



The effect of market confidence on a financial system from the perspective of fractional calculus: Numerical investigation and circuit realization



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ABSTRACT

Modeling and analysis of financial systems have been interesting topics among researchers. The more precisely we know dynamic of systems, the better we can deal with them. This way, in this paper, we investigate the effect of market confidence on a financial system from the perspective of fractional calculus. Market confidence, which is a significant concern in economic systems, is considered, and its effects are comprehensively investigated. The system has been studied through numerical simulations and analyses, such as the Lyapunov exponents, bifurcation diagrams, and phase portrait. It is shown that the system enters chaos through experiencing a cascade of period doublings, and the existence of chaos is verified. Finally, an analog circuit of the chaotic system is designed and implemented to prove its feasibility in real-world applications. Also, through the circuit implementation, the effects of different factors on the behavior of the systems are investigated.

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1. Introduction

In the last decades, modeling of economic systems has been the subject of intense research [1–3]. Many studies have investigated the nonlinear behavior of these systems [4]. Significantly, chaos as a particular phenomenon has been observed in these systems and formed a new subject in this research field [5]. A detailed study of the chaotic behavior of systems can reveal regularity hidden in the seemingly random economic phenomena [6–12]. This way, it pro-

vides a perspective of understanding the complexity of economic systems as some of their internal structures, rather than accidental or external behaviors [13].

Fractional calculus is an effective method for expressing nonlinear dynamics in the fields of physics and engineering [14–16]. In fact, the features of the fractional calculus are beneficial to describe more accurately many real-world phenomena in various fields such as digital circuits [17], viscoelastic studies [18], ferroelectric materials [19], biology [20], cryptography [21], economics [22], etc. Fractional calculus can describe memory effects, and many variables in economics systems possess memory effects. As it is obvious, our past economic decisions may affect our present and future ones [23]. Thereby, exploiting fractional calculus in the modeling of economic and financial systems is a rational and argumentative approach [24,25].

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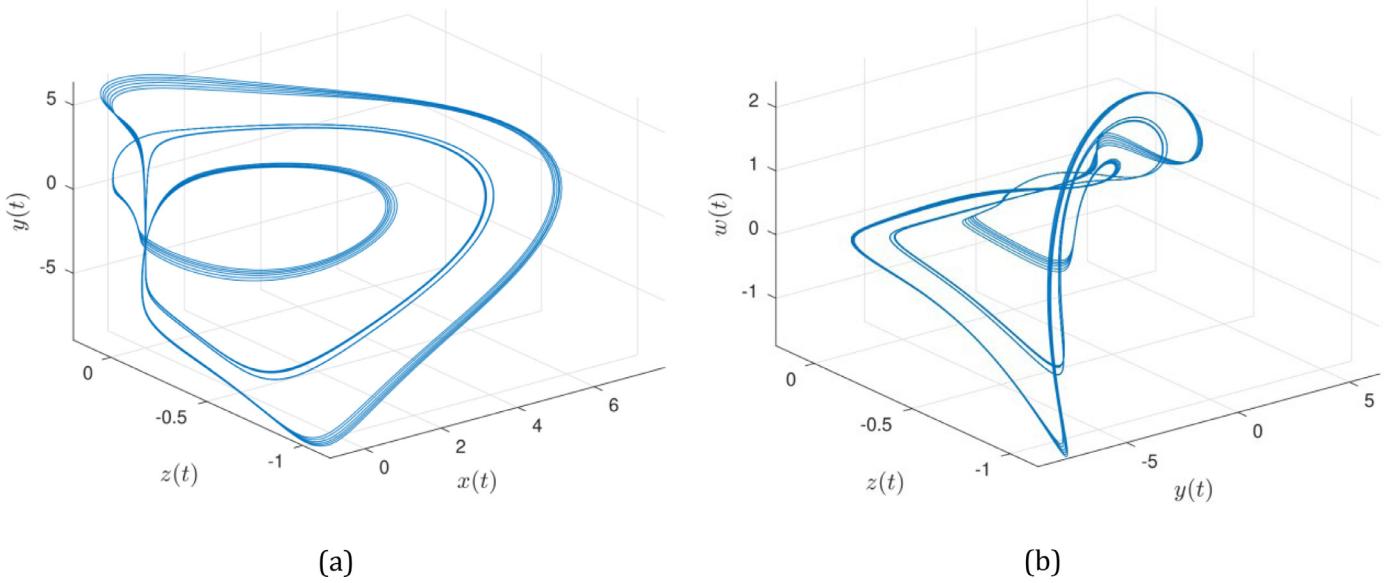


Fig. 1. Phase portraits of the financial system (3). (a) state space (x, y, z) and (b) state space (y, z, w) .

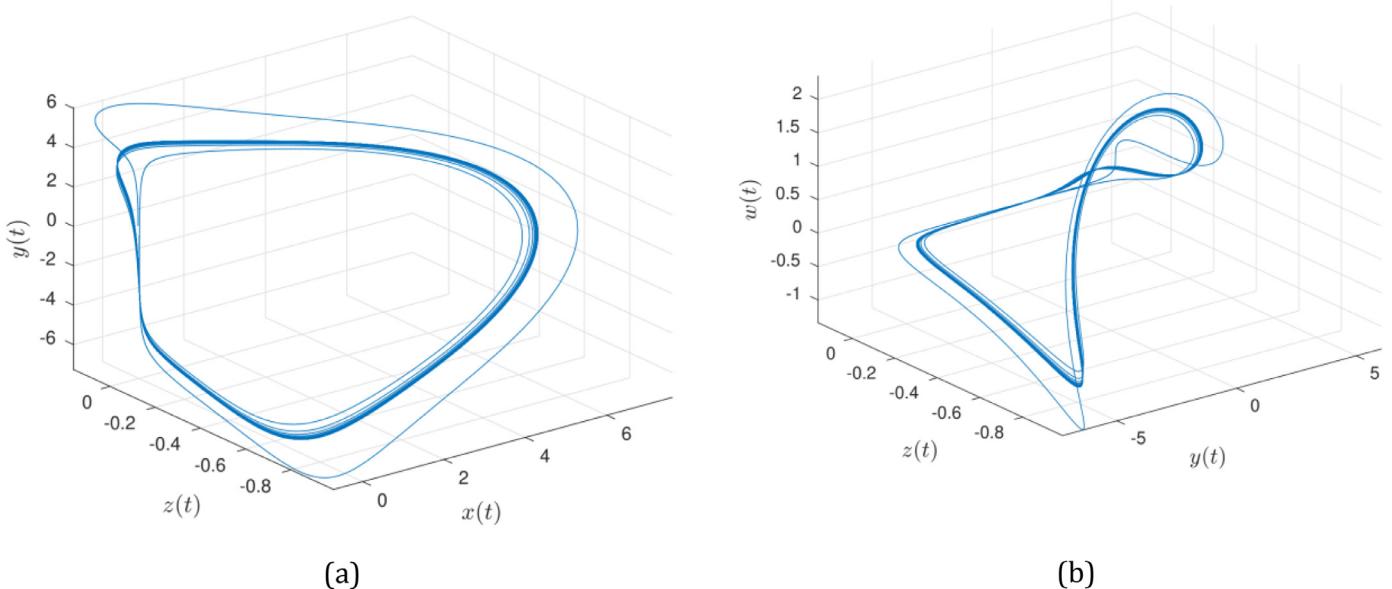


Fig. 2. Phase portraits of the fractional financial system (5). (a) state space (x, y, z) and (b) state space (y, z, w) .

On the other hand, one of the most critical factors in rescuing or influencing the economic crisis is confidence. Hence, another critical issue that should be considered in the modeling and study of economic and financial systems is confidence, the confidence of the people [26]. When people lose their confidence in economic systems, it will bring about a vicious cycle and chain of reaction, including reducing production, canceling investment, and stopping consuming [27,28]. This situation will result in less investment, less employment, less confidence, more stock of products, more deficit, and more layoffs. If there exist enough investment confidence and opportunities, consumers will not save too much. Also, the confidence in economic systems will be increased by low interest rates and be lowered by alarming interest rates [29]. Governments must do their best to provide a balancing platform that

promotes investments and savings. Investment confidence plays a significant role in fostering investment, and consequently, in the behavior of systems. This way, it is meaningful to consider market confidence in the modeling of financial systems. Nevertheless, there are a few studies that consider chaotic behavior in finance systems with market confidence [30].

