrium has $\mathbb{P}[A] = 0$ (in the figure the quorum is optimal for $\bar{m}^n + \frac{1}{2}\gamma = -0.2$).

It is clear than in all stable equilibria with $\mathbb{P}[A] > 0$, the probability of acceptance is increasing for low N and decreasing for high N. There thus exists a value of N for

[11]Figure 5 uses $\bar{m}^y = 0.2 + \frac{1}{2}\gamma$, $\alpha = 2$, $\gamma \in \{-0.9, 0, 0.9\}$, $q = 0.5$, $\underline{y} = 0.3$ and $\overline{y} = 0.8$. Note that when $\gamma = 0$, N ranges from 0.25 to 0.75.

which $\mathbb{P}[A]$ attains its maximum. Denote this value by $\hat{N}$. Under the conditions of the proposition, $\hat{N}$ for the simple-hearted voters is given by

$$\hat{N} = \begin{cases} \frac{qY}{2Y-q} & \text{if } Y \geq q, \\ \min\left\{\frac{qY+2Y^2\sqrt{\frac{q}{Y}-1}}{2Y-q}, \frac{q-\bar{y}Y}{1-\bar{y}}, 1\right\} & \text{if } Y < q. \end{cases}$$

The expression in the second line follows from setting the derivative of p_b to zero and noting that the maximum should be attained before the majority constraint takes over or the propensity to vote exceeds 1. Loosely speaking, a no-pressure group should *decrease* $\bar{m}^n$ when N is below $\hat{N}$ and *increase* $\bar{m}^n$ when N is higher than $\hat{N}$. This is in line with intuition: when the propensity to vote of no-voters is rather high, the quorum is likely to be met. To ensure that the participating no-voters form the majority, a no-pressure group should encourage no-voters to vote. When on contrary the propensity to vote is rather low, the quorum will probably not be met. A no-pressure group should now lower the propensity to vote even further to decrease the probability that the quorum is met.

5. Conclusion

In this paper we studied the impact of the quorum on referendum outcomes. Although a quorum is potentially useful to attain the population majority outcome, this crucially depends on the ability of setting the quorum at the appropriate level. Insufficient knowledge or a lack of political power to do so tend to favor the status quo. Moreover, when voters care more about the outcome when they are participating, there can be a second equilibrium in which the referendum is always invalid. Pressure groups opposing the proposal should also strategically aim for an invalid outcome when turnout is expected to be low.

This paper thus adds another critique concerning the use of referenda to the list of Nurmi (1998). Without resorting to compulsory voting, the choice is between imposing a quorum and accepting its possible distortions on the one hand and not imposing a quorum and accepting the possible non-representativeness of the participating voters on the other. Clearly, if a low turnout is expected, a referendum is not the ideal tool for decision making. Also topics for which minority groups have some strong opinions should be excluded from opinions. When the turnout on both sides is expected to be at least moderate a referendum can be appropriate. The results of this paper suggest that in this case imposing a quorum is more harmful than not imposing one. This argument for abolishing the quorum complements the arguments of Felsenthal and Machover (1997) who show that the highest degree of democratic participation is achieved, i.e. the opinion of the average voter achieves its maximum impact, in the absence of a quorum. Without a quorum, each side can only reach its aim by convincing voters of its position and of the necessity to vote. This is clearly more in line with democratic principles than giving one side the possibility to abuse the rules of the game.

However, in a recent referendum in Portugal about easing restrictions on abortion, the Catholic Church did *not* urge voters to stay at home. Interestingly, late polls suggested a significant majority of proponents, with as only doubt "whether enough voters will turn out for the result to be constitutionally binding" (The Economist 2007). This would have been the ideal case to discourage opponents from participating. Although this would just have been strategically exploiting the referendum rules, reactions on their campaign in Italy might have made the Catholic Church to act closer in line with the democratic principles underlying referenda.

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Appendix A. Different Intensities of Voters' Preferences

Suppose that the proposal should only be accepted if at least a fraction $\hat{y}$ of the population is in favor. We call this the *optimal outcome*. The referendum design is broadened by also allowing for a qualified majority among the referendum participants. Let the qualified majority θ denote the required fraction of participating voters in favor of the non-default outcome. Define $q^* = \hat{y}Y + (1-\hat{y})N$ and $\theta^* = \hat{y}Y/q^*$. The following proposition considers simple-hearted voters, analogue results hold for calculating or affectionate voters.

