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Abstract

We prove the existence of symmetric pure Cournot equilibria with heterogeneous goods under the following condition: Each firm reacts to a rise in competitors' output in such a way that its market price does not rise. This condition is not related to whether goods are strategic substitutes or complements. We establish that maximum and minimum equilibrium prices are decreasing as more firms enter if competitors' outputs enter inverse demand aggregated; for non-aggregative demand prices may increase. Total quantity increases only if each firm's market price is more affected by its own output than competitors' outputs.

JEL classification: C62, D43, L13

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1 Introduction

Since Cournot's early contribution his model of oligopoly has received more and more attention, and nowadays is a basic building block of applied work on a wide range of topics involving imperfect competition. Its usefulness depends on two features: First, existence and uniqueness of equilibria at the market stage must be easily established, and second, comparative statics results should be readily available. In the context of homogeneous goods both these aspects have been treated extensively, whereas for heterogeneous goods there are much fewer results available.

It is possible to ascertain the existence of pure Cournot equilibria under the assumption that profits are concave, using the general result that games with concave payoffs possess pure equilibria (see Friedman 1991). This condition is not easily translatable into assumptions about demand and production costs, therefore there have been many attempts to identify those features that guarantee the existence of equilibria. The first strand of the literature identified conditions on demand that could be fruitfully exploited: Novshek (1985), Kukushkin (1994) and Corchón (1994, 1996) assume that goods are strategic substitutes, while Vives (1990) assumes that goods are strategic complements. Therefore it is assumed that firms' reaction functions are either decreasing or increasing. The second strand initially imposed assumptions only on costs and proved the existence of symmetric equilibria: McManus (1962, 1964), and Roberts and Sonnenschein (1976) assumed that costs were linear or convex, i.e. had nonincreasing returns to scale. Amir and Lambson (1998) showed for homogeneous goods that it was possible to allow for limited increasing returns to scale in production, resulting in a condition that combines both the demand and cost functions. It is interesting to note that it is not by chance that these two strands of the literature exist, since fundamentally each strand uses one of the two stability conditions by Hahn (1962), which impose different kinds of regularity on the model.

Most of the above authors have only covered the case of homogeneous goods. Kukushkin (1994) and Corchón (1994, 1996) deal with additive aggregation, i.e. where the *sum* of competitors' outputs is relevant, but assume strategic substitutes; Vives (1990) allows for general non-homogeneous goods, but under strategic complements. Spence (1976) indicates how to prove existence of Nash equilibria for a special class of inverse demand functions with heterogeneous goods. Our work is the first to address the question of existence of equilibria with heterogenous goods in a general context that does make not use of the assumption of strategic substitutes or complements. Rather, it is based on the second strand of literature and directly generalizes Amir and Lambson's (1998) work to heterogeneous goods.

We impose the condition that firms react to a rise in competitors' quantities by adjusting their own production in such a way that their market price does not rise (condition A). Doing so, output may increase or decrease, but must not decrease too strongly. This condition is formulated without making use of differentiability or convexity assumptions, rather it is formulated in lattice-theoretic terms as a single-crossing condition. If goods are substitutes, and under standard regularity conditions, we show that this condition implies the existence of symmetric pure Cournot equilibria even when outputs are heterogeneous.

Concerning uniqueness, asymmetric equilibria can be ruled out if we add the additional weak assumption that own market price reacts more to changes in own output than in competitors' outputs (condition B). Multiple symmetric equilibria can be excluded only under much stronger assumptions.

Comparative statics on demand or cost variables for Cournot oligopoly have been analyzed by many authors, among them Frank (1965), Dixit (1986), Corchón (1994, 1996), while comparative statics with respect to the number of firms have been discussed by Frank (1965), Ruffin (1971), Seade (1980), Szidarovsky and Yakowitz (1982), Corchón (1994, 1996), and recently by Amir and Lambson (1998). The case of heterogeneous goods has been treated by Dixit (1986) for two firms, and by Corchón (1994, 1996) for additive aggregation, but they as most authors have imposed the assumption of strategic substitutes from the outset, which is irrelevant for most comparative-statics conclusions. In fact, Amir and Lambson have shown for homogeneous goods that the only relevant condition for decreasing equilibrium prices and increasing equilibrium total quantity is that there are no strong increasing returns to scale.

One of the fundamental conclusions of this literature is that stability of equilibrium is closely connected to "non-paradoxical" comparative statics results. Our analysis for heterogeneous goods, which is based on lattice-theoretic monotone comparative statics methods, makes precise predictions for maximal and minimal equilibria that do not rely on stability, while we show that comparative statics results for arbitrary equilibria continue to depend decisively on the stability of equilibrium.

In this work we will concentrate exclusively on the comparative statics of entry. Our main result is that if competitors' quantities enter inverse demand in some aggregated form, then equilibrium prices do not increase as more firms enter the industry. We also show by means of an example that this result is not extendable to general heterogenous goods, i.e. equilibrium prices may rise even if there are no increasing returns to scale and equilibrium is stable.

Total equilibrium output may rise or fall even if prices are decreasing, but will rise under the same condition that we already used to rule out asymmetric equilibria. Individual output rises or falls depending on whether goods are strategic complements or substitutes, while profits always decrease.

The rest of our paper continues as follows: Section 2 sets out the model, and section 3 introduces the main condition. Existence results are presented in section 1, and related conditions and some examples are discussed in sections 5 and 6. Section 7 presents our comparative statics results, and section 8 concludes. All proofs are in the appendix.

2 The Model

There are n firms with identical finite production capacities[1] $0 < K < \infty$ and identical production cost functions $c : [0, K] \to \mathbb{R}_+$, which are assumed to be lower semi-continuous.

Denote firm i's output quantity by x_i, and by x_{-i} the vector of outputs of the other firms. Inverse demand of firm i ($i = 1..n$) is given by a continuous function $p : [0, K]^n \to \mathbb{R}_+$, with $p_i = p(x_i, x_{-i})$,[2] which is symmetric in the other firms' outputs: Let $\tilde{x}_{-i}$ be any permutation of x_{-i}, then $p(x_i, \tilde{x}_{-i}) = p(x_i, x_{-i})$ for all $(x_i, x_{-i}) \in [0, K]^n$. That is, firms are completely symmetric in that, apart from identical production technologies, all demand functions have the same functional form and all competitors' goods enter each demand function symmetrically. Assume that p is nonincreasing in x_i and x_{-i}, i.e. in particular goods are substitutes, and strictly decreasing in x_i where inverse demand is positive.

Firm i's profits are, for $x_i \in [0, K]$ and given $x_{-i} \in [0, K]^{n-1}$,

$$\Pi(x_i, x_{-i}) = x_i p(x_i, x_{-i}) - c(x_i), \ i = 1 \ldots n. \tag{1}$$

We will now reformulate profits Π as a function of firm i's *market price* p_i and the other firms' outputs[3]. As shown in the appendix, there is a set $X \subset \mathbb{R}_+ \times [0, K]^{n-1}$ such that we can express firm i's output as a function $\chi : X \to [0, K]$ of own market price p_i and the others' output that is continuous and nonincreasing in $(p_i, x_{-i}) \in X$, and strictly decreasing in p_i. Also, there is a new constraint set $\pi(x_{-i})$ that is non-empty, closed, compact, and nonincreasing in x_{-i}. Firm i's maximization problem can

[1]Alternatively, as is often done, one may assume that inverse demand falls below marginal cost (given any output of rivals) or even becomes zero for outputs larger than a certain limit. All these assumptions ensure that firms' outputs are bounded.

[2]By $p(x_i, x_{-i})$ we do mean that x_i is the first argument of p, i.e. that for all i own quantity x_i enters firm i's inverse demand differently from other firms' quantities x_j, $j \neq i$.