As a substantial part of a robust economic system, financial systems must be gotten further attention and researches. Though up to now, studies have made impressive progress in this field of study, some problems related to the behavior of economic are yet to be resolved. For instance, a few studies there are in the literature about economic models that possess market confidence. Hence, more study is required on fractional-order chaotic financial systems with market confidence to achieve a comprehensive

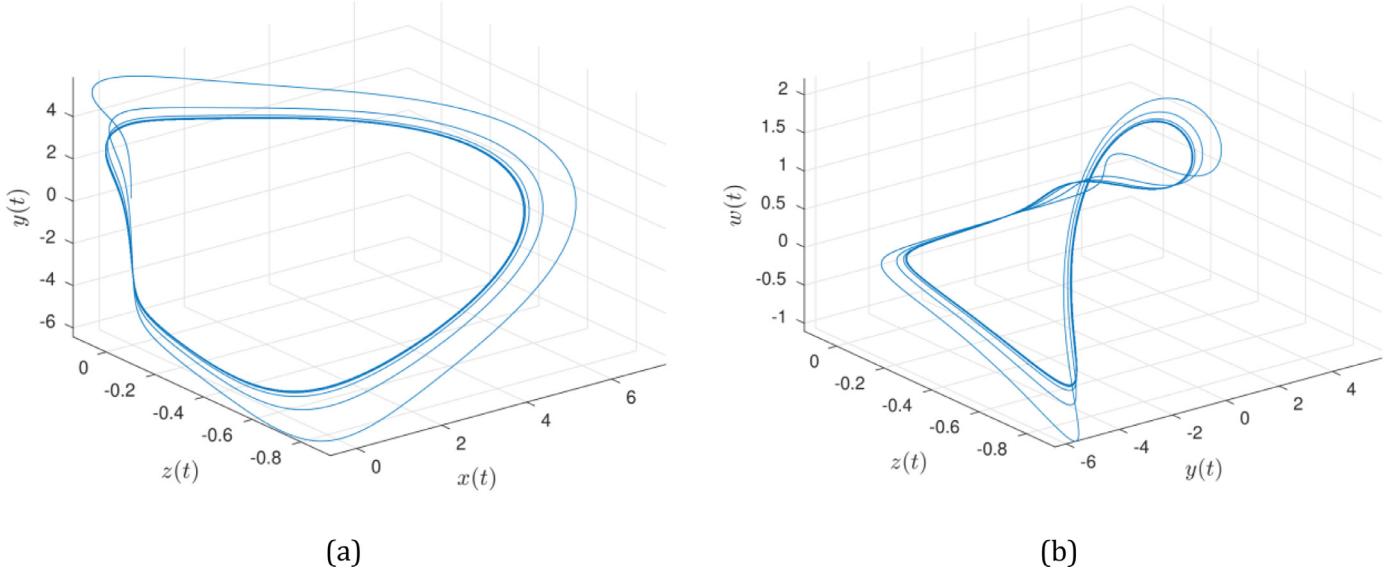


Fig. 3. Phase portraits of the fractional financial system (11). (a) state space \$(x, y, z)\$ and (b) state space \$(y, z, w)\$.

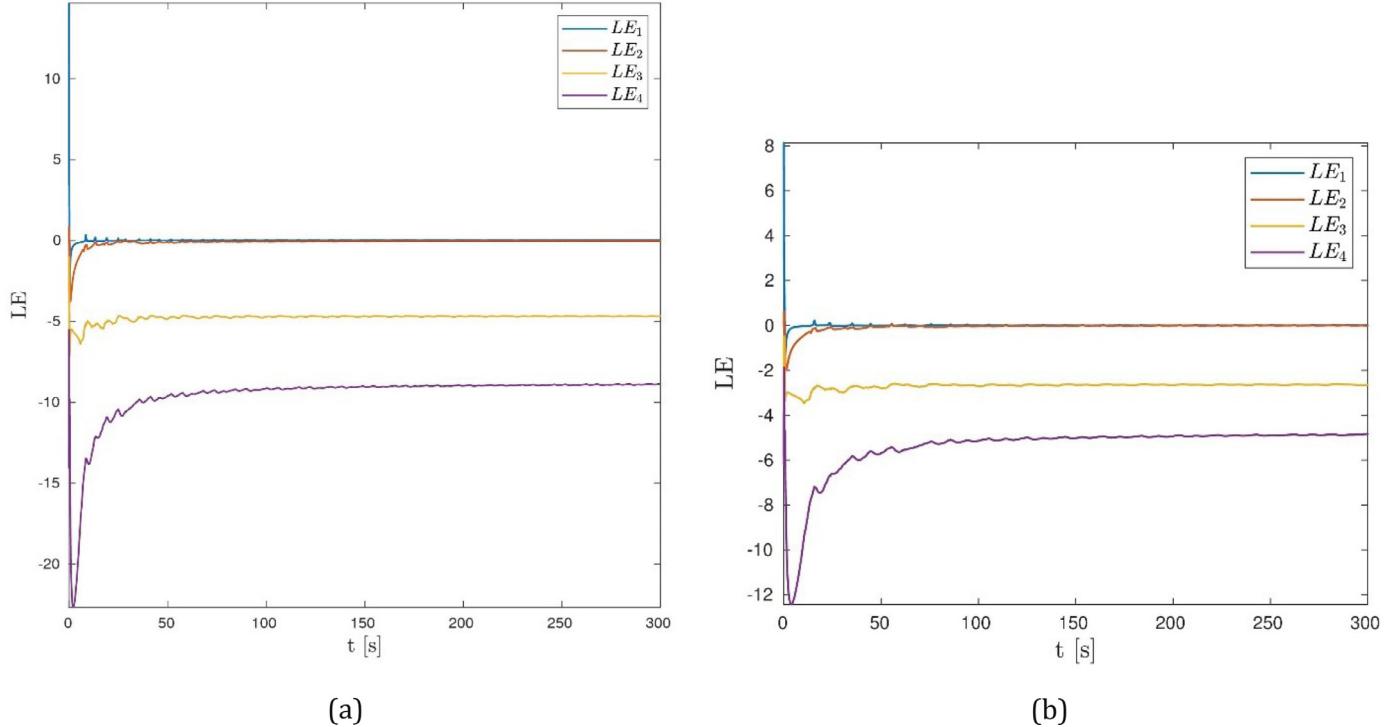


Fig. 4. Dynamics of the LEs of the fractional financial system in (a) Riemann-Liouville and (b) Liouville-CaputoseNSE.

understanding of these systems. These issues motivate the present research. The financial systems with market confidence may show many nonlinear dynamical behaviors, such as bifurcation, chaos, and are thoroughly investigated in this study. Besides, the implementation of chaotic systems is a significant area of interest in the field of chaos [31–33]. Thus, in order to prove its real existence, the chaotic behavior of the system is also realized through electronic circuits.

The article is planned as follows: [Section 2](#) details the mathematical model and dynamical analysis of a fractional-order chaotic financial system with market confidence. Nonlinear dynamics of the system are studied through various tools, including phase portraits, bifurcation diagrams, and Lyapunov exponent spectrum in

[Section 2](#). In [Section 3](#), a circuit implementation is designed to show the behavior of the system in the real-world application, followed by conclusions, presented in [Section 4](#).

2. Mathematical preliminaries

Definition 1. Let $\alpha \in \mathbb{R}_+$ and $n = \lceil \alpha \rceil$. The fractional operator in Riemann-Liouville is given by

$${}_{a}^{RL}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n, \quad (1)$$

where a and t are the operating limits ${}_{a}^{RL}D_t^\alpha$ and $\Gamma(\cdot)$ is the Euler Gamma function.

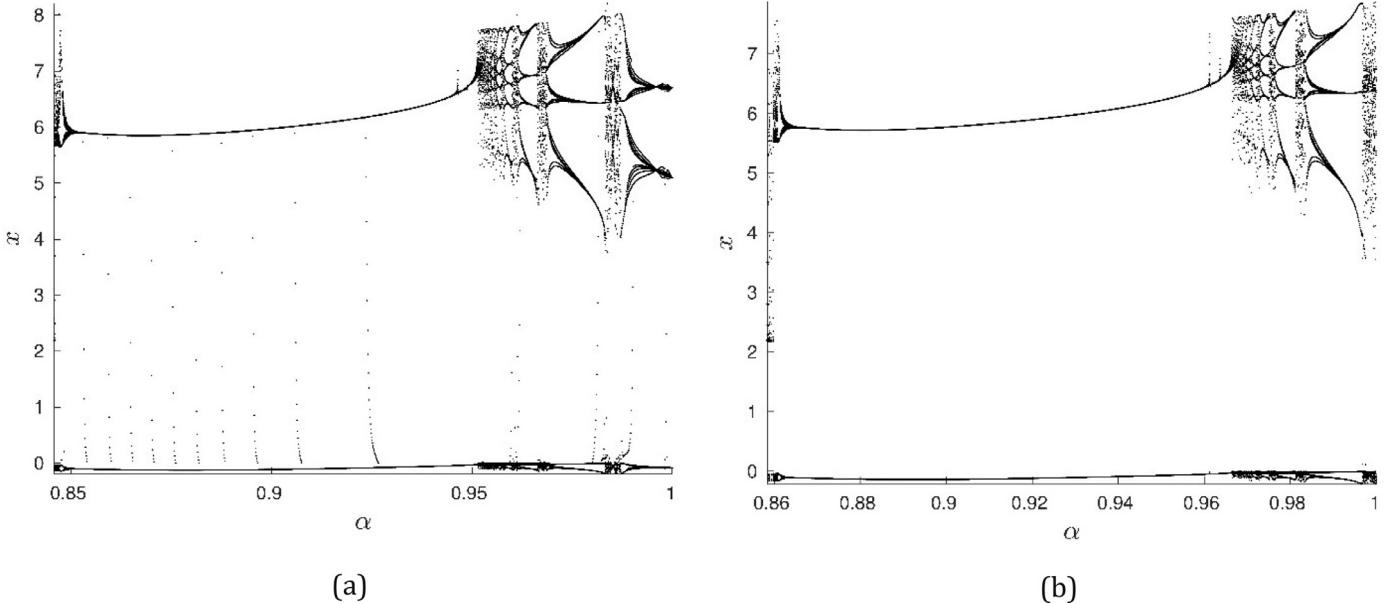


Fig. 5. Bifurcation diagrams for the fractional financial system in (a) Riemann-Liouville and (b) Liouville-Caputo sense.