Proposition 8. (Intensities of Voters' Preferences)
Assume that $\gamma = 0$ *and* $\bar{m}^y, \bar{m}^n \in (-\frac{\alpha}{2}, \frac{\alpha}{2})$.
i) When $\bar{m}^y = \bar{m}^n$*, the optimal outcome is only achieved in the unique equilibrium of the referendum with a required majority of* θ^**, a quorum of at most* q^* *and default outcome* $D \in \{A, R\}$.
ii) When $\bar{m}^y \neq \bar{m}^n$*, the optimal outcome is only achieved in the unique equilibrium of a referendum with either a qualified majority of at most* θ^* *and quorum* q^* *or a referendum with qualified majority* θ^* *and a quorum of at most* q^**. In both cases the default outcome is* $D = R$ *if* $\bar{m}^y > \bar{m}^n$ *and* $D = A$ *otherwise.*

Statement i) follows by noting that $\theta^* = \hat{y}$ and that the participation rate equals q^*. Statement ii) follows by noting that the participation constraint or the (qualified) majority constraint (or both) should be exactly binding when a fraction $\hat{y}$ of the population is in favor. A sufficiently low quorum or qualified majority is always met when the other constraint is satisfied.

Note that in the paper the required majority among referendum participants is set at 50%. Although allowing for a qualified majority would introduce other referendum designs with the same outcome, focussing on a majority of 50% is the most neutral from a political point of view.

Appendix B. A Non-Optimal Quorum when $\gamma \neq 0$

First consider the model with calculating voters, so $\gamma < 0$. Define $\hat{p}$ as the probability for which the propensities to vote of yes-voters and no-voters are equal. Using the definitions of $Y(\hat{p})$ and $N(\hat{p})$ gives

$$\hat{p} = \frac{\alpha}{2\gamma}\big(N(0) - Y(0)\big).$$

The uniform distribution of the moral pressures has the convenient property that the sum of the propensities to vote s is constant

$$s = Y(p) + N(p) = \frac{1}{2} + \frac{\bar{m}^y + \gamma p}{\alpha} + \frac{1}{2} + \frac{\bar{m}^n + \gamma - \gamma p}{\alpha} = 1 + \frac{\bar{m}^y + \bar{m}^n - \gamma}{\alpha}.$$

When only the majority constraint is binding, the equilibrium condition $p_m - \phi(\overline{y} - N(p_m)/s) = 0$ gives

$$p_m = \frac{\phi\big(\overline{y} - \frac{N(0)}{s}\big)}{1 - \frac{\phi\gamma}{\alpha s}}.$$

When only the quorum constraint is binding, the equilibrium condition $p_q - \phi(\overline{y} - (q - N(p_q))/(Y(p_q) - N(p_q))) = 0$ defines a second order polynomial equation in p_q with solutions

$$p_q^{\pm} = \frac{1}{2}(\hat{p} + \xi) \pm \sqrt{\frac{1}{4}\big(\hat{p} + \xi\big)^2 + \frac{\phi\alpha}{2\gamma}\big(\overline{y}Y(0) + (1 - \overline{y})N(0) - q\big)}.$$

Similarly, when both conditions are binding, the equilibrium condition $p - \phi((q - N(p_b))/(Y(p_b) - N(p_b)) - N(p_b)/s) = 0$ defines a second order polynomial in p_b with solutions

$$p_b^{\pm} = \frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} \pm \sqrt{\left(\frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\right)^2 \hat{p}^2 + \frac{\phi\alpha}{2\gamma}\frac{q - \frac{2Y(0)N(0)}{s}}{1 - \frac{\phi\gamma}{\alpha s}}}.$$

The equilibrium probabilities are related to the quorum in the following proposition.

Proposition 9. (Calculating Voters and a Not-Optimally Set Quorum)
Suppose $\gamma < 0$ *with* $1 + \frac{\phi\gamma}{\alpha s} > 0$.

i) Suppose $\bar{m}^y \geq \bar{m}^n + \gamma(1-2\xi)$ *and* $\frac{N(1)}{s} > \underline{y}$*. Then*

$$\mathbb{P}[A] = \begin{cases} p_m & \text{if } q \leq \frac{2Y(p_m)N(p_m)}{s}, \\ p_q^- & \text{if } \frac{2Y(p_m)N(p_m)}{s} \leq q \leq \overline{y}Y(0) + (1-\overline{y})N(0), \\ 0 & \text{if } \overline{y}Y(0) + (1-\overline{y})N(0) \leq q. \end{cases}$$

ii) Suppose $\bar{m}^y < \bar{m}^n + \gamma(1-2\xi)$ *and* $N(0) < Y(0)$*. Then*

$$\mathbb{P}[A] = \begin{cases} p_m & \text{if } q \leq \overline{y}Y(p_m) + (1-\overline{y})N(p_m), \\ p_b^+ & \text{if } \overline{y}Y(p_m) + (1-\overline{y})N(p_m) \leq q \leq \frac{1}{2}s, \\ p_q^- & \text{if } \frac{1}{2}s \leq q \leq \overline{y}Y(0) + (1-\overline{y})N(0), \\ 0 & \text{if } \overline{y}Y(0) + (1-\overline{y})N(0) \leq q. \end{cases}$$

The proposition requires $\gamma > -\alpha s/\phi$. The first statement assumes that the quorum is correctly set. This case is depicted in Figure 3. The condition $N(1)/s > \underline{y}$ implies that $p_m < 1$. Similar to the model with simple-hearted voters, $p_m > \xi$ (this is made formal in the proof of the proposition). The second statement assumes that the default outcome is incorrectly set. Although the equilibrium probability p_m is positive, it is below ξ.