[3]Note that the variable to be maximized over (output or price) is irrelevant as long as afterwards each firm 'commits' to a fixed production quantity or at least competitors believe that it is so.

then be expressed as

$$\max_{p_i \in \pi(x_{-i})} \tilde{\Pi}(p_i, x_{-i}) = \chi(p_i, x_{-i}) p_i - c(\chi(p_i, x_{-i})), \tag{2}$$

resulting in the price best response[4] $P(x_{-i})$.

If we consider profits $\tilde{\Pi}(p_i, x_{-i})$ "on the diagonal" where all competitors produce the same amount $y \in [0, K]$, we can define

$$\hat{\Pi}(p_i, y) = \tilde{\Pi}(p_i, y, \ldots, y). \tag{3}$$

Some special cases are, in order of increasing specialization, what we will call "*Competitor aggregation*", "*industry aggregation*",[5] and *homogeneous goods.* Under competitor aggregation there are functions $\hat{p} : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $f : [0, K]^{n-1} \to \mathbb{R}_+$, where f is strictly increasing, such that

$$p(x_i, x_{-i}) = \hat{p}(x_i, f(x_{-i})), \tag{4}$$

where the competitors' quantities are aggregated into one number. One example is *additive aggregation* with $f(x_{-i}) = Y_i = \Sigma_{j \neq i} x_j$.

Under industry aggregation there exists a function $\bar{p} : \mathbb{R}_+ \to \mathbb{R}_+$ and $\delta \in \mathbb{R}_+$ such that

$$p(x_i, x_{-i}) = \bar{p}(x_i + \delta Y_i), \tag{5}$$

and if goods are homogeneous then $\delta = 1$.

For industry aggregation it is easy to see that $\chi(p_i, x_{-i}) = D(p_i) - \delta Y_i$, where $D = \bar{p}^{-1}$ is the demand function, and the profit maximization problem becomes

$$\max_{p_i \in [\bar{p}(K+\delta Y_i), \bar{p}(\delta Y_i)]} \tilde{\Pi}(p_i, x_{-i}) = (D(p_i) - \delta Y_i) p_i - c(D(p_i) - \delta Y_i),$$

where for identical outputs by firm i's competitors,

$$\hat{\Pi}(p_i, y) = (D(p_i) - \delta(n-1)y) p_i - c(D(p_i) - \delta(n-1)y).$$

[4] We adopt this formulation to avoid confusion with the standard (quantity) best response or reaction function $x_i = r(x_{-i})$.

[5] I would like to thank Karl Schlag for proposing these terms.

3 The Condition

Our main condition on profits is of a type that has recently been shown to be of central importance in any exercise of comparative statics: In a lattice-theoretic context, Milgrom and Shannon (1994) have shown that the set of maximizers of the parametric maximization problem $\max_{x \in S} f(x,t)$, where $S \subset \mathbb{R}$, is nondecreasing in (t, S) *if and only if f satisfies the weak single crossing property in* (x,t): for all $x' > x$ and $t' > t$ we have that

$$f(x',t) - f(x,t) \geq (>)\, 0 \;\Rightarrow\; f(x',t') - f(x,t') \geq (>)\, 0.$$

In the context of game theory this result can be applied to best response maps, and we do so after our change of variables from own quantity to own price described above. Underlying our results is the following condition:

Condition A: $\hat{\Pi}(p_i, y)$ satisfies the dual[6] weak single crossing property in (p_i, y), i.e. for all $p_i' > p_i$ and $y' > y$ we have that

$$\hat{\Pi}(p_i', y) - \hat{\Pi}(p_i, y) \leq (<)\, 0 \;\Rightarrow\; \hat{\Pi}(p_i', y') - \hat{\Pi}(p_i, y') \leq (<)\, 0. \tag{6}$$

Even though this condition seems to be extremely abstract, its interpretation is very simple and economically intuitive: Condition A means that, *starting from a situation where all other firms produce identical quantities, if the other firms raise their outputs by the same amount, it will be advantageous for firm i to adjust its output only in such a way that the resulting market price is not higher than before.* Doing so, own output may increase or decrease, depending on whether goods are strategic substitutes or complements. This condition is a very natural condition to consider when one is interested in how equilibrium prices changes with entry of new firms in a setting where all firms are equal; here we will show that it even implies existence of equilibrium, subject to some regularity conditions.

[6]This is a "dual" single-crossing property because the inequality signs in the definition are reversed.

Note that condition A imposes the dual single crossing property only on the "diagonal", i.e. where competitors all produce the same quantity. This is sufficient for the following existence result since we are only interested in symmetric equilibria, while we will have to state a condition covering the whole space of outputs to deal with asymmetric equilibria.

In addition, condition A is formulated for identical increases in output for all competitors. This is equivalent to formulating the corresponding condition in terms of an increase in just one competitor's quantity as long as inverse demand is symmetric in competitors' outputs, while it is more general if inverse demand is not symmetric. Condition A therefore even applies to cases where firms are identical but inverse demands are not symmetric in all competitors' outputs. One example of this is a situation where each firm only has two neighbors, as in a "circular city" model. In this paper we will concentrate on the symmetric case.

It is important to note that condition A rules out the existence of *avoidable fixed cost*, i.e. fixed costs that are not incurred if nothing is produced: If they were present, firm i might prefer to stop producing at all (in effect raising own price), instead of lowering its own price, if the other firms raise their outputs. Any other upward jump in production cost is similarly excluded.

Condition A applies no matter whether inverse demand and production costs are differentiable or not. Since it is an ordinal condition, it is not surprising that there is no *equivalent* condition in terms of derivatives even if demand or cost are differentiable. Using the *method of dissection* discussed in Milgrom and Shannon (1994, p. 167), we can find a sufficient differential condition that is slightly stronger than condition A (see appendix 9.2). If Π^{ii} and Π^{ij} are the second partial derivatives of the profit function of firm i with respect to outputs, and p^i and p^j the partial derivatives of the inverse demand of firm i with respect to x_i and x_j, condition A is implied by

Condition AD: For all i, $x_i \in [0, K]$*, and* $x_{-i} = (y, ..., y) \in [0, K]^{n-1}$,

$$\Pi^{ii}(x_i, x_{-i}) - \frac{p^i(x_i, x_{-i})}{p^j(x_i, x_{-i})}\Pi^{ij}(x_i, x_{-i}) \leq 0. \tag{7}$$

In the cases of industry aggregates or homogeneous goods, condition AD reduces to the condition (as we discuss in section 6) $p' - c'' \leq 0$. Here condition AD has the following interpretation: There are *at most weakly increasing returns to scale*, or *profit margins* $p - c'$ *are falling in own output.*[7] The relation between conditions AD and A is as follows: If output by the other firms increases marginally, and if firm i *reduces* output such that its market price remains constant, then firm i's profits decrease by the profit margin $(p - c')$, which is a first-order effect; since by condition AD profit margins are decreasing in own output, this decrease in profits can only be counterbalanced by an *increase* in own output and a resulting lower market price, therefore firm i will not want to drive prices up. Best response output may go up or down since there are two opposite movements in output involved.

An interesting implication of condition AD is that it implies a bound on the slopes of quantity best responses $r_i(x_{-i})$,

$$\frac{\partial}{\partial x_j} r_i(x_{-i}) = -\frac{\Pi^{ij}}{\Pi^{ii}} \geq -\frac{p^j}{p^i}.$$

Apart from condition AD there are various other conditions that imply condition A and that may sometimes be easier to verify. Some we will discuss in the next section, but the one most easily verified is the following:[8]

Condition D: $\tilde{\Pi}(p_i, x_{-i})$ *has (weakly) decreasing differences in* (p_i, x_{-i})*, i.e.*

$$\tilde{\Pi}(p_i', x_{-i}') - \tilde{\Pi}(p_i, x_{-i}') \leq \tilde{\Pi}(p_i', x_{-i}) - \tilde{\Pi}(p_i, x_{-i}) \tag{8}$$

[7] An equivalent interpretation, due to Amir and Lambson (1998) for homogenous goods is that, "inverse demand or price decreases faster (...) at any given output level than does marginal cost at all lower output levels."