Definition 2. Let $\alpha \in \mathbb{R}_+$ and $n = \lceil \alpha \rceil$, the Caputo differential operator of order α is defined as Eq. (2):

$${}_{\text{a}}^{\text{C}}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{d^n}{dt^n} f(\tau) (t-\tau)^{n-\alpha-1} d\tau, \quad n-1 < \gamma < n, \quad (2)$$

for $a \leq x \leq t$.

3. Chaotic financial system with market confidence

Xin and Zhang [30] proposed a chaotic dynamic model by considering market confidence. This system was presented in a four-dimensional form and it was obtained from the nonlinear financial model developed by Huang and Li [3]. To achieve the novel financial model, the authors proposed four assumptions where the interest rate (x), the market confidence (w), the investment command (y), and the price index (z) were involved. The above-mentioned system is given as Eq. (3):

$$\begin{aligned} \dot{x} &= z + (y - a)x + m_1 w, \\ \dot{y} &= 1 - by - x^2 + m_2 w, \\ \dot{z} &= -x - cz + m_3 w, \\ \dot{w} &= -xyz, \end{aligned} \quad (3)$$

where x , y , z , a , b , and c have the same meanings. Besides, m_1 , m_2 , and m_3 are the impact factors.

Fig. 1 shows the numerical simulation setting $a = 2.1$, $b = 0.01$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$, and initial conditions $x(0) = -0.01$, $y(0) = 0.5$, $z(0) = 0.004$, and $w(0) = -0.003$. This was carried out with a step size $h = 1 \times 10^{-2}$ during 300s.

3.1. Fractional chaotic financial system

In this section, we get a fractional representation for the system (3), by generalizing the classical operator d/dt . Therefore, the fractional representation is written as Eq. (4) and it includes the term memory effect

$$\begin{aligned} {}_0^{\text{RL}}D_t^\alpha(x) &= z + (y - a)x + m_1 w, \\ {}_0^{\text{RL}}D_t^\alpha(y) &= 1 - by - x^2 + m_2 w, \\ {}_0^{\text{RL}}D_t^\alpha(z) &= -x - cz + m_3 w, \\ {}_0^{\text{RL}}D_t^\alpha(w) &= -xyz, \end{aligned} \quad (4)$$

where Q denotes the fractional-order in any sense.

3.2. Fractional chaotic financial system in Riemann-Liouville sense

By considering Eq. (1), the fractional financial system in Riemann-Liouville is given as

$$\begin{aligned} {}_0^{\text{RL}}D_t^\alpha(x) &= z + (y - a)x + m_1 w, \\ {}_0^{\text{RL}}D_t^\alpha(y) &= 1 - by - x^2 + m_2 w, \\ {}_0^{\text{RL}}D_t^\alpha(z) &= -x - cz + m_3 w, \\ {}_0^{\text{RL}}D_t^\alpha(w) &= -xyz, \end{aligned} \quad (5)$$

3.2.1. Numerical scheme for fractional financial model in Riemann-Liouville sense

To get numerical solutions for the Riemann-Liouville derivative (1), the Grünwald–Letnikov (GL) [34] approximation is carried out

$${}_{\text{a}}^{\text{GL}}D_t^\gamma f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{\frac{t-a}{h}} (-1)^j \binom{\alpha}{k} f(t - kh), \quad (6)$$

where k is the time increment.

For a wide class of systems, the fractional derivative (1) can be approximated by the Eq. (6). Therefore, we start from a general form as Eq. (7)

$${}_{\text{a}}^{\text{GL}}D_t^\gamma x(t) = f(x(t), t), \quad (7)$$

whose numerical solution was presented in [35] and it is given as follows

$$x(t_k) = f(x(t_k), t_k) h^\alpha - \sum_{j=0}^k c_j^{(\alpha)} x(t_{k-j}), \quad (8)$$

where $c_j^{(\alpha)}$, ($j = 0, 1, \dots$) are the binomial coefficients computed by the following expression

$$c_0^{(\alpha)} = 1, c_j^{(\alpha)} = \left(1 - \frac{1+\alpha}{j}\right) c_{j-1}^{(\alpha)}. \quad (9)$$

We consider the Eqs. (8) and (9) to get a numerical solution for the fractional chaotic financial system in the Riemann-Liouville

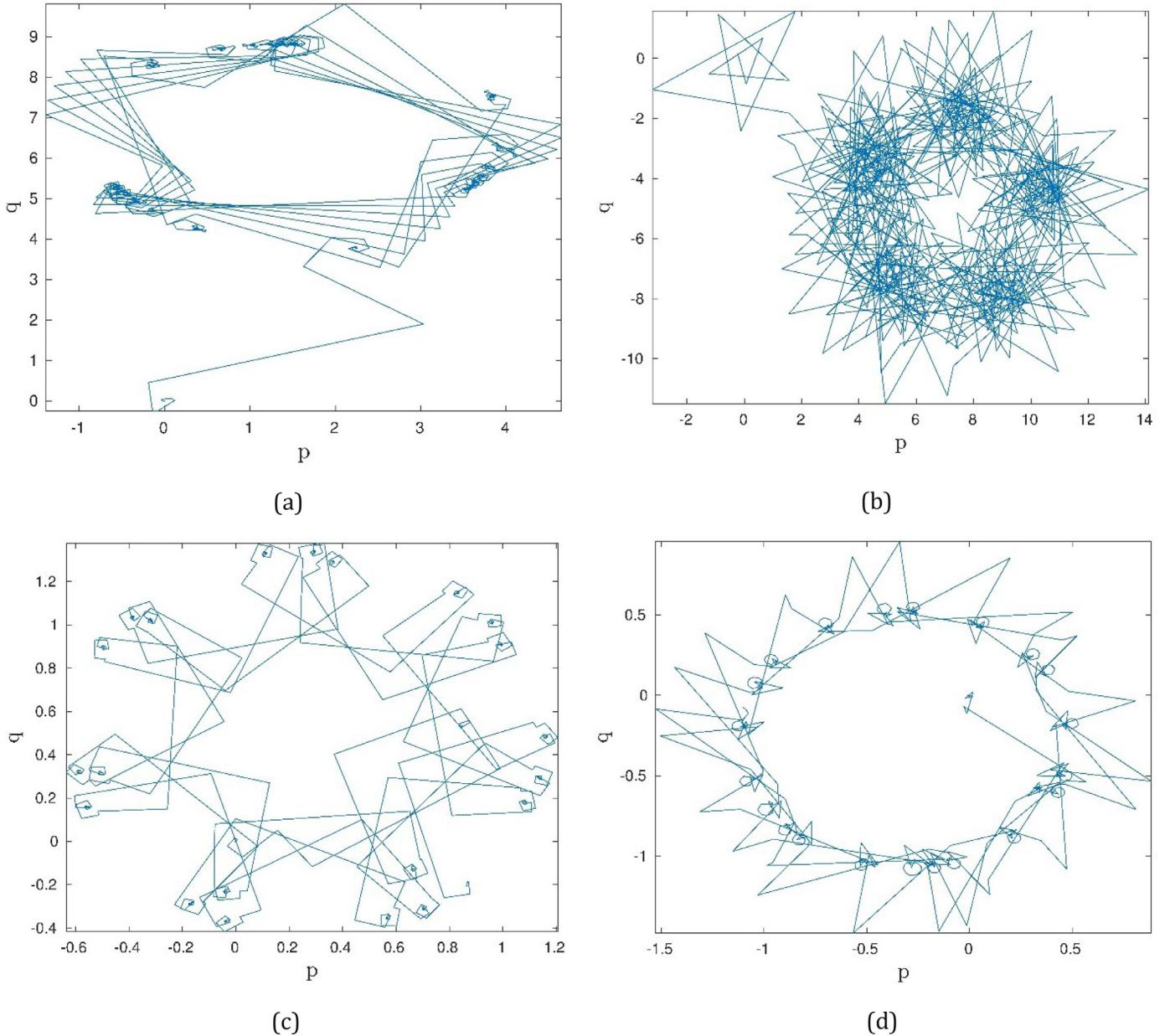


Fig. 6. Variables p and q in the fractional financial system (5) for $\alpha = 0.95$. (a), (c), and (d) show strongly chaotic behaviors for $\phi_1(t) = x, \phi_3(t) = z$, and $\phi_4(t) = w$, respectively; (b) depicts chaotic dynamics with $\phi_2(t) = y$.

sense (5) as Eq. (10)

$$\begin{aligned} x(t_k) &= (z(t_k) + (y(t_k) - a)x(t_k) + m_1 w(t_k))h^\alpha - \sum_{j=0}^k c_j^{(\alpha)} x(t_{k-j}), \\ y(t_k) &= (1 - b y(t_k) - x(t_k)^2 + m_2 w(t_k))h^\alpha - \sum_{j=0}^k c_j^{(\alpha)} y(t_{k-j}), \\ z(t_k) &= (-x(t_k) - c(t_k)z(t_k) + m_3 w(t_k))h^\alpha - \sum_{j=0}^k c_j^{(\alpha)} z(t_{k-j}), \\ w(t_k) &= (-x(t_k)y(t_k)z(t_k))h^\alpha - \sum_{j=0}^k c_j^{(\alpha)} w(t_{k-j}), \end{aligned} \tag{10}$$

Fig. 2 depicts the numerical results setting the following parameters $\alpha = 2.1$, $b = 0.01$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$ and initial conditions $x(0) = -0.01$, $y(0) = 0.5$, $z(0) = 0.004$, and $w(0) = -0.003$. The simulation was achieved in 300s with a step size $h = 1 \times 10^{-2}$ and fractional order $\alpha = 0.95$.