Now consider the model with affectionate voters, $\gamma > 0$. The only candidates for the equilibrium probabilities are again p_m, $p_q^{\pm}$ and $p_b^{\pm}$. Before stating the proposition that relates these probabilities with the quorum, two critical values of the quorum q are needed

$$q_q = \frac{\gamma}{2\phi\alpha}(\hat{p}+\xi)^2 + \overline{y}Y(0) + (1-\overline{y})N(0),$$

$$q_b = \frac{2Y(0)N(0)}{s} - \frac{\gamma}{\phi\alpha}\frac{\left(1-2\frac{\phi\gamma}{\alpha s}\right)^2}{2-2\frac{\phi\gamma}{\alpha s}}\hat{p}^2.$$

From the definition of $p_q^{\pm}$ it can be seen that p_q^+ and p_q^- only exist for $q \leq q_q$. Similarly, p_b^+ and p_b^- only exist for $q \geq q_b$.

Proposition 10. (Affectionate Voters and a Not-Optimally Set Quorum)
Suppose $\gamma > 0$ *with* $1 - \frac{\phi\gamma}{\alpha s} > 0$.
i) Suppose $\bar{m}^y \geq \bar{m}^n + \gamma(1-2\xi)$ *and* $Y(0) < N(0)$*. Then*

$$\mathbb{P}[A] = \begin{cases} 1 & \text{if } \frac{N(1)}{s} \leq \underline{y} & \text{and } q \leq \underline{y}Y(1) + (1-\underline{y})N(1), \\ p_m & \text{if } \frac{N(1)}{s} \geq \underline{y} & \text{and } q \leq \frac{2N(p_m)Y(p_m)}{s}, \\ p_q^+ & \text{if } \frac{N(1)}{s} \leq \underline{y} & \text{and } \underline{y}Y(1) + (1-\underline{y})N(1) \leq q \leq q_q, \\ & \text{or if } \frac{N(1)}{s} \geq \underline{y} & \text{and } \frac{2N(p_m)Y(p_m)}{s} \leq q \leq q_q, \\ p_q^- & \text{if } \frac{1}{2}s < q \leq q_q, & \\ p_b^+ & \text{if } 1-2\frac{\phi\gamma}{\alpha s} \geq 0 & \text{and } q_b \leq q < \frac{1}{2}s, \\ & \text{or if } 1-2\frac{\phi\gamma}{\alpha s} \leq 0 & \text{and } \frac{2Y(0)N(0)}{s} \leq q < \frac{1}{2}s, \\ p_b^- & \text{if } 1-2\frac{\phi\gamma}{\alpha s} \geq 0 & \text{and } q_b \leq q \leq \frac{2Y(0)N(0)}{s}, \\ 0 & \text{if } \frac{2Y(0)N(0)}{s} \leq q. & \end{cases}$$

ii) Suppose $\bar{m}^y < \bar{m}^n + \gamma(1-2\xi)$ *and* $\frac{N(0)}{s} < \overline{y}$. *Then*

$$\mathbb{P}[A] = \begin{cases} p_m & if \quad q \le \overline{y}Y(p_m) + (1-\overline{y})N(p_m), \\ p_b^+ & if \;\; Y(\xi) - N(\xi) \ge -2\frac{\gamma}{\alpha}\xi \\ & \quad and \;\; q_b \le q \le \overline{y}Y(p_m) + (1-\overline{y})N(p_m), \\ p_b^- & if \;\; Y(\xi) - N(\xi) \ge -2\frac{\gamma}{\alpha}\xi \;\;\; and \;\; q_b \le q \le \frac{2Y(0)N(0)}{s}, \\ & or \;\; if \;\; Y(\xi) - N(\xi) \le -2\frac{\gamma}{\alpha}\xi \\ & \quad and \;\; \overline{y}Y(p_m) + (1-\overline{y})N(p_m) \le q \le \frac{2Y(0)N(0)}{s}, \\ 0 & if \;\; \frac{2Y(0)N(0)}{s} \le q. \end{cases}$$