[8] Here $x_{-i}' \geq x_{-i}$ means that $x_j' \geq x_j$ for all $j \neq i$.

for all $p_i' \geq p_i \geq 0$ *and* $x_{-i}' \geq x_{-i} \in [0, K]^{n-1}$.

This condition is strictly stronger than condition A (Milgrom and Shannon 1994). If inverse demand and production costs are twice continuously differentiable, this is equivalent to (see appendix 9.3), for all $j \neq i$,

$$\frac{\partial^2}{\partial p_i \partial x_j} \tilde{\Pi}(p_i, x_{-i}) \leq 0,$$

or

$$\Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij} + \left(p_i + xp^i - c'\right)\frac{p^j p^{ii} - p^i p^{ij}}{p^i p^j} \leq 0. \tag{9}$$

The last term in (9) disappears if goods are industry aggregates ($p^j p^{ii} - p^i p^{ij} = 0$), or at interior best responses ($p_i + xp^i - c' = 0$), leading to condition AD.

4 Existence of Equilibria

We will now state our main result on the existence of symmetric pure Cournot equilibria. In addition to condition A stated above, we need some regularity conditions to ensure that the decision problem of each firm has an optimal solution:

Condition R (Regularity): 1. Production capacity K is limited: $0 < K < \infty$*;*

2. production cost $c(x_i)$ *is lower semi-continuous;*

3. inverse demand $p(x_i, x_{-i})$ *is continuous in* (x_i, x_{-i}).

Multiple symmetric equilibria can be ranked according to equilibrium quantities (or prices). If there is a symmetric equilibrium where quantities are smaller (higher) than in any other symmetric equilibrium, this equilibrium is called *minimal* (*maximal*).

Theorem 1 *Assume that inverse demand is nonincreasing in all arguments (goods are substitutes), and strictly decreasing in own output while* *inverse demand is positive. Under conditions R and A there exist maximal and minimal symmetric pure Cournot equilibria.*

From a technical point of view, at the heart of theorem 1 lies the fact that under condition A the price best response $P(x_{-i})$ has *nonincreasing* maximal and minimal selections, which allows for the construction of *nondecreasing* maps from the space of prices into itself. Tarsky's (1955) theorem, which states that any nondecreasing map from an interval into itself has a fixed point, can then be applied to show that maximal and minimal fixed points exist. These result in maximal and minimal symmetric pure strategy Cournot equilibria.

The equilibrium is unique if and only if the maximal and minimal equilibria are identical, but this cannot be established without further assumptions. Under the strong assumption that profits are quasiconcave, *multiple symmetric* equilibria can be excluded if one assumes that the slopes of quantity best reactions are smaller than $1/(n-1)$, which follows in particular if all equilibria are stable (see appendix 9.4.2). On the other hand, using stronger versions of condition A and a weak additional condition B, one can exclude the existence of *asymmetric* equilibria.

Let us state two conditions related to condition A, both of which are strictly stronger and involve the whole space of competitors' outputs $[0,K]^{n-1}$: For all $i = 1...n$,

Condition AS: $\tilde{\Pi}(p_i, x_{-i})$ *satisfies the dual strict single crossing property in* (p_i, x_{-i}) *for all* $x_{-i} \in [0,K]^{n-1}$*: For all* $p_i' > p_i$ *and* $x_{-i}' > x_{-i}$,

$$\tilde{\Pi}(p_i', x_{-i}) - \tilde{\Pi}(p_i, x_{-i}) \leq 0 \Rightarrow \tilde{\Pi}(p_i', x_{-i}') - \tilde{\Pi}(p_i, x_{-i}') < 0. \tag{10}$$

Condition ASD: $\partial\tilde{\Pi}/\partial p_i$ *exists, and is strictly decreasing in* x_j *for all* $j \neq i$*, where* $p(x_i, x_{-i})$ *is positive and for all* $x_{-i} \in [0,K]^{n-1}$.

Condition AS means that a firm will not raise *any* best response market price as a reaction to an increase in competitors' outputs (as opposed to just maximum and minimum best response prices under condition A). Condition ASD implies that a firm will *strictly decrease* its market price as a reaction to an increase in competitors' outputs and is stronger than conditions A and AS, and even stronger than weakly or strictly decreasing differences of profits (see Edlin and Shannon 1998a).

Let x_{-ij} be the vector of outputs of firms $k \neq i, j$. The additional conditions on inverse demands are:

Condition BW (weak): For all $j \neq i$*, all* $(x_i, x_j, x_{-ij}) \in [0, K]^n$*, and all* $\varepsilon > 0$*,* $p(x_i + \varepsilon, x_j, x_{-ij}) \leq p(x_i, x_j + \varepsilon, x_{-ij})$*.*

Condition BS (strict): For all $j \neq i$*, all* $(x_i, x_j, x_{-ij}) \in [0, K]^n$*, and all* $\varepsilon > 0$*,* $p(x_i + \varepsilon, x_j, x_{-ij}) \leq p(x_i, x_j + \varepsilon, x_{-ij})$*, where the inequality is strict when* $p(x_i, x_j + \varepsilon, x_{-ij}) > 0$*.*

If inverse demand is differentiable these conditions correspond to $p^i \leq p^j$ and $p^i < p^j$, respectively. Conditions BW and BS mean that each firm's changes in quantity influence its own market price more than the same changes in other firms' quantities, which is a very reasonable assumption as firms are symmetric. Note that the case of homogeneous goods, where $p^i = p^j = p'$, falls under condition BW. In fact, both these conditions follow from utility maximization of a representative consumer: If inverse demands are derived from maximizing a (strictly) concave utility function U, where at the optimum $p_i = \partial U / \partial x_i$, then the condition $p^i \leq p^j$ ($p^i < p^j$) follows from the (strict) definiteness of the Hessian and the symmetry of the demand functions.[9]

With these conditions, we have the following proposition:

Proposition 2 *Asymmetric equilibria do not exist if either 1. or 2. holds:*

1. *Conditions AS and BS hold.*
2. *Conditions ASD and BW hold.*

[9]Note that $p^i = \partial^2 U / \partial x_i^2$, $p^j = \partial^2 U / \partial x_i \partial x_j$, and that the determinant of every 2x2 minor of the Hessian must be non-negative (positive):

$$\left(p^i\right)^2 - \left(p^j\right)^2 \geq (>)\ 0.$$

For homogeneous goods we must assume condition ASD (including the assumption that profits are differentiable) to rule out asymmetric equilibria, while for heterogeneous goods the weaker condition AS is enough. On the other hand, condition AS must be accompanied with the slightly stricter condition BS.

5 Related conditions

In the following we will discuss the conditions which have been used so far to establish existence of Cournot equilibrium and their relation to condition A. In general, strong conditions on payoffs, like concavity or quasiconcavity, yield existence in arbitrary games, see Friedman (1977, 1991), but are difficult to translate into economically meaningful statements about demand or cost.

Spence (1976) presents a class of demand functions with a special functional structure where Cournot equilibria can be found maximizing a certain 'wrong' surplus function. Here the question of existence of Nash equilibria reduces to the question of existence of maxima of this function. Slade (1994) finds a necessary and sufficient condition for this relation between equilibria and maxima to exist, and shows that for homogeneous goods such functions exist if and only if demand is linear, while there are more general cases for heterogeneous goods.