3.3. Fractional chaotic financial system in Liouville-Caputo sense

Fractional chaotic financial system in the Liouville-Caputo sense can be defined as follows:

$$\begin{aligned} {}_0^C D_t^\alpha(x) &= z + (y - a)x + m_1 w, \\ {}_0^C D_t^\alpha(y) &= 1 - by - x^2 + m_2 w, \\ {}_0^C D_t^\alpha(z) &= -x - cz + m_3 w, \\ {}_0^C D_t^\alpha(w) &= -xyz, \end{aligned} \tag{11}$$

3.3.1. Numerical scheme for fractional financial model in Liouville-Caputo sense

Let us consider a fractional differential equation with the Caputo derivative as Eq. (12)

$${}^C_0D_t^\alpha x(t) = f(t, x(t)). \quad (12)$$

If we apply the fractional calculus fundamental theorem, then the above-mentioned equation is expressed as a fractional integral

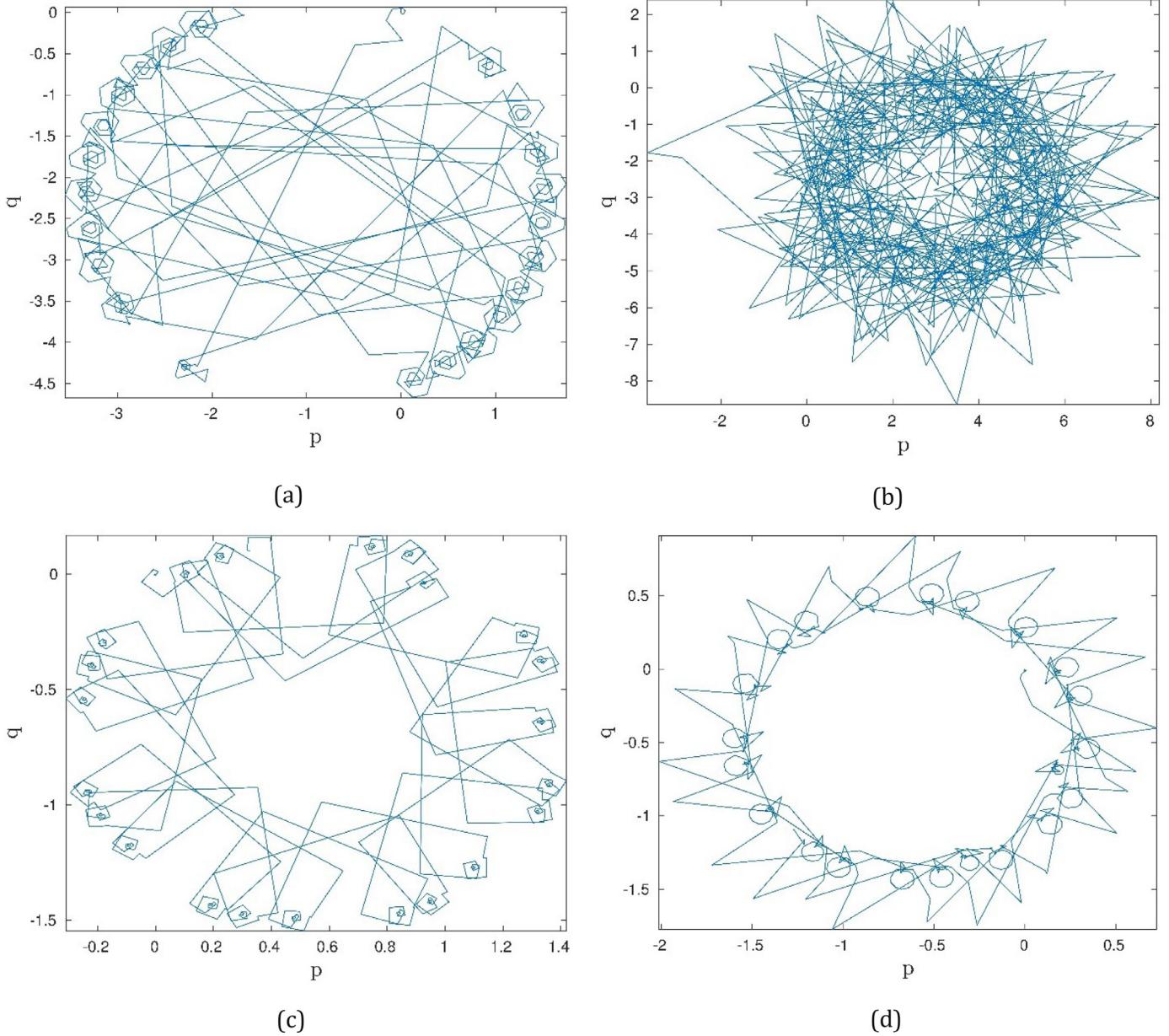


Fig. 7. Variables p and q in the fractional financial system (11) for $\alpha = 0.95$. (a), (b), (c), and (d) show strongly chaotic behaviors for $\phi_1(t) = x, \phi_2(t) = y, \phi_3(t) = z$, and $\phi_4(t) = w$, respectively.

equation of the form

$$x(t) - x(0) = \frac{1}{\Gamma(\alpha)} \int_0^t f(u, x(u)) (t-u)^{\alpha-1} du. \quad (13)$$

Reformulating the Eq. (13) and approximating $f(u, x(u))$ by the two-step Lagrange polynomial interpolation on an interval $[t_k, t_{k+1}]$, Eq. (14) can be obtained as follows:

$$\begin{aligned} x_{n+1}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^n & \left(\frac{f(t_k, y_k)}{h} \int_{t_k}^{t_{k+1}} (t-t_k)(t_{k+1}-t)^{\alpha-1} dt \right. \\ & \left. - \frac{f(t_{k-1}, y_{k-1})}{h} \int_{t_k}^{t_{k+1}} (t-t_k)(t_{k+1}-t)^{\alpha-1} dt \right). \end{aligned} \quad (14)$$

Following with the numerical scheme proposed by, we can compute the numerical simulations for the system (11) as Eq. (15):

$$x_{n+1}(t) = x(0)$$

$$\begin{aligned} & + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^n \left(\frac{f_1(t_k, x_k, y_k, z_k, w_k)}{\alpha+1} ((n+1-k)^\alpha (n-k+2+\alpha) \right. \\ & \left. - (n-k)^\alpha (n-k+2+2\alpha)) \right. \\ & \left. - \frac{f_1(t_{k-1}, x_{k-1}, y_{k-1}, z_{k-1}, w_{k-1})}{\alpha+1} \right. \\ & \times \left. ((n+1-k)^{\alpha+1} - (n-k)^\alpha (n-k+1+\alpha)) \right), \\ y_{n+1}(t) = y(0) & + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^n \left(\frac{f_2(t_k, x_k, y_k, z_k, w_k)}{\alpha+1} ((n+1-k)^\alpha (n-k+2+\alpha) \right. \\ & \left. - (n-k)^\alpha (n-k+2+2\alpha)) \right. \\ & \left. - \frac{f_2(t_{k-1}, x_{k-1}, y_{k-1}, z_{k-1}, w_{k-1})}{\alpha+1} \right. \end{aligned}$$

$$\begin{aligned}
& \times ((n+1-k)^{\alpha+1} - (n-k)^\alpha (n-k+1+\alpha)) \Big), \\
z_{n+1}(t) &= z(0) \\
& + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^n \left(\frac{f_3(t_k, x_k, y_k, z_k, w_k)}{\alpha+1} ((n+1-k)^\alpha (n-k+2+\alpha) \right. \\
& - (n-k)^\alpha (n-k+2+2\alpha)) \\
& - \frac{f_3(t_{k-1}, x_{k-1}, y_{k-1}, z_{k-1}, w_{k-1})}{\alpha+1} \\
& \times ((n+1-k)^{\alpha+1} - (n-k)^\alpha (n-k+1+\alpha)) \Big), \\
w_{n+1}(t) &= w(0) \\
& + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{k=0}^n \left(\frac{f_4(t_k, x_k, y_k, z_k, w_k)}{\alpha+1} ((n+1-k)^\alpha (n-k+2+\alpha) \right. \\
& - (n-k)^\alpha (n-k+2+2\alpha)) \\
& - \frac{f_4(t_{k-1}, x_{k-1}, y_{k-1}, z_{k-1}, w_{k-1})}{\alpha+1} \\
& \times ((n+1-k)^{\alpha+1} - (n-k)^\alpha (n-k+1+\alpha)) \Big), \quad (15)
\end{aligned}$$

where

$$\begin{aligned}
f_1(t_{k-1}, x_{k-1}, y_{k-1}, z_{k-1}, w_{k-1}) &:= z + (y-a)x + \\
m_1 w, f_2(t_{k-1}, x_{k-1}, y_{k-1}, z_{k-1}, w_{k-1}) &:= 1 - by - x^2 + \\
m_2 w, f_3(t_{k-1}, x_{k-1}, y_{k-1}, z_{k-1}, w_{k-1}) &:= -x - cz + \\
m_3 w, f_4(t_{k-1}, x_{k-1}, y_{k-1}, z_{k-1}, w_{k-1}) &:= -xyz,
\end{aligned}$$