The proposition requires $\gamma < \alpha s/\phi$. The first statement assumes that the default outcome is correctly set. This case is depicted in Figure 3. The condition $Y(0) < N(0)$ excludes the case where yes-voters have always the highest propensity to vote. When $N(1)/s < \underline{y}$, the majority constraint is always satisfied for a low quorum. Otherwise the equilibrium probability p_m is below 1 though above ξ. For both cases, the quorum constraint becomes binding when the q increases. There are two possible equilibria, p_q^+ and p_q^-. A necessary condition for their existence is $Y > N$, so they should be higher than $\hat{p}$. They should be lower than p_m, since equilibria with a higher probability are not possible. It follows that p_q^+ exists from the point where it equals $\min\{1, p_m\}$ until q_q, while p_q^- exists when the quorum is higher than $\frac{1}{2}s$ but at most q_q. When the probability of acceptance is below $\hat{p}$, it follows that $Y < N$. This shows that both constraints are binding. The equilibrium with p_b^+ exists until $\frac{1}{2}s$, since it then equals $\hat{p}$. When it starts from $p_b^+ = 0$, the p_b^- equilibrium does not exist. When p_b^+ exists from q_b onwards, $p_b^+ > 0$ and the p_b^- equilibrium exists between q_b and $2Y(0)N(0)/s$. For a higher quorum the equilibrium with $\mathbb{P}[A] = 0$ exists.

The second statement assumes that the default outcome is incorrectly set. Similar to the simple-hearted voters, p_m is below ξ. There exists a range with three equilibria when γ is not too small.

When pressure groups can affect the turnout of voters, the equilibria are found by using Propositions 9 and 10 and rearranging the conditions.

Appendix C. Proofs

Proof of Proposition 1.

This proposition is proved in the main text. □

Proof of Proposition 2.

Assume that $\bar{m}^y \ge \bar{m}^n + \gamma(1-2\xi)$ (the proof of statement ii) with $\bar{m}^y < \bar{m}^n + \gamma(1-2\xi)$ follows in the same way). An equilibrium is characterized by $p - \mathbb{P}[A] = 0$ and the analysis can be confined to $p \in [0,1]$. Note that $Y(p) = \frac{1}{2} + (\bar{m}^y + \gamma p)/\alpha$ is strictly decreasing in

p and $N(p) = \frac{1}{2} + (\bar{m}^n + \gamma - \gamma p)/\alpha$ strictly increasing. The participation rate equals

$$yY(p) + (1-y)N(p) = 1 + \frac{y(\bar{m}^y + \gamma p) + (1-y)(\bar{m}^n + \gamma - \gamma p)}{\alpha}.$$

The first step is to determine the quorum values for which the population majority outcome can occur. When the proposal should be accepted if and only if $y \geq 0$, it follows that $\mathbb{P}[A] = \xi$. When $\bar{m}^y > \bar{m}^n + \gamma(1-2\xi)$, so that $Y(\xi) > N(\xi)$, there is already a majority of yes-voters for $y < \frac{1}{2}$. To ensure that the proposal is only accepted for $y \geq \frac{1}{2}$, the quorum constraint should be exactly binding for $y = \frac{1}{2}$. This implies that the quorum should be $q^* = \frac{1}{2}Y(\xi) + \frac{1}{2}N(\xi)$. When $\bar{m}^y = \bar{m}^n + \gamma(1-2\xi)$, so that $Y(\xi) = N(\xi)$, the fractions of participating voters in favor and against are identical to the population fraction. Any quorum below q^* is always met and the majority constraint correctly determines the outcome.

The second step is to establish that only the population majority outcome can occur for the found quorum values. Suppose first that the quorum is q^*.

When $Y(p) > N(p)$ the participation rate is increasing in y. Since for $y = \frac{1}{2}$ it equals q^*, the quorum constraint is only met when $y \geq \frac{1}{2}$. Since in this case also the majority constraint is met, the probability of accepting the proposal is ξ.

When $Y(p) = N(p)$ the participation rate is constant and equal to q^*. The fractions of participating voters in favor and against are identical to the population fractions. The quorum is always met and the majority constraint only when $y \geq \frac{1}{2}$, so $\mathbb{P}[A] = \frac{1}{2}$.

When $Y(p) < N(p)$ the participation rate is decreasing in y. Since for $y = \frac{1}{2}$ it equals q^*, this means that the quorum constraint can only be met for $y < \frac{1}{2}$. However, for these cases the majority constraint is violated and $\mathbb{P}[A] = 0$.