Most work has concentrated on economically meaningful conditions on demand or cost, or both. Unsurprisingly, practically all are related with either one or the other of Hahn's (1962) pair of stability conditions,

$$p' - c'' \le 0, \quad p' + xp'' \le 0, \tag{11}$$

or in our notation

$$\Pi^{ii} \le \Pi^{ij}, \quad \Pi^{ij} \le 0, \quad j \ne i. \tag{12}$$

For homogeneous goods, McManus (1962, 1964) and Roberts and Sonnenschein (1976) prove existence of symmetric pure Cournot equilibrium assuming that production costs were convex, while Szidarovsky and Yakowitz (1977) additionally assume that inverse demand is concave. Kukushkin (1993), assuming convex cost, shows existence of pure symmetric equilibria if outputs are discrete variables.[10] If demand is non-increasing, from the assumption of convex costs follows that $p' - c'' < 0$ or $\Pi^{ii} < \Pi^{ij}$ $(i \neq j)$, i.e. the first of Hahn's stability conditions. Amir and Lambson (1998), again for homogeneous goods, directly assume this, and prove existence of pure symmetric Cournot equilibria. Their work is important in several respects: It shows that the assumption of convex costs can be relaxed, and that the relevant condition $p' - c'' < 0$ is lattice-theoretic in nature. As one can see from condition AD, our condition A is a direct generalization to heterogenous goods of this condition.

The second of Hahn's stability conditions, $p' + xp'' \leq 0$ or $\Pi^{ij} \leq 0$, means that marginal revenue does not increase if competitors raise their outputs, i.e. that goods are *strategic substitutes* in the terminology of Bulow *et al.* (1985). More generally, goods being strategic substitutes is equivalent to profits $\Pi(x_i, x_{-i})$ having weakly decreasing differences in outputs (x_i, x_j) for $j \neq i$. This condition is not related to our condition A.

For homogeneous goods, Novshek (1985) shows that (possibly non-symmetric) Cournot equilibria exist if goods are strategic substitutes. Van Long and Soubeyran (1999) prove existence and uniqueness of Cournot equilibria under strategic substitutes *and* convex cost. For general aggregative games, i.e. "competitor aggregation", Corchón (1994 , 1996) imposes a generalization of Hahn's conditions, which can be written as $\Pi^{ii} < \Pi^{ij} < 0$, and proves existence through the concavity of payoffs, while Kukushkin (1994) only assumes strategic substitutes.

For *strategic complements*, i.e. weakly increasing differences of $\Pi(x_i, x_{-i})$ in outputs (x_i, x_j) for $j \neq i$, or $\Pi^{ij} \geq 0$ $(j \neq i)$ under differentiability, Vives (1990), shows existence of pure Cournot equilibria for heterogeneous goods (symmetric with symmetric firms) in the general context of supermodular games.

The most intuitive way to characterize both strands of literature is

[10] *Mixed* equilibria always exist if outputs are discrete.

to express all conditions used in terms of slopes of (quantity) reaction functions $r(x_{-i})$: Assuming that profits are twice differentiable, we obtain $r'(x_{-i}) = -\Pi^{ij}/\Pi^{ii}$. The first strand of literature effectively assumes that this slope is bounded below by -1, while strategic substitutes (complements) imply that $r' \leq (\geq) 0$. Our condition AD in general implies $r' \geq -p^j/p^i$, which is equal to -1 under homogeneous goods, and larger than -1 under condition BS.

While condition A generalizes the first part of the literature, its relation with the second group is not straightforward. For homogeneous goods the assumption of strategic complements implies condition A if profits are (at least locally) concave. Since profits are locally concave at interior best responses, this result captures the fact that reaction functions certainly have slope larger than -1 if they are nondecreasing. For heterogeneous goods this relationship is not clear.

Figures 1-3 summarize the relations between the various conditions mentioned above according to their implications on the slopes of best responses, which for simplicity are assumed to be differentiable. Most conditions only apply to homogeneous goods or "competitor aggregation".

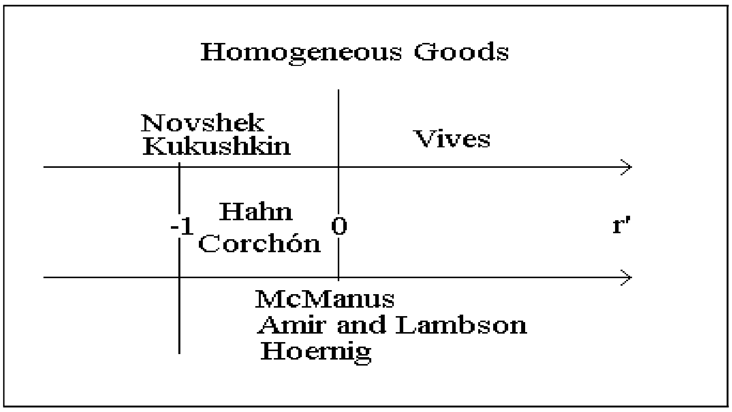


Figure 1

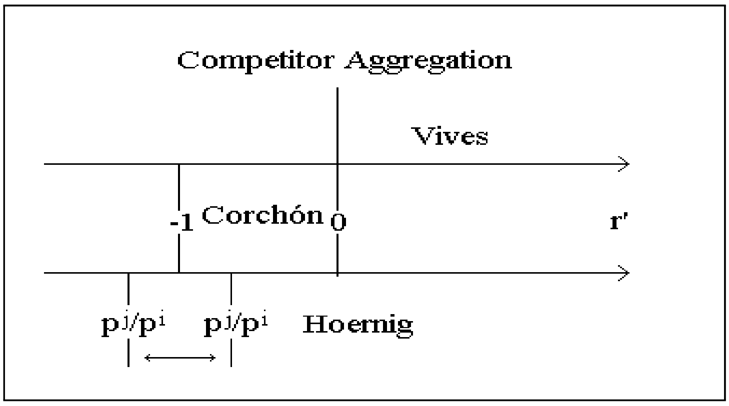


Figure 2

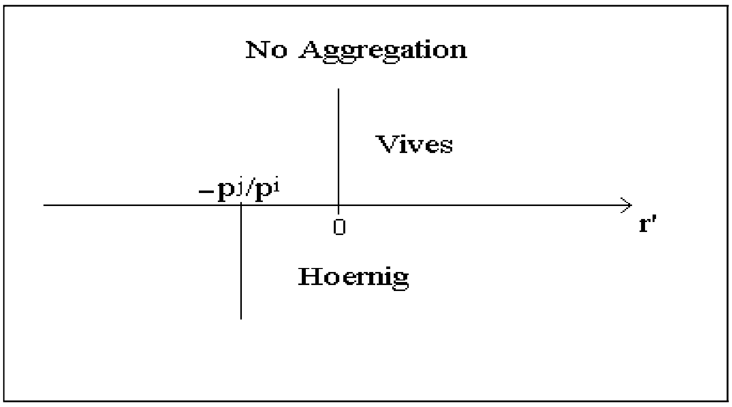


Figure 3

6 Examples

6.1 Linear Cournot Oligopoly

Consider the class of Cournot models with heterogeneous linear demand and linear cost functions, with $p_i = A - Bx_i - CY_i$, where $Y_i = \Sigma_{j \neq i} x_j$, and $B > 0$. Goods are substitutes for $C > 0$, and homogeneous if $C = B$. Condition AD is always satisfied since $\Pi^{ii} = -2B$, $\Pi^{ij} = -C$, and $\Pi^{ii} - (p^i/p^j)\,\Pi^{ij} = -B < 0$. On the other hand, condition D is equivalent to strategic substitutes if goods are substitutes: $\Pi^{ij} = -C$, and $\frac{\partial^2}{\partial p_i \partial Y_i}\tilde{\Pi} = -C/B$, and both expressions are negative. Assuming w.l.o.g. that marginal costs are zero, equilibrium prices $p_{(n)} = AB/\left(2B + (n-1)\,C\right)$ are decreasing in the number of firms if goods are substitutes.