Fig. 3 depicts the numerical results setting the following parameters $\alpha = 2.1$, $b = 0.01$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$ and initial conditions $x(0) = -0.01$, $y(0) = 0.5$, $z(0) = 0.004$, and $w(0) = -0.003$. The simulation was achieved in 300s with a step size $h = 1 \times 10^{-2}$ and fractional order $\alpha = 0.95$.

4. Criteria to determine chaos

In order to analyze chaotic behaviors in the fractional systems (5) and (11), estimation of the Lyapunov exponents, bifurcation maps, and 0-1 test were carried out.

4.1. Estimation of the Lyapunov exponents

The get Lyapunov exponents for the fractional financial system in Riemann-Liouville and Liouville-Caputo sense, we use the Benettin-Wolf algorithm presented in [36]. Even when there is not a criterion to choose the normalization moment variable h norm, we carried out several numerical simulations to choose it.

Fig. 4 shows the numerical results by applying this algorithm. Here, one can see that we have only one positive exponent and three negatives. Therefore, the systems (5) and (11) can be considered chaotic. The final value of the LEs presented in **Table 1** were carried out with the following parameters: $a = 2.1$, $b = 0.01$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$ and initial conditions $x(0) = -0.01$, $y(0) = 0.5$, $z(0) = 0.004$, $w(0) = -0.003$.

Table 1
LEs for the fractional financial systems (5) and (11).

Financial system	α	λ_1	λ_2	λ_3	λ_4
Riemann-Liouville sense (5)	0.95	0.0022	-0.0540	-4.6589	-8.7405
Liouville-Caputo sense (11)	0.95	0.0079	-0.0127	-2.5937	-4.6544

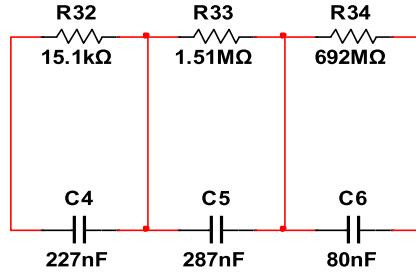


Fig. 8. Fractional-order chain for $q = 0.95$.

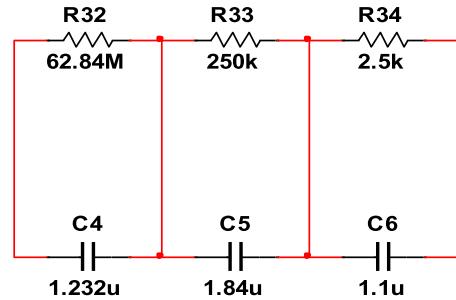


Fig. 9. Fractional-order chain for $q = 0.9$.

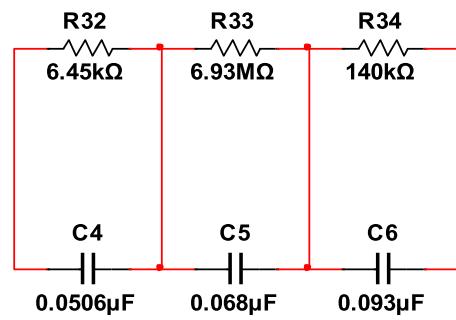


Fig. 10. Fractional-order chain for $q = 0.85$.

4.2. Bifurcation maps

Fig. 5 shows bifurcation maps for the financial systems (5) and (11) respectively. In these maps, one can see regularity windows as well as period-doubling cascade where the order α is near to 0.95 whereas for the system (11) they are present when α is approximately 0.96. These bifurcations give place to chaotic behaviors. Besides, in the regularity windows presented by two maps, it is possible to see that there are pitchfork bifurcations. See **Figs. 5a** and **b**, where $\alpha \in (0.95, 1)$.

4.3. 0-1 Test

The 0-1 test provides the following 2-dimensional system as **Eq. (16)**

$$\begin{aligned}
p(t+1) &= p(t) + \phi(t+1) \cos(c \cdot n), \\
q(t+1) &= q(t) + \phi(t+1) \sin(c \cdot n),
\end{aligned} \quad (16)$$

derived from $\phi(t)$ for $t = 0, 1, 2, \dots, T$. Besides, $c \in (0, 2\pi)$ is fixed.

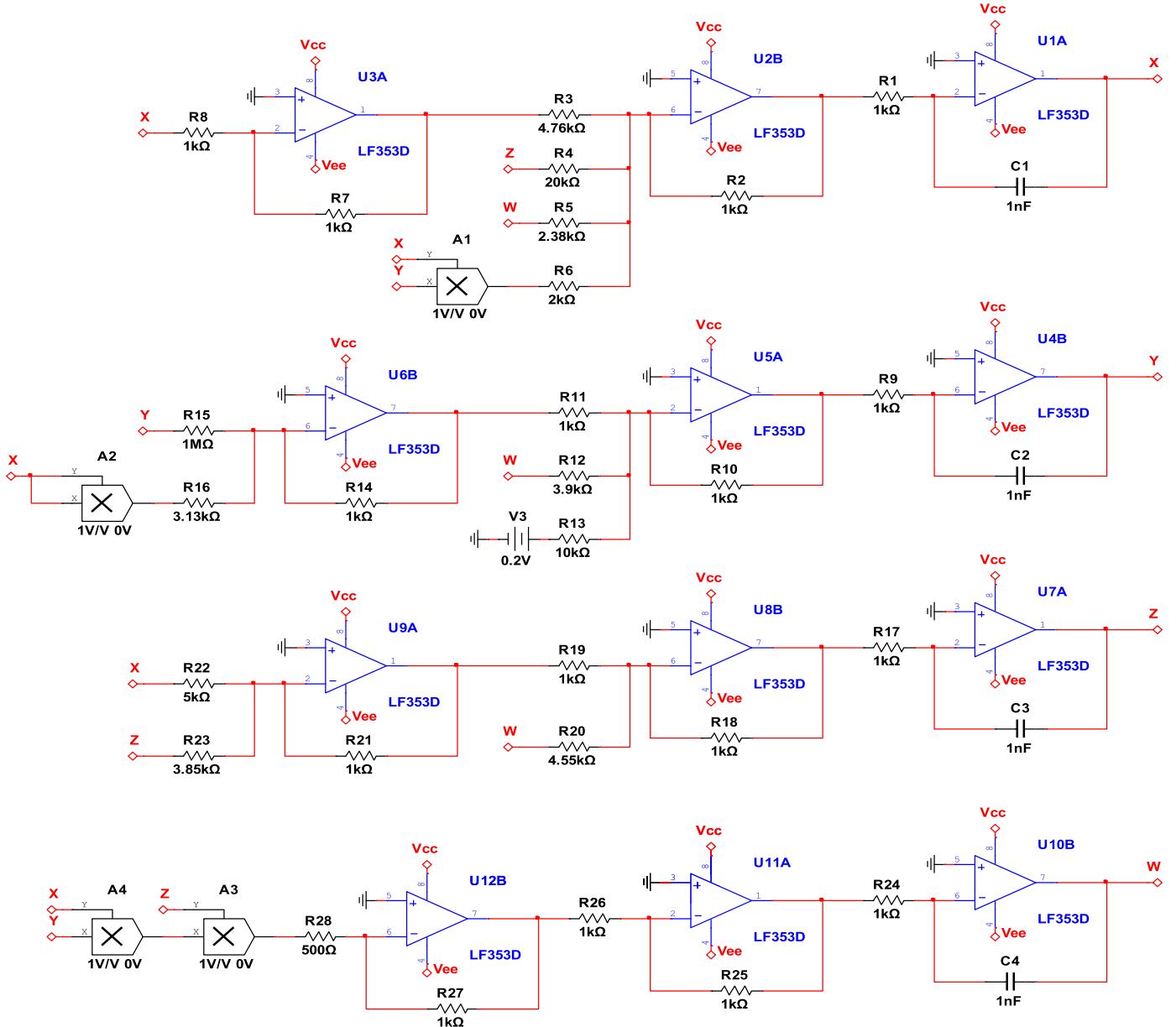


Fig. 11. Circuit realization of the integer-order financial system.