To summarize $p - \mathbb{P}[A] = p - \xi \mathbb{1}_{\{Y(p) \geq N(p)\}}$ (see also Figure 1). Remember that Y is decreasing in p while N is increasing and that $Y(0) \geq N(0)$, hence any solution p^* of $p - \mathbb{P}[A] = 0$ thus satisfies $Y(p^*) \geq N(p^*)$. Since $p - \mathbb{P}[A]$ is strictly increasing on $[0,1]$, any solution is necessarily unique. The claim in statement i) now follows by noting that $p^* = \xi$ is a solution with $Y(p^*) = N(p^*)$. The claim in statement ii) follows by noting that $p^* = \xi$ is a solution with $Y(p^*) > N(p^*)$.

Now suppose that $Y(\xi) = N(\xi)$ and that the quorum is below q^*. Since $Y(p) > N(p)$ for $p < \xi$, the quorum will be met for $y < \frac{1}{2}$ and $\mathbb{P}[A] > \frac{1}{2}$. This shows that $p - \mathbb{P}[A] < 0$ for $p < \xi$. Likewise it follows that $p - \mathbb{P}[A] > 0$ for $p > \xi$. The equilibrium found above is thus unique. □

Proof of Proposition 3.

Assume that $\bar{m}^y \geq \bar{m}^n + \gamma(1-2\xi)$ (the proof with $\bar{m}^y < \bar{m}^n + \gamma(1-2\xi)$ follows in the same way). In the same way as in the proof of Proposition 2, the quorum values for which the population majority outcome occur are found. It an identical way it also follows that $p - \mathbb{P}[A] = p - \xi \mathbb{1}_{\{Y(p) \geq N(p)\}}$. However, now Y is increasing in p while N is decreasing (see also Figure 2). The proof of statement i) follows by noting that $p^* = \xi$ is a solution with $Y(p^*) = N(p^*)$ when $\bar{m}^y = \bar{m}^n + \gamma(1-2\xi)$, and a solution with $Y(p^*) > N(p^*)$ when $\bar{m}^y > \bar{m}^n + \gamma(1-2\xi)$.

The proof of statement ii) follows by noting that since $p - \mathbb{P}[A]$ is strictly increasing for p such that $Y(p) \geq N(p)$, any other equilibrium should satisfy $Y(p) < N(p)$. But for these p the probability of acceptance $\mathbb{P}[A]$ is zero, so that $p - \mathbb{P}[A] = p$. This shows that $p = 0$ is the only candidate for a solution. This is only possible if $Y(0) < N(0)$, so if $Y(0) - N(0) = (\bar{m}^y - \bar{m}^n - \gamma)/\alpha < 0$. This gives the condition for uniqueness. □

Proof of Proposition 4.
In equilibrium $p = \mathbb{P}[A]$ for all voters. The propensity to vote of a yes-voter i with parameters $(\bar{m}_i, \alpha_i, \gamma_i)$ is $\frac{1}{2}+(\bar{m}_i+\gamma_i p)/\alpha_i$, which follows from the assumption that $(\bar{m}_i, \alpha_i, \gamma_i) \in \hat{\mathcal{P}}$. The average propensity to vote is given by

$$Y = \int_{\hat{\mathcal{P}}} \frac{1}{2} + \frac{\bar{m}_i + \gamma_i p}{\alpha_i} \mathrm{d}\Phi^y(\bar{m}_i, \alpha_i, \gamma_i) = \frac{1}{2} + \bar{m}^y + \gamma p.$$

Similarly, $N = \frac{1}{2} + \bar{m}^n + \gamma - \gamma p$. The proofs of Propositions 1-3 go through with the found expressions for Y and N when α is taken to be 1. □

Proof of Proposition 5.
i) In this case $Y \geq N$. First consider $Y > N$. When the quorum is sufficiently small, the probability of acceptance is determined by the majority constraint. Since $N/s < \frac{1}{2}$ and by assumption $N/s > \underline{y}$, it follows that $p_m \in (0,1)$. The majority constraint and the quorum constraint coincide for $q = 2YN/s$. The quorum constraint is the only binding constraint until it can never be satisfied, so until $q = \overline{y}Y + (1-\overline{y})N$. For a higher quorum the probability of acceptance is 0. Now consider $Y = N$. This implies that $p_m = \xi$. The quorum is always satisfied as long as $q \leq Y = N = \frac{1}{2}s$. A higher quorum can never be satisfied.
ii) In this case $Y < N$. Since $N/s > \frac{1}{2}$ and by assumption $N/s < \underline{y}$, it follows that $p_m \in (0,1)$. When the quorum is so low that it is always satisfied, i.e. below $\overline{y}Y + (1-\overline{y})N$, the probability of acceptance is determined by the majority constraint. When q increases, both constraints are binding until the majority and the quorum constraint can not be simultaneously met. This happens when $(q-N)/(Y-N) = N/s$, which is identical to $q = 2YN/s$. For a higher quorum the probability of acceptance is 0. □

Proof of Proposition 6.
The proof follows by similar reasoning as the proof of Proposition 5. The only technical detail is that for $N \leq \frac{1}{2}q$ a proportion y that satisfies the quorum constraint also satisfies the majority constraint. Clearly $(1-y)N \leq \frac{1}{2}(1-y)q$. That $\frac{1}{2}(1-y)q \leq yY$ follows from $q \leq yY + (1-y)N \leq yY + \frac{1}{2}(1-y)q \leq yY + \frac{1}{2}(1-y)q + yq$ and moving all terms involving q to the left hand side. □

Proof of Proposition 7.