6.2 Industry Aggregation

Here the demand function is of the type $\bar{p}\left(x + \delta Y\right)$, and under differentiability condition AD becomes

$$\Pi^{ii} - \frac{1}{\delta}\Pi^{ij} = \bar{p}' - c'' \leq 0,$$

which is equal to the condition $\bar{p}' - c'' \leq 0$ under homogeneous goods. Note that the lower bound on the slope of the quantity reaction function becomes $r'\left(Y\right) \geq -\delta$. If $\delta < 1$ there are no asymmetric equilibria since for their existence it is necessary that the slope is -1 or smaller.

6.3 Non-aggregative demand

Here we give an example that shows that there are reasonable assumptions on consumers' preferences that give rise to inverse demand functions that do not allow aggregation of competitors' quantities (demand is *non-aggregative*).

Let the utility of a representative consumer be quasi-linear, and depend on a numeraire good y and n other goods $x_1, ..., x_n$ in the following form:

$$U\left(y, x_1, .., x_n\right) = y + \Sigma_{i=1}^{n}\left(x_i - x_i^2/2\right) + \Sigma_{i=1}^{n}\Sigma_{j>i}\ln\left(1 - x_i x_j\right),$$

$U\left(.\right)$ is a *generalized quadratic* utility function (Spence 1976).

At the consumer's optimum we have $\partial U/\partial x_i = p_i$ for $i = 1 \ldots n$, therefore the inverse demand functions are defined on $x \in [0,1]^n$, with

$$p\left(x_i, x_{-i}\right) = \frac{\partial U}{\partial x_i} = 1 - x_i - \Sigma_{j \neq i}\frac{x_j}{1 - x_i x_j}$$

while $\partial U/\partial x_i > 0$, and zero otherwise. With zero production cost, condition AD is satisfied "on the diagonal" $x_j = \bar{x}$, $j \neq i$ since p is twice differentiable and it can be shown that

$$\Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij} = -\frac{(n-1-x_i^2)\,\bar{x}^2 + (1-2x_i\bar{x})^2}{(1-x_i\bar{x})^2} < 0.$$

In the next section we give an example of a non-aggregative inverse demand function that does not fall in Spence's class.

7 Effects of Entry

A long-standing point of interest has been the question whether Cournot equilibrium approaches a competitive equilibrium as more firms enter the market. It has become common to call a Cournot equilibrium *quasi-competitive* if equilibrium total quantity is increasing or price is decreasing in the number of firms. It it easy to see that for heterogeneous goods there is not necessarily a strict inverse relation between total quantity and market prices even if goods are symmetric. Since the sum of outputs makes less sense as a measure of quasi-competitiveness for heterogeneous goods precisely because outputs are not of the same good, we argue that the more useful measure is whether market prices are decreasing.

One should note that with heterogeneous goods the entry of a new competitor raises the number of goods (and welfare if consumers value variety), which in general may have surprising effects. As we will see in the following, under competitor aggregation the conventional wisdom (equilibrium prices decrease after entry) prevails, while for more general forms of heterogeneity this need no longer be true.

At first we will restrict attention to competitor aggregation. Assume there is a countable number of identical firms that may enter in the market[11]. Let $f : [0, K]^\infty \to F \subset \mathbb{R} \cup \{\infty\}$ be continuous, strictly

[11] Since we are not interested in determining a free entry equilibrium, fixed cost of entry are irrelevant.

increasing, and symmetric in its arguments: Let $\tilde{x}_{-i}$ be a permutation of $x_{-i} \in [0, K]^{\infty}$, then $f(\tilde{x}_{-i}) = f(x_{-i})$. Let inverse demand be given by

$$p = p(x_i, f(x_{-i})), \tag{13}$$

where $x_{-i} \in [0, K]^{\infty}$ and $p : [0, K] \times F \to \mathbb{R}_+$ is continuous and nondecreasing in (x_i, f_i), and strictly decreasing in its first argument while inverse demand is positive. For competitor aggregation, industry aggregation and homogeneous goods the simplest definition of f is $f(x_{-i}) = \Sigma_{j \neq i} x_j$.

Under condition A, the existence of symmetric Cournot equilibria follows from theorem 1, therefore the following comparative statics conclusions are not empty.

Theorem 3 *Under competitor aggregation the following holds:*

1. Under condition A maximal and minimal equilibrium prices are nonincreasing in the number of firms n.

2. Under condition ASD, and if $p(x_i, x_{-i})$ is strictly decreasing in x_j for all $j \neq i$ while $p_i > 0$, then maximal and minimal equilibrium prices are strictly decreasing in the number of firms n as long as they are positive.

Some remarks are in order: As noted above, even if $\tilde{\Pi}$ is differentiable, condition ASD, the condition that $\partial\tilde{\Pi}/\partial p_i$ exists and is strictly decreasing in x_j for all $j \neq i$, is strictly stronger than condition A or even strictly decreasing differences of $\tilde{\Pi}$ in (p_i, x_{-i}) (see Edlin and Shannon 1998a).

Second, our method also can say something about the comparative statics of symmetric equilibria that are *interior* , i.e. characterized by first-order conditions, if they are *stable* equilibria in the usual definition as asymptotically stable equilibria under some classes of adjustment mechanisms. Hahn (1962) and Seade (1980) gave sufficient conditions for stability, while Seade also gave sufficient conditions for instability of

Cournot equilibria: Equilibria with n firms are stable if the slopes of best reactions lie between the following bounds:

$$-1 < \frac{\partial}{\partial x_j} r\left(x_{-i}\right) = -\Pi^{ij}/\Pi^{ii} < 1/\left(n-1\right),\ j \neq i, \tag{14}$$

and instable if

$$\frac{\partial}{\partial x_j} r\left(x_{-i}\right) = -\Pi^{ij}/\Pi^{ii} > 1/\left(n-1\right),\ j \neq i. \tag{15}$$

Define $g\left(x,n\right) = f\left(x,...,x,0,...\right)$, where n firms all produce x, and the others nothing.

Corollary 4 *Under competitor aggregation, let inverse demand and production cost be twice continuously differentiable, and let $g\left(x,n\right)$ be differentiable in n with partial derivative $g_n > 0$. If condition AD holds then at interior equilibria the equilibrium price is nonincreasing in the number of firms if the equilibrium under consideration is stable. If equilibrium price is increasing then the equilibrium is unstable.*

In the appendix we show, making explicit use of the aggregation, that

$$\frac{dp_{(n)}}{dn} = \frac{g_n p_2}{\Pi^{ii} + \left(n-1\right)\Pi^{ij}} \left(\Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij}\right), \tag{16}$$

where the second factor is non-positive by condition AD, and the first one is positive if the equilibrium is stable. Therefore $dp_{(n)}/dn$ is non-positive under condition AD and stability.

In the following schematic portrait of the fixed point map $\psi\left(p\right)$ determining equilibrium prices, which was used in the proof of theorem 1, the maximal and minimal equilibria (fixed points) are stable, and equilibrium prices decrease when we shift the map downwards to the dotted curve; the interior fixed point corresponds to an unstable equilibrium and indeed equilibrium prices increase.[12]

[12] We show in appendix 9.4.2 that equilibria are unstable if the fixed point map cuts the diagonal from below.