For a continuous-time system, the 2-dimensional mean square displacement is given as follows

$$M_c(t) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T ([p(t + \tau) - p(\tau)]^2 + [q(t + \tau) - q(\tau)]^2) d\tau, \quad (17)$$

where the growth rate is as Eq. (18):

$$K = \lim_{t \rightarrow \infty} \frac{\log(M_c(t))}{\log(t)}. \quad (18)$$

In the above-mentioned equation, $K \approx 1$ means that the system is chaotic and $K \approx 0$ that the system is regular [37]. By approximating the Eq. (17) by means time series sampled, Eq. (19) can be obtained as follows:

$$M_c(n) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N ([p_{\tau_s}(j+n) - p_{\tau_s}(j)]^2 + [q_{\tau_s}(j+n) - q_{\tau_s}(j)]^2) \tau_s^2. \quad (19)$$

Table 2
Growth rates for the fractional financial system.

Dynamical system	x	y	z	w
Fractional financial system (5)	0.9702	0.8903	0.9686	0.9664
Fractional financial system (11)	0.9643	0.9281	0.9669	0.9688

This test was carried out for the systems given in Eqs. (5) and (11) with a sample time $\tau_s = 0.015$. Table 2 shows the growth rates for the fractional financial systems in Riemann-Liouville and Liouville-Caputo sense. Here, we can conclude that both systems are chaotic due to that growth rates are near to 1. Figs. 6 and 7 show the 0-1 test for Riemann-Liouville and Caputo models for fractional order $\alpha = 0.95$.

5. Circuit realization

Because fractional calculus does not allow the direct calculation of the differential operators in time domain, to design a circuit we

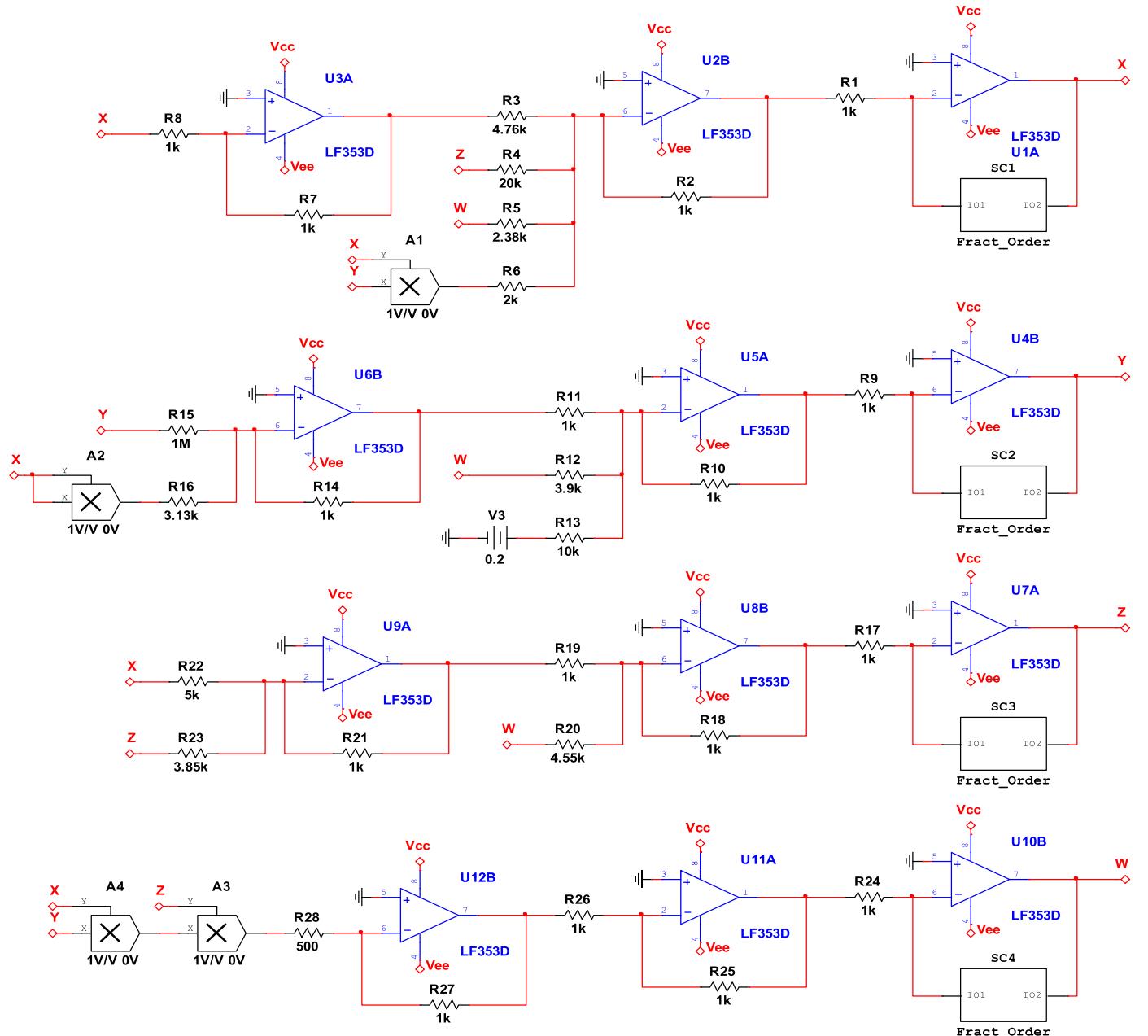


Fig. 12. Circuit realization of the fractional-order financial system.

usually adopt the method of approximation conversion from time domain to frequency domain. Based on Bode diagrams, Liu [38] developed an effective algorithm to approximate the fractional-order transfer functions by utilizing frequency-domain techniques. Based on this, we deduced $1/s^q$ ($q = 0.1 \sim 0.99$) with discrepancies 2dB and 3dB through linear approximation in frequency domain. Here, we utilize the approximation of $1/s^q$ ($q = 0.1 \sim 0.9$) with discrepancy of 2dB to design the analog circuits. According to [39], Eqs. (20-22) can be obtained as follows:

$$\frac{1}{s^{0.95}} = \frac{1.2834s^2 + 18.6004s + 2.0833}{s^3 + 18.4738s^2 + 2.6574s + 0.003} \quad (20)$$

$$\frac{1}{s^{0.9}} = \frac{2.2675(s + 1.292)(s + 215.4)}{(s + 0.01292)(s + 2.154)(s + 359.4)} \quad (21)$$

$$\frac{1}{s^{0.85}} = \frac{2.743s^2 + 294.7s + 2810.6}{s^3 + 184.8s^2 + 897.5s + 114.5} \quad (22)$$

A circuit design where resistors and capacitors are connected in parallel was proposed in [40] to realize fractional calculus. When $q = 0.95$, $q = 0.9$, and $q = 0.85$, the fractional order chain circuit units are shown in Figs. 8-10.

We select LF353D as the amplifier and AD633JN as the multiplier to design the fractional-order circuits. In order to restrict the change of state variables to the operating voltage of the analog circuit, the state variables are reduced by 4, 5, 2 and 2 times, namely let $(x, y, z, w) \rightarrow (4X, 5Y, 2Z, 2W)$.