The arguments of the proof are similar to previous ones. In case $Y \geq q$, $p_m < 1$ if $N/s > \underline{y}$ for $N = qY/(2Y-q)$. This is implied by $Y > \frac{1}{2}q/\overline{y}$.

When $Y < \frac{1}{2}q$, the majority constraint and the quorum constraint cannot be met simultaneously. It is clear that $yY \leq \frac{1}{2}yq$. That $\frac{1}{2}yq \leq (1-y)N$ follows from $q \leq yY + (1-y)N < \frac{1}{2}yq + (1-y)N \leq \frac{1}{2}yq + (1-y)q + (1-y)N$ and moving all terms involving q to the left hand side. □

Proof of Proposition 8.

This proposition is proved in the text. □

Proof of Proposition 9.

In proving the proposition, the following relations between $\hat{p}$, ξ and p_m are used

$$\hat{p} - \xi = -\frac{\bar{m}^y - \bar{m}^n - \gamma(1-2\xi)}{2\gamma},$$
$$p_m - \hat{p} = \frac{\bar{m}^y - \bar{m}^n - \gamma(1-2\xi)}{2\gamma(1-\frac{\phi\gamma}{\alpha s})},$$
$$p_m - \xi = \frac{\bar{m}^y - \bar{m}^n - \gamma(1-2\xi)}{2\frac{\alpha s}{\phi}(1-\frac{\phi\gamma}{\alpha s})}.$$

i) First consider $\bar{m}^y > \bar{m}^n + \gamma(1-2\xi)0$. The derived relations above show that $\hat{p} > p_m > \xi$, so $Y(p_m) > N(p_m)$ and $Y(\xi) > N(\xi)$. Note that no equilibria with $p^* > p_m$ can occur. The condition that $N(1)/s \geq \underline{y}$ guarantees that $p_m < 1$. The majority constraint is the only binding constraint until it crosses with the quorum constraint, which happens for $q = 2Y(p_m)N(p_m)/s$. When the quorum constraint takes over, it does so until it can never be satisfied, which happens for $q = \overline{y}Y(0) + (1-\overline{y})N(0)$. When $\frac{1}{2}(\hat{p}+\xi) > p_m$ it is clear that p_q^+ cannot be an equilibrium. That this is the case follows by using $\frac{1}{2}(\hat{p}+\xi) - p_m = \frac{1}{2}(\hat{p} - p_m) + \frac{1}{2}(\xi - p_m)$ and the derived relations above so that

$$\frac{1}{2}(\hat{p}+\xi) - p_m = -\frac{1}{4\gamma}\frac{\bar{m}^y - \bar{m}^n - \gamma(1-2\xi)}{1-\frac{\phi\gamma}{\alpha s}}\left(1+\frac{\phi\gamma}{\alpha s}\right) > 0.$$

When the quorum is above $\overline{y}Y(0) + (1-\overline{y})N(0)$, the quorum constraint can never be satisfied and $p^* = 0$.

Now suppose $\bar{m}^y = \bar{m}^n + \gamma(1-2\xi)$. Then $\hat{p} = p_m = \xi$, and $2Y(p_m)N(p_m)/s = \frac{1}{2}s$. So, $p^* = \xi$ for $q \leq \frac{1}{2}s$. Since $\frac{1}{2}(\hat{p}+\xi) - p_m = 0$, the equilibrium with p_q^+ does not exist for a higher quorum. The equilibrium probability is p_q^- until q is raised so high that it becomes 0.

ii) The relations derived above show that $\xi > p_m > \hat{p}$, so $Y(p_m) < N(p_m)$ and $Y(\xi) < N(\xi)$. The condition $N(0) < Y(0)$ implies $\hat{p} > 0$ so that $p_m > 0$. When the quorum constraint is sufficiently small $p^* = p_m$ is the equilibrium. The quorum constraint becomes binding when $(q - N(p_m))/(Y(p_m) - N(p_m)) = \overline{y}$. Since $((1-2\phi\gamma/\alpha s)/(2-2\phi\gamma/\alpha s))\hat{p} <$