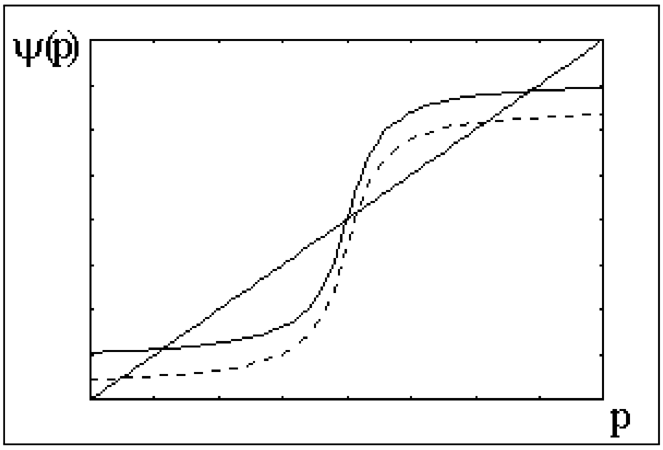


Figure 4

It can be shown that if $\hat{\Pi}(p_i, y)$ satisfies the (*non-dual*) weak single-crossing property in (p_i, y), then extremal equilibrium prices are *nondecreasing*, which demonstrates that condition A is critical for establishing quasi-competitiveness (see the proof of theorem 3).

Fourth, theorem 3 and corollary 4 do not extend to the case of non-aggregative demands, as the following example shows: Assume there are n firms with production capacity $K \geq 1/2$ and zero production cost. Inverse demands are $p(x_i, x_{-i}) = 1 - x_i - \Sigma_{j \neq i}\left(1 - e^{-x_i x_j}\right)$ where this expression is positive, and otherwise $p(x_i, x_{-i}) = 0$. Symmetric equilibrium outputs are given by the first-order constraint (sufficient second-order conditions are satisfied)

$$2 - 2x - n + (n-1)\left(1 - x^2\right)e^{-x^2} = 0,$$

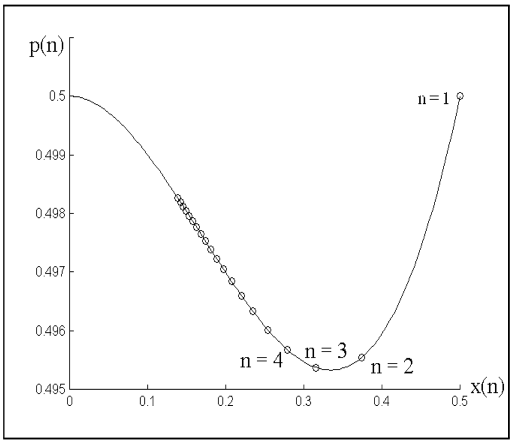
which for each value of $n \geq 1$ has exactly one solution $x_{(n)} \leq 1/2$. Therefore for each n there is exactly one symmetric equilibrium which at the same time is minimal, maximal and interior. On the diagonal $x_j = \bar{x}$ $(j \neq i)$ condition AD is fulfilled since

$$\Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij} = -x\bar{x} < 0,$$

and the equilibrium is stable according to the above definition since

$$\Pi^{ii} + (n-1)\,\Pi^{ij} = -2 - 2(n-1)\,x\left(2 - x^2\right)e^{-x^2} < 0.$$

Still, equilibrium prices fall until $n = 3$, and then rise:



The reason for this maybe perplexing result is that the fixed-point maps defining successive equilibria are shifted *upwards* instead of downwards for $n > 3$ since the reduction in individual output best responses caused by the entrant is outweighed by a corresponding reduction in *competitors'* outputs (see the argument in appendix 9.4.4). Note that the traditional comparative statics analysis based on the implicit function theorem, which in appendix 9.4.5 we extend to the case of competitor aggregation, is not applicable here since inverse demand cannot be written as a *differentiable* function of the number of firms.

There are three other variables of interest whose equilibrium values vary with the number of firms: Total output, individual outputs, and firm profits. In supermodular games, i.e. games with strategic complements, equilibrium strategies and individual payoffs rise with the number of players, see Topkis (1998), theorem 4.2.3. In Cournot oligopoly the comparative statics of individual quantities, total quantities and profits each depend on a different condition.

We state the comparative statics results about quantities for competitor aggregation. Let us assume that inverse demand and production cost are twice continuously differentiable, and concentrate on interior equilibria, i.e. equilibria characterized by first order conditions. This is sufficient for our purposes since we want to make the simple point that conditions A or AD do not drive the results.

Corollary 5 *Under condition AD, consider any interior equilibrium where equilibrium prices $p_{(n)}$ are decreasing in the number of firms n.*

1. Under condition BS total equilibrium output $Q_{(n)}$ is strictly increasing in n.

2. Individual equilibrium quantities $x_{(n)}$ are decreasing (increasing) in n if goods are strategic substitutes (complements).

Without proof we note that to both comparative statics results there corresponds an own differencing condition on profits: For total quantity, it is that

$$\hat{\Pi}(Q, x_{-i}) = \Pi(Q - \Sigma_{j \neq i} x_j, x_{-i})$$

has nondecreasing differences in (Q, x_j) for all $j \neq i$, and for individual quantities that $\Pi(x_i, x_{-i})$ has nonincreasing (nondecreasing) differences in (x_i, x_j) for all $j \neq i$. These could of course be generalized to single-crossing conditions.

On the other hand, condition A is sufficient to show that individual profits are decreasing:

Corollary 6 *Under condition A individual profits $\Pi_{(n)}$ in maximal and minimal equilibria are nonincreasing in the number of firms n, and strictly decreasing if inverse demand $p(x_i, f_i)$ is strictly decreasing in f_i.*

Total firm profits, i.e. the sum of profits of all firms in the industry, may be increasing or decreasing.

The different effects of an increase in the number of firms on prices and quantities are summarized in the following two figures, where '+' ('-') means that the variable is going up (down):

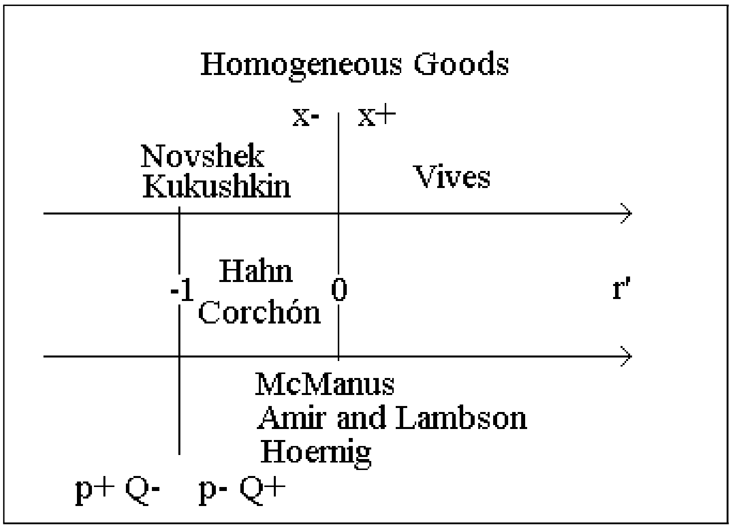


Figure 5

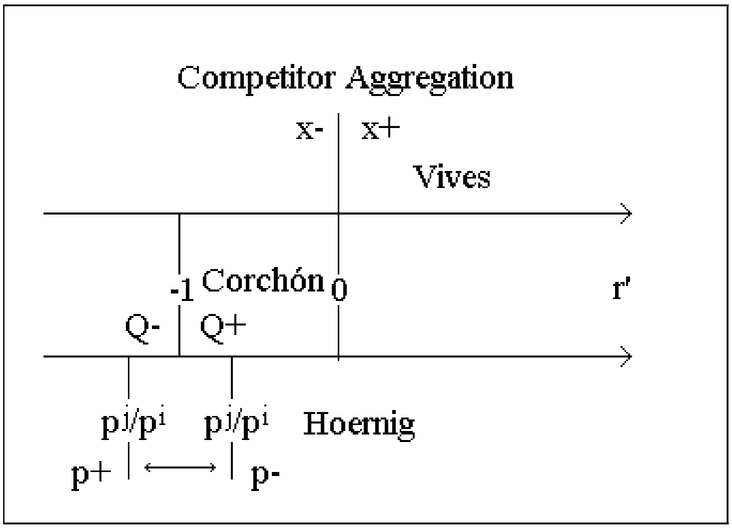


Figure 6

7.1 Related conditions

Most of the early literature on quasi-competitiveness, i.e. Frank (1965), Ruffin (1971), Seade (1980), Szidarovsky and Yakowitz (1982), all for homogeneous goods, assume both $\Pi^{ij} = p' + xp'' < 0$ (strategic substitutes) and $p' - c'' < 0$ (condition AD for homogeneous goods), and show that equilibrium market prices decrease as more firms enter. Corchón