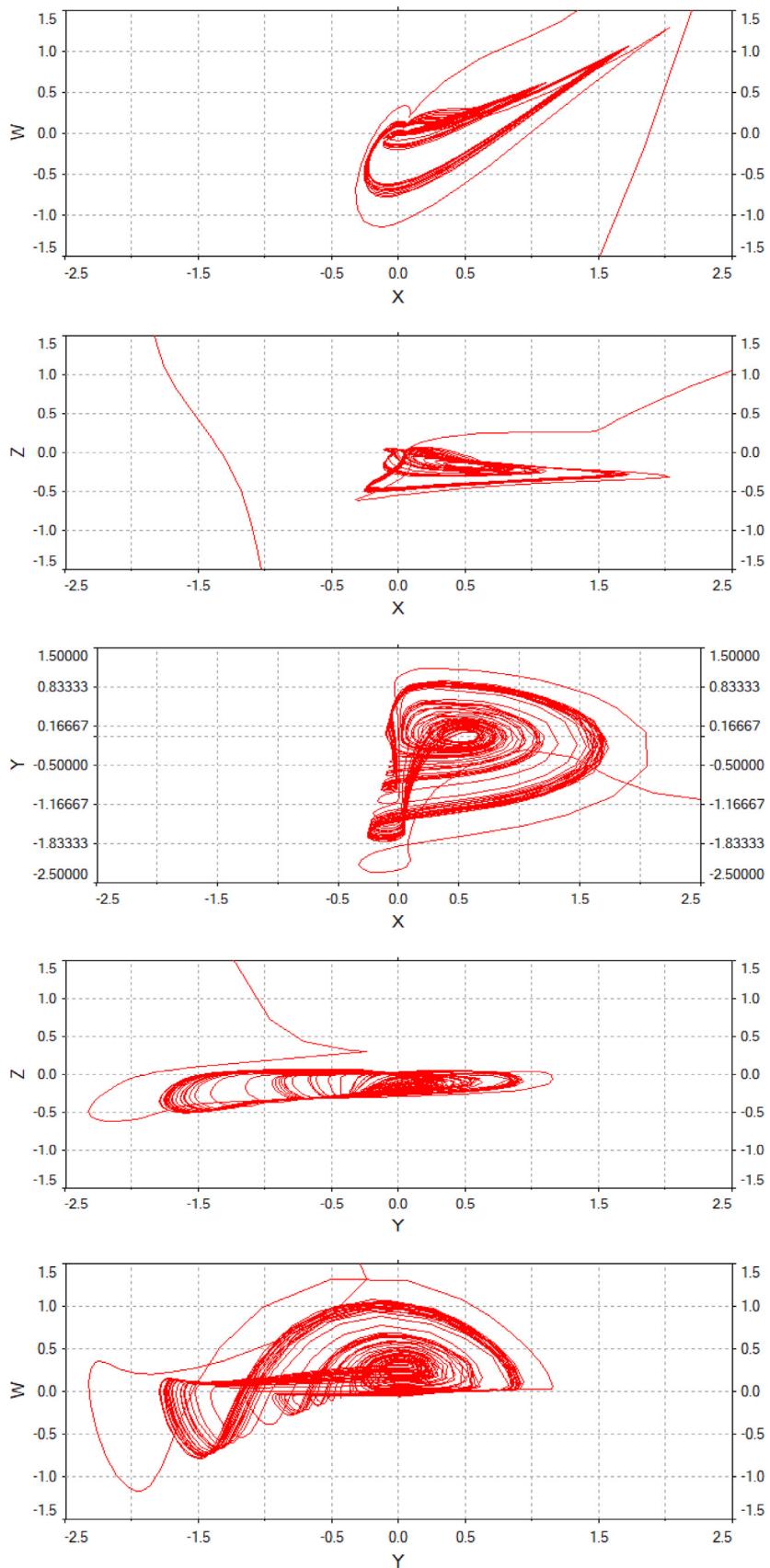
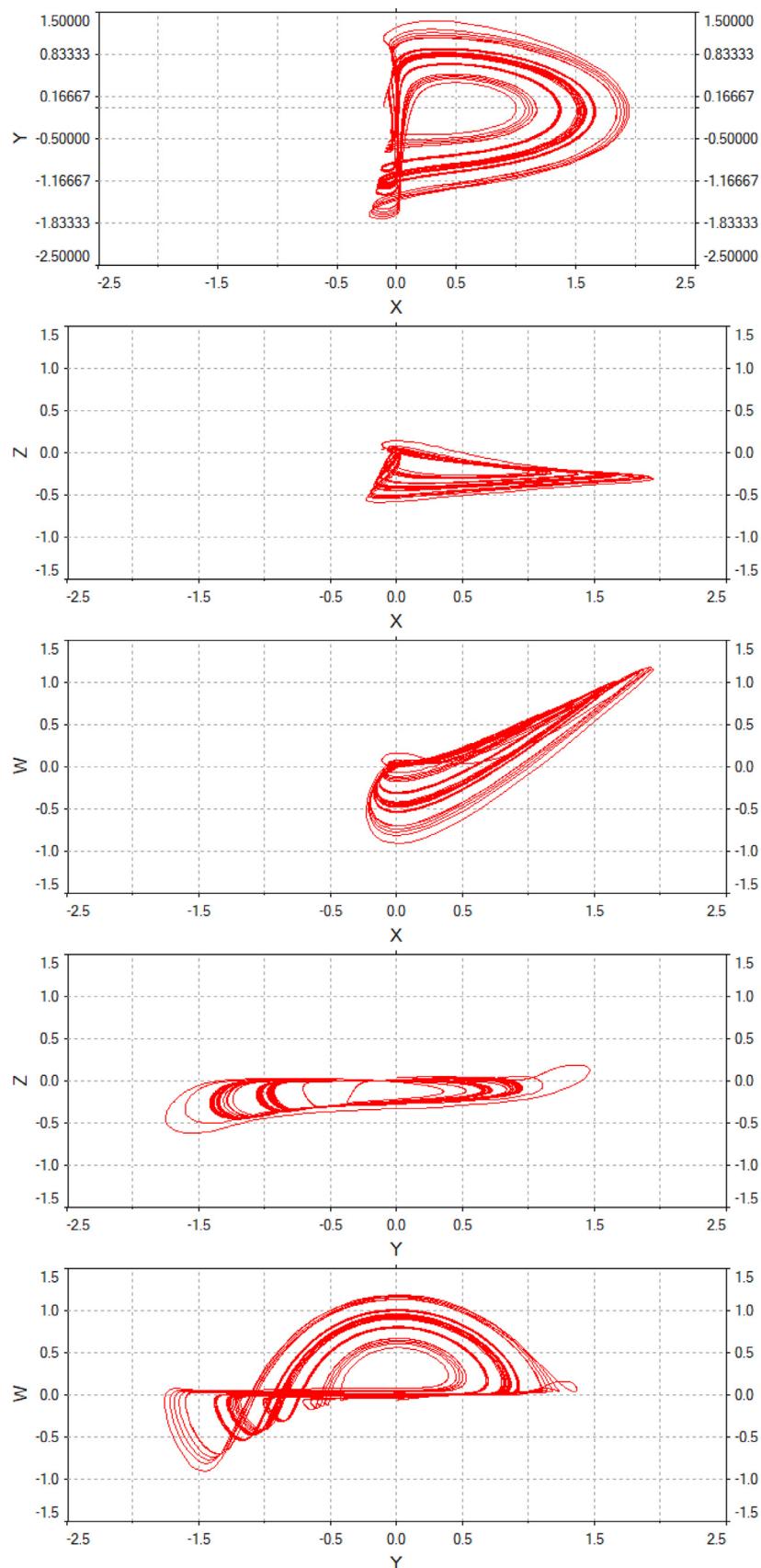


Fig. 13. Phase portraits of the finance system for $q = 1$.

**Fig. 14.** Phase portraits of the finance system for $q = 0.95$.

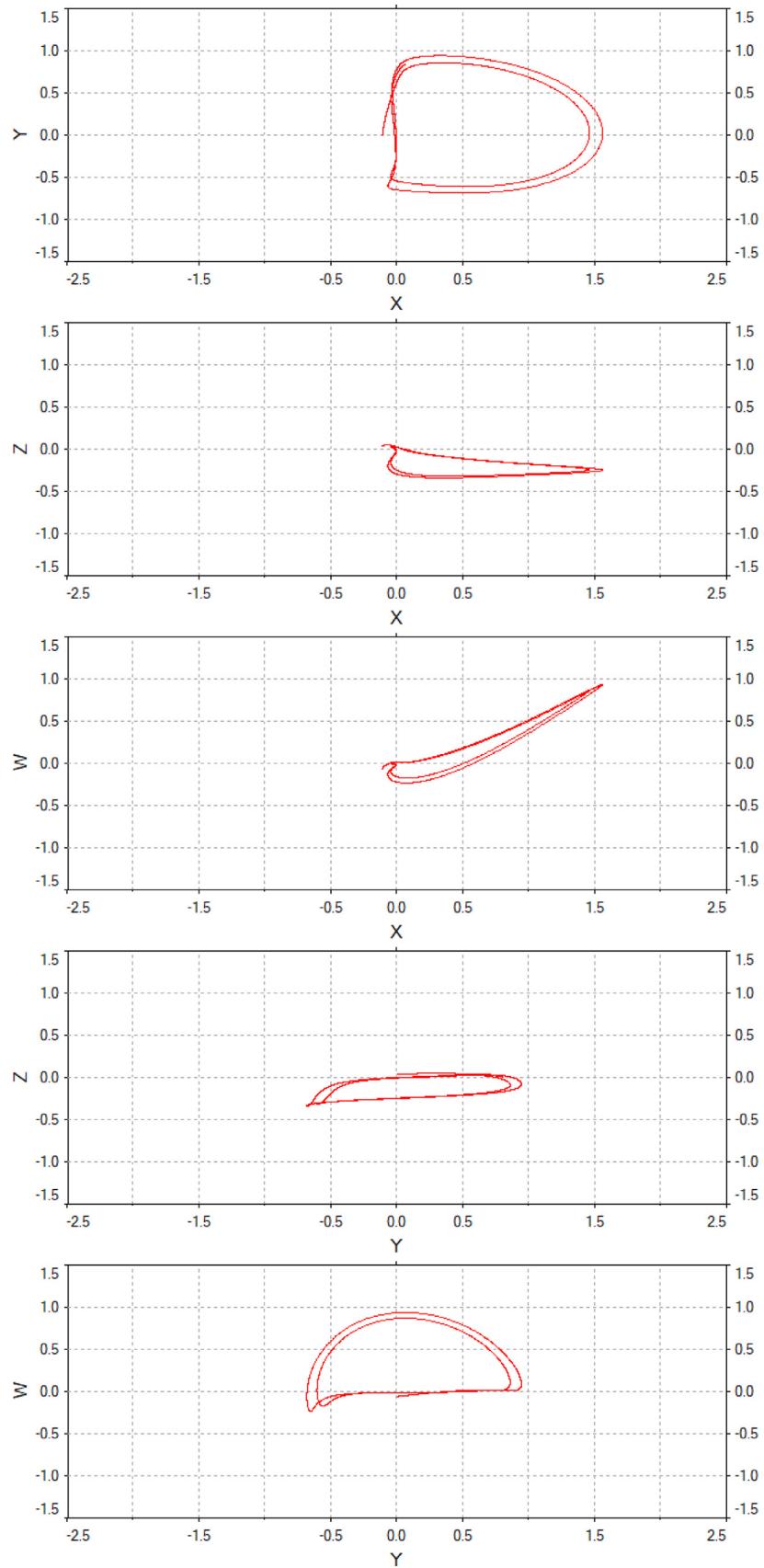


Fig. 15. Phase portraits of the finance system for $q = 0.9$.