$\hat{p} < p_m$ it is clear that p_b^+ is the equilibrium that takes over from p_m and that p_b^- does not exist. For $q = \frac{1}{2}s$, it holds that $q = \frac{1}{2}s^2/s$ and thus $\frac{1}{2}(\phi\alpha/\gamma)(q - 2Y(0)N(0)/s) = \frac{1}{2}(\phi\alpha/\gamma s)(\frac{1}{2}(Y(0) + N(0))^2 - 2Y(0)N(0)) = \frac{1}{4}(\phi\alpha/\gamma s)(Y(0) - N(0))^2 = (\phi\gamma/\alpha s)\hat{p}^2$. The value of p_b^+ in $q = \frac{1}{2}s$ equals

$$\frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} + \sqrt{\left(\frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\right)^2 \hat{p}^2 + \frac{\phi\alpha}{2\gamma}\frac{q - \frac{2Y(0)N(0)}{s}}{1 - \frac{\phi\gamma}{\alpha s}}}$$

$$= \frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} + \sqrt{\frac{1}{4}\frac{\left(1 - \frac{\phi\gamma}{\alpha s} - \frac{\phi\gamma}{\alpha s}\right)^2}{\left(1 - \frac{\phi\gamma}{\alpha s}\right)^2}\hat{p}^2 + \frac{\left(1 - \frac{\phi\gamma}{\alpha s}\right)\frac{\phi\gamma}{\alpha s}}{\left(1 - \frac{\phi\gamma}{\alpha s}\right)^2}\hat{p}^2}$$

$$= \frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} + \sqrt{\frac{1}{4}\frac{1}{\left(1 - \frac{\phi\gamma}{\alpha s}\right)^2}\hat{p}^2} = \frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} + \frac{1}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} = \hat{p}.$$

Since $\hat{p} = \frac{1}{2}(\alpha/\gamma)(N(0) - Y(0)) > 0$ the p_b^+ equilibrium exists when $q \leq \frac{1}{2}s$. For a higher quorum $Y > N$ and the quorum constraint is the only binding constraint. Since $\frac{1}{2}(\hat{p} + \xi) > \hat{p}$, only p_q^- can be an equilibrium. To find p_q^- in $q = \frac{1}{2}s$, first rewrite $q = \frac{1}{2}s = -\frac{1}{2}(N(0) - Y(0)) + N(0) = -(N(0) - Y(0))(\overline{y} - \xi/\phi) + N(0)$, then $\frac{1}{2}(\phi\alpha/\gamma)(\overline{y}Y(0) + (1 - \overline{y})N(0) - q) = -\frac{1}{2}(\alpha/\gamma)(N(0) - Y(0))\xi = -\hat{p}\xi$ so that

$$\frac{1}{2}(\hat{p} + \xi) - \sqrt{\frac{1}{4}(\hat{p} + \xi)^2 + \frac{\phi\alpha}{2\gamma}\Big(\overline{y}Y(0) + (1 - \overline{y})N(0) - q\Big)}$$

$$= \frac{1}{2}(\hat{p} + \xi) - \sqrt{\frac{1}{4}(\hat{p} + \xi)^2 - \hat{p}\xi} = \frac{1}{2}(\hat{p} + \xi) - \frac{1}{2}(\xi - \hat{p}) = \hat{p}.$$

The p_q^- equilibrium exists until the quorum can never be satisfied, which is the case for $q = \overline{y}Y(0) + (1 - \overline{y})N(0)$. For a higher quorum $p^* = 0$. □

Proof of Proposition 10.
From $2N(0)Y(0)/s \leq 2(\frac{1}{2}s\frac{1}{2}s)/s = \frac{1}{2}s$ it follows that $q_b < \frac{1}{2}s$. That $q_q \geq \frac{1}{2}s$ follows from

$$\frac{\gamma}{2\phi\alpha}(\hat{p} + \xi)^2 = \frac{\gamma}{2\phi\alpha}(\hat{p} - \xi)^2 + 4\frac{\gamma}{2\phi\alpha}\hat{p}\xi = \frac{\gamma}{2\phi\alpha}(\hat{p} - \xi)^2 + \big(N(0) - Y(0)\big)(\overline{y} - \frac{1}{2})$$

so that $q_q = \frac{1}{2}(\gamma/\phi\alpha)(\hat{p} - \xi)^2 + \frac{1}{2}(Y(0) + N(0)) \geq \frac{1}{2}s$. The inequality is strict when $\hat{p} \neq \xi$, so when $\bar{m}^y \neq \bar{m}^n + \gamma(1 - 2\xi)$.

i) First consider $\bar{m}^y > \bar{m}^n + \gamma(1 - 2\xi)$. The relations derived at the beginning of the previous proof show that $p_m > \xi > \hat{p}$, so that $Y(p_m) > N(p_m)$ and $Y(\xi) > N(\xi)$.