(1994, 1996) generalizes these conditions and results to general aggregative games; in the special case of Cournot competition his conditions are

$$\Pi^{ii} < \Pi^{ij} < 0,$$

where the first inequality corresponds to $p' - c'' < 0$ for homogeneous goods.

Not until in Amir and Lambson (1998) it became clear that the only condition relevant for quasi-competitiveness with homogeneous goods is $p' - c'' < 0$. As their work is based on lattice theory they can identify the conditions that are critical for their conclusions, and avoid unnecessary ones like strategic substitutes and concavity.[13] Our condition AD is a generalization to heterogeneous goods of the classical condition $p' - c'' < 0$, and condition A applies in more general contexts.[14]

Now we will give a simple example under homogeneous goods that shows that under condition A equilibrium prices go down, while under its 'opposite' they go up. Let inverse demand be given by $p(Q) = 3 - 2Q$, therefore goods are strategic substitutes.

First assume that marginal cost is constant with $c(x) = \frac{1}{2}x$. Then equilibrium price is $p_n = (6+n)/2(1+n)$ which is strictly decreasing and converges to $c'(0) = \min_x c(x)/x = 1/2$, and therefore to the competitive outcome. Condition A is satisfied since $p' - c'' = -2 < 0$.

For strongly increasing returns to scale, for example $c(x) = \frac{1}{2}x - \frac{22}{20}x^2$ (for $x \leq 1/2$) equilibrium price $p_n = (5n-3)/(10n-1)$ is strictly *increasing* in the number of firms and converges to $c'(0) = 1/2 >$

[13] De Meza (1985) and Villanova, Paradis and Viader (1999) exhibit examples where n-firm oligopoly prices are decreasing (therefore quasi-competitive according to the definition used here) but are higher than the monopoly price. This outcome is due to the assumption of increasing returns to scale in production that only set in for large output quantities.

[14] An issue that is related but different from quasi-competitiveness is the issue of *convergence* of equilibrium to the 'competitive price'. Ruffin (1971), following McManus (1964) and Frank (1965), shows that the Cournot equilibrium price converges to the competitive equilibrium price if and only if there are no increasing returns to scale.

$\min c(x)/x = 0$. Condition A rules this case out since $p' - c'' = -2 + 44/20 = 1/5 > 0$.

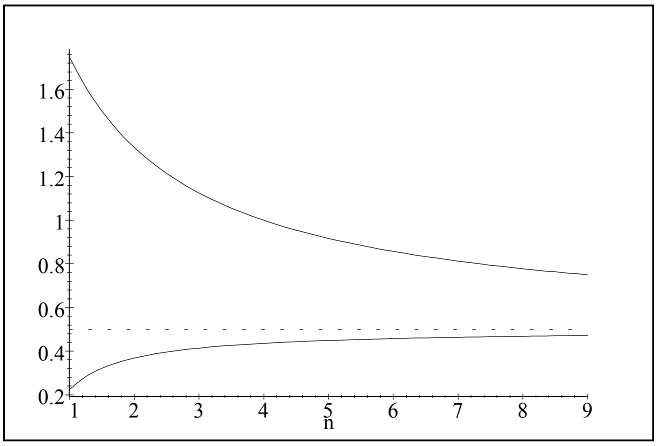


Figure 7

8 Conclusions

Using a lattice theory-based approach, we established the existence of pure symmetric Cournot equilibria for homogeneous or heterogeneous goods under a simple condition that generalizes the condition of weakly increasing returns used in the literature for the case of homogeneous goods. We were able to rule out asymmetric equilibria using a weak additional condition.

Under our main condition maximal and minimal equilibrium prices are decreasing in the number of firms if competitors' quantities enter inverse demand as an aggregate, but may be increasing if inverse demand is non-aggregative. We obtain the same result for stable interior equilibria. Total quantity increases with the number of firms under the same additional condition as above, while individual quantities increase (decrease) if goods are strategic complements (substitutes). Individual firm profits are decreasing after entry. These results show quite clearly that each comparative statics result depends on a different critical condition, and therefore model builders striving for generality should attempt to only include the assumptions that drive the comparative statics results that they really need.

One topic for further research is to extend our methods (and maybe some results) to cases where product heterogeneity is not symmetric, as e.g. in Hotelling models, or to models with exogenous or endogenous quality differences.

Second, similar results will certainly be obtained by applying corresponding conditions to models of heterogenous price competition, which to some extent already have been treated.

9 Appendix

9.1 Change of variable in the profit function

In this appendix we discuss thoroughly the properties of the function $\chi(p_i, x_{-i})$ that is used to change the decision variable from own quantity to market price in the profit function. The important points are: monotonicity and continuity of $\chi(p_i, x_{-i})$, and convexity, closedness and monotonicity of the constraint set $\pi(x_{-i})$.

Let minimum and maximum prices be $p_K = p(K, .., K)$ and $p_0 = (0, .., 0)$, and the interval of possible prices with outputs by the other firms fixed

$$\pi(x_{-i}) = [p(K, x_{-i}), p(0, x_{-i})], \; x_{-i} \in [0, K]^{n-1}. \tag{17}$$

The set $\pi(x_{-i})$ is the new constraint set for the maximization over p_i, obviously non-empty, compact and convex, and is descending (nonincreasing) in x_{-i} (in the strong set order, see Milgrom and Shannon (1994)) since $p(., x_{-i})$ is nonincreasing in x_{-i}.

The range of possible combinations between market price and the others' outputs is

$$X = \left\{(p_i, x_{-i}) \in [p_K, p_0] \times [0, K]^{n-1} \,|p_i \in \pi(x_{-i})\right\}. \tag{18}$$

Let $\bar{x}(x_{-i})$ be the maximum output that firm i will produce given that the other firms are already producing x_{-i}, either because this output is equal to capacity, or because inverse demand becomes zero:

$$\bar{x}(x_{-i}) = \min\{K, \min\{x \in [0, K] \,|p(x, x_{-i}) = 0\}\} \geq 0. \qquad (19)$$

Then since p is strictly decreasing and continuous on $x_i \in [0, \bar{x}(x_{-i})]$, we can express firm i's output as a function of market price and the others' output $\chi : X \to [0, K]$ that is continuous and nonincreasing in $(p_i, x_{-i}) \in X$, and strictly decreasing in p_i with image $[0, \bar{x}(x_{-i})]$ for each $x_{-i} \in [0, K]^{n-1}$.[15]

9.2 Dissection condition

In this appendix we derive a differential condition that is sufficient for condition A to hold. Assume that inverse demand and production costs are twice continuously differentiable. We apply the *method of dissection* described in Milgrom and Shannon (1994, p. 167). This method works as follows: The effect of an increase in own price p_i on profits is "dissected" into two parts, a *beneficial* effect due to a price increase (higher revenue per unit and lower total production cost due to the associated decrease in demand), and a *costly* effect due to the effect of the decrease in demand on revenue. To these effects are associated the price variables p^+ and p^-, respectively. Profits are written as

$$U\left(p^-, p^+, y\right) = p^+ \chi\left(p^-, y, .., y\right) - c\left(\chi\left(p^+, y, .., y\right)\right),$$

where

$$\frac{\partial U}{\partial p^-} = p^+ \chi^p < 0, \quad \frac{\partial U}{\partial p^+} = \chi - c' \chi^p > 0,$$

where $\chi^p = \partial \chi / \partial p_i = 1/p^i < 0$ (superscripts denote partial derivatives). Then $\hat{\Pi}(p_i, y)$ satisfies the dual weak single crossing property in (p_i, y), i.e. condition A, if $\left(\partial U / \partial p^-\right) / \left|\partial U / \partial p^+\right|$ is nonincreasing in y. We have,

[15] This is an application of a continuous version of the implicit function theorem.

replacing p^+ and p^- by p_i,

$$\begin{aligned}
\frac{d}{dy}\frac{\partial U/\partial p^-}{|\partial U/\partial p^+|} &= \frac{d}{dy}\frac{p_i \chi^p}{\chi - c'\chi^p} \\
&= (n-1)\frac{p_i \chi^{pj}(\chi - c'\chi^p) - p_i\chi^p(\chi^j - c''\chi^j\chi^p - c'\chi^{pj})}{(\chi - c'\chi^p)^2} \\
&= -\frac{(n-1)\,p_i(\chi^p)^2\chi^j}{(\chi - c'\chi^p)^2}\left(\frac{1}{\chi^p} - c'' - \frac{\chi}{\chi^p}\frac{\chi^{pj}}{\chi^p\chi^j}\right) \leq 0,
\end{aligned}$$

where $\chi^j = \partial\chi/\partial x_j = -p^j/p^i \leq 0$, $\chi^{pj} = \partial^2\chi/\partial p_i\partial x_j = (p^jp^{ii} - p^ip^{ij})\,/\,(p^i)^3$, and we have used the symmetry of inverse demand with respect to opponents' outputs. Therefore condition A holds if

$$\begin{aligned}
\frac{1}{\chi^p} &- c'' - \frac{\chi}{\chi^p}\frac{\chi^{pj}}{\chi^p\chi^j} = p^i - c'' + x_i\frac{p^jp^{ii} - p^ip^{ij}}{p^j} \\
&= \left(2p^i + x_ip^{ii} - c''\right) - \frac{p^i}{p^j}\left(p^j + x_ip^{ij}\right) \\
&= \Pi^{ii} - \frac{p^i}{p^j}\Pi^{ij} \leq 0,
\end{aligned}$$

where

$$\begin{aligned}
\Pi^{ii} &= \tfrac{\partial^2}{\partial x_i^2}\Pi(x_i, x_{-i}) = 2p^i(x_i, x_{-i}) + x_ip^{ii}(x_i, x_{-i}) - c''(x_i), \\
\Pi^{ij} &= \tfrac{\partial^2}{\partial x_i\partial x_j}\Pi(x_i, x_{-i}) = p^j(x_i, x_{-i}) + x_ip^{ij}(x_i, x_{-i}).
\end{aligned}$$

9.3 The differential version of Condition D

Condition D holds if and only if

$$\begin{aligned}
\tfrac{\partial^2}{\partial p_i\partial x_j}\tilde{\Pi}(p_i, x_{-i}) &= (1 - c''\chi^p)\,\chi^j + (p_i - c')\,\chi^{pj} \\
&= -\frac{p^j}{(p^i)^2}\left(p^i - c''\right) + (p_i - c')\,\chi^{pj} \\
&= -\frac{p^j}{(p^i)^2}\left(\Pi^{ii} - \tfrac{p^i}{p^j}\Pi^{ij} + \left(p_i + xp^i - c'\right)\frac{\chi^{pj}}{\chi^p\chi^j}\right) \\
&\leq 0.
\end{aligned}$$

In the special cases of industry aggregation or homogeneous goods we have $\chi^{pj} = 0$, as can easily be seen:

$$\chi^{pj} = \left[\left(\delta \bar{p}' \right) \bar{p}'' - \bar{p}' \left(\delta \bar{p}'' \right) \right] / \left(\bar{p}' \right)^3 = 0.$$

9.4 Proofs

9.4.1 Existence of Equilibrium

Proof. (Theorem 1).

1. First we show that price best responses are well-defined given the regularity conditions. Given any vector of outputs $x_{-i} \in [0, K]^{n-1}$ of the other firms, firm i's maximization problem is

$$\max_{p_i \in \pi(x_{-i})} \tilde{\Pi} \left(p_i, x_{-i} \right) = \chi \left(p_i, x_{-i} \right) p_i - c \left(\chi \left(p_i, x_{-i} \right) \right),$$

where χ is continuous in p_i, c is lower semi-continuous, and

$$\pi \left(x_{-i} \right) = \left[p \left(K, x_{-i} \right), p \left(0, x_{-i} \right) \right]$$

is a non-empty compact set. Then $\tilde{\Pi}$ is an upper semi-continuous function of p_i on the compact set $\pi \left(x_{-i} \right)$ and therefore attains its maximum. Thus the price best response $P \left(x_{-i} \right)$ exists, where $P : [0, K]^{n-1} \rightarrow [p_K, p_0]$ is a correspondence that is symmetric in x_{-i} since $p \left(x_i, x_{-i} \right)$ is symmetric in x_{-i}. Now restrict P to the 'diagonal' $x_{-i} = (y, ..., y)$, $y \in [0, K]$, and define

$$\bar{P} \left(y \right) = P \left(y, .., y \right), \; y \in [0, K].$$

2. Maximal and minimal price best responses in $\bar{P} \left(y \right)$ are nonincreasing in y: The constraint set $\pi \left(x_{-i} \right)$ is descending, or decreasing in the strong set order, since both $p \left(K, x_{-i} \right)$ and $p \left(0, x_{-i} \right)$ are nonincreasing in x_{-i}. This follows from the assumptions that goods are substitutes and that $p \left(x_i, x_{-i} \right)$ is continuous in x_{-i}. Invoking this fact and condition A, by Milgrom and Shannon' s (1994) monotonicity theorem the set of maximizers $\bar{P} \left(y \right)$ is decreasing in y in the strong set order. This implies in particular that maximum and minimum selections of $\bar{P}$ exist and are

nonincreasing in y. Let $\tilde{P} : [0, K] \to [p_K, p_0]$ be a maximum or minimum selection, then $\tilde{P}$ is a nonincreasing function.

3. Continuation 1 (Fixed point in prices): Consider prices at identical outputs for *all* firms: Let

$$\tilde{p}(x) = p(x, x, .., x), \ x \in [0, K].$$

Then $\tilde{p}$ is nonincreasing since p is nonincreasing in own output and because goods are substitutes. It is strictly decreasing while positive, and maps $[0, K]$ onto $[p_K, p_0]$. Let $\bar{x}$ be the largest symmetric output that all firms might produce in equilibrium (for all larger outputs less than capacity market price is zero),

$$\bar{x} = \min \{K, \min \{x \in [0, K] \, | \tilde{p}(x) = 0\}\},$$

then the restricted $\tilde{p} : [0, \bar{x}] \to [p_K, p_0]$ is strictly decreasing and one-to-one, and has a strictly decreasing inverse $\tilde{\chi} : [p_K, p_0] \to [0, \bar{x}] \subset [0, K]$.

4. Construct a fixed point map: The map

$$\psi(p) = \tilde{P}(\tilde{\chi}(p)), \ p \in [p_K, p_0] \tag{20}$$

is a nondecreasing