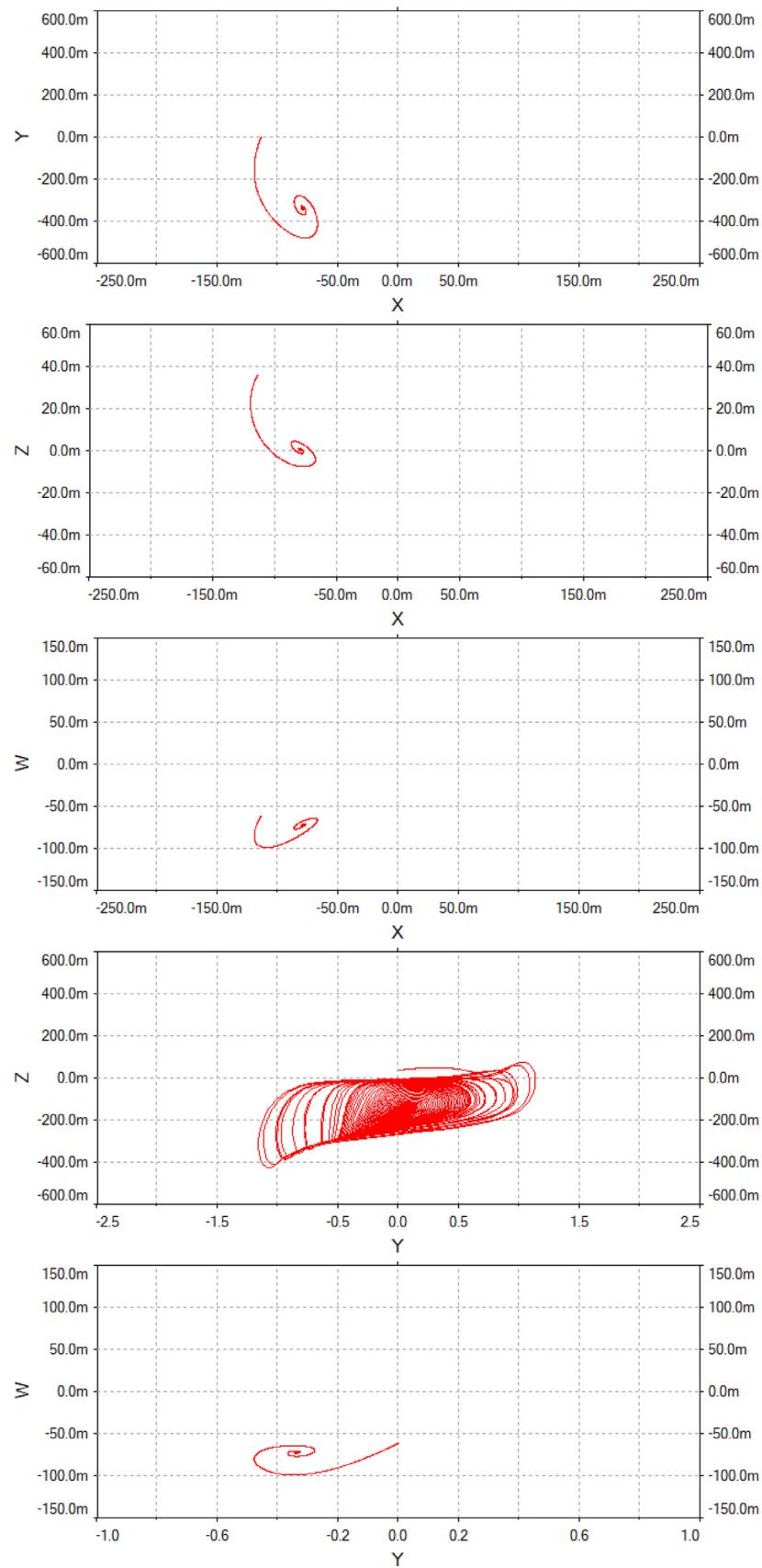


Fig. 16. Phase portraits of the finance system for $q = 0.85$.

As a result, the financial system with market confidence can be rewritten as Eq. (23):

$$\begin{cases} \frac{d^q X}{dt^q} = \frac{1}{2}Z + 5XY - aX + \frac{1}{2}m_1W, \\ \frac{d^q Y}{dt^q} = \frac{1}{5} - bY - \frac{16}{5}X^2 + \frac{2}{5}m_2W, \\ \frac{d^q Z}{dt^q} = -2X - cZ + m_3W, \\ \frac{d^q W}{dt^q} = -20XYZ, \end{cases} \quad (23)$$

By replacing a , b , c , m_1 , m_2 , and m_3 with their values, the system (4) can be shown as Eq. (24):

$$\begin{cases} \frac{d^q X}{dt^q} = 0.5Z + 5XY - 2.1X + 4.2W, \\ \frac{d^q Y}{dt^q} = 0.2 - 0.01Y - 3.2X^2 + 2.56W, \\ \frac{d^q Z}{dt^q} = -2X - 2.6Z + 2.2W, \\ \frac{d^q W}{dt^q} = -20XYZ, \end{cases} \quad (24)$$

The schematic circuits of the integer-order and fractional-order financial system with market confidence are given by Figs. 11 and 12. The circuit equation is as Eq. (25):

$$\begin{cases} \frac{d^q X}{dt} = \frac{R_2}{R_1 R_4}Z + \frac{R_2}{R_1 R_6}XY - \frac{R_2 R_7}{R_1 R_3 R_8}X + \frac{R_2}{R_1 R_5}W, \\ \frac{d^q Y}{dt} = \frac{R_{10}}{R_9 R_{13}}0.2 - \frac{R_{10} R_{14}}{R_9 R_{11} R_{15}}Y - \frac{R_{10} R_{14}}{R_9 R_{11} R_{16}}X^2 + \frac{R_{10}}{R_9 R_{12}}W, \\ \frac{d^q Z}{dt} = -\frac{R_{18} R_{21}}{R_{17} R_{19} R_{22}}X - \frac{R_{18} R_{21}}{R_{17} R_{19} R_{23}}Y + \frac{R_{18}}{R_{17} R_{20}}W \\ \frac{d^q W}{dt} = \frac{R_{25} R_{27}}{R_{24} R_{26} R_{28}}XYZ \end{cases} \quad (25)$$

The input supplies are as $V_{cc} = +15V$, $V_{ee} = -15V$. The values of the electronic components in Figs. 11 and 12 are chosen to match the known parameters of the system (23) as follows:

$R_1 = R_9 = R_{17} = R_{24} = 100k\Omega$, $R_2 = R_7 = R_8 = R_{10} = R_{11} = R_{14} = R_{18} = R_{19} = R_{21} = R_{27} = 1k\Omega$, $R_{13} = R_{15} = R_{25} = R_{26} = 10k\Omega$, $R_3 = 4.76k\Omega$, $R_4 = 20k\Omega$, $R_5 = 2.38k\Omega$, $R_6 = 2k\Omega$, $R_{12} = 3.90k\Omega$, $R_{16} = 3.13k\Omega$, $R_{20} = 4.55k\Omega$, $R_{23} = 3.85k\Omega$, $R_{28} = 500\Omega$.

The proposed circuits are designed by using Electronic Work Bench (EWB). Figs. 13–16 show the obtained phase portraits in (X, Y) -plane, (X, Z) -plane, (X, W) -plane, (Y, Z) -plane and (Y, W) -plane, respectively.

6. Conclusion

A fractional-order financial system by considering the effects of market confidence was studied. Using phase portraits, Lyapunov exponents, and bifurcation diagrams, the dynamical behavior of the proposed system was investigated. It was demonstrated that the proposed system goes to chaos through experiencing a cascade of period doublings, and the existence of chaos is verified. Then, using the Electronic Work Bench, circuit implementation was successfully performed for integer and fractional-order financial systems to demonstrate the existence of the chaos in the system. Through circuit realization, it was clearly demonstrated that by changing the value fractional-order derivative, the system shows different responses. As a future suggestion, the complexity analysis could be conducted for the fractional-order chaotic financial system with market confidence. Also, the presented finance model can be investigated by a variable-order fractional derivative.

Declaration of Competing Interest

This statement is to certify that no conflict of interest exists in the submission of this manuscript. Also, the manuscript is approved for publication by all authors. I would like to declare that the work described was an original research that has not been published previously, and not under consideration for publication elsewhere. All the authors listed have approved the manuscript that is enclosed.

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