When $N(1)/s \leq \underline{y}$, the majority constraint is always satisfied and the equilibrium probability is 1 until the quorum constraint is crossed for the quorum $\underline{y}Y(1) + (1 - \underline{y})N(0)$. When $N(1)/s \geq \underline{y}$ it follows that $p_m < 1$. The majority constraint is binding until $(q - N(p_m))/(Y(p_m) - N(p_m)) = N(p_m)/s$, which is the stated condition. When the quorum constraint becomes binding p_q^+ is the equilibrium since $\frac{1}{2}(\hat{p} + \xi) < p_m$ implies that p_q^- only exists for lower probabilities then $\frac{1}{2}(\hat{p} + \xi)$. So, p_q^+ stops to exist at q_q. Note that for this quorum the minimum of p_q^+ is achieved which equals $\frac{1}{2}(\hat{p}+\xi)$. Since this is bigger than $\hat{p}$, indeed $Y > N$. By assumption $N(0) > Y(0)$ so that $\hat{p} > 0$ and p_q^+ exists until q_q. From here p_q^- decreases when q *decreases*. In the previous proof it was shown that $p_q^- = \hat{p}$ for $q = \frac{1}{2}s$. This shows that $Y > N$ so that p_q^- exists for $q > \frac{1}{2}s$.

Note that the equilibrium $p^* = \hat{p}$ does not exist! The only quorum candidate would be $q = \frac{1}{2}s$. But for this quorum $Y(\hat{p}) = N(\hat{p}) = \frac{1}{2}s$, so the quorum is always met. But, if only the quorum constraint binds, p_m is the only equilibrium candidate, but $p_m > \hat{p}$.

When the quorum decreases from $\frac{1}{2}s$, both constraints are binding. When $p < \hat{p}$ it follows that $Y < N$, hence only p_b^+ and p_b^- are equilibrium candidates. In the previous proof it was shown that $p_b^+ = \hat{p}$ for $q = \frac{1}{2}s$, so that $Y < N$. The minimum value of p_b^+ is attained in q_b and equals $((1 - 2\phi\gamma/\alpha s)/(2 - 2\phi\gamma/\alpha s))\hat{p}$. The equilibrium with p_b^+ does not exist on the whole interval from q_b to $\frac{1}{2}s$ if $1 - 2\phi\gamma/\alpha s < 0$. In this case it only exists when $q > 2Y(0)N(0)/s$. When it does exists on the whole interval, p_b^- exists from q_b to $N(0)Y(0)/s$. In both cases, $p^* = 0$ when q is so big that the majority constraint and the quorum constraint cannot be satisfied simultaneously. This is the case for $q \geq 2Y(0)N(0)/s$.

Now consider $\bar{m}^y = \bar{m}^n + \gamma(1 - 2\xi)$. The relations derived in the previous proof show that $\hat{p} = \xi = p_m$. Note also that $2N(p_m)Y(p_m)/s = q_q = \frac{1}{2}s$ (see the expression for q_q derived at the beginning of this proof), so the $p_q^\pm$ part does not exist. Note also that $q_b < \frac{1}{2}s$, which shows that the p_b^- arm does exist.

ii) The relations derived in the previous proof show that $\hat{p} > \xi > p_m$, so that $Y(p_m) < N(p_m)$ and $Y(\xi) < N(\xi)$.

Since by assumption $N(0)/s < \overline{y}$, it follows that $p_m > 0$. This is the only equilibrium until the quorum constraint becomes binding in $q = \overline{y}Y(p_m) + (1 - \overline{y})N(p_m)$. The equilibrium with $p^* = p_b^+$ can only exist when $((1 - 2\phi\gamma/\alpha s)/(2 - 2\phi\gamma/\alpha s))\hat{p} < p_m$, so when

$$\frac{1}{2}\left(1 - 2\frac{\phi\gamma}{\alpha s}\right)\hat{p} - \phi\left(\overline{y} - \frac{N(0)}{s}\right) = \frac{1}{2}\left(1 - 2\frac{\phi\gamma}{\alpha s}\right)\hat{p} - \xi + \frac{\phi\gamma}{\alpha s}\hat{p} = \frac{1}{2}\hat{p} - \xi < 0.$$

When this is the case, the p_b^* equilibrium exists from q_b until $\overline{y}Y(p_m) + (1 - \overline{y})N(p_m)$. Note that $p_b^+ > 0$ since $\hat{p} > 0$. When the p_b^+ equilibrium exists, the p_b^- equilibrium takes over from q_b, otherwise directly from $\overline{y}Y(p_m) + (1 - \overline{y})N(p_m)$. It exists until p_b^- is zero, which happens at $2Y(0)N(0)/s$. For a higher quorum the majority and the quorum constraint are mutually exclusive and $p^* = 0$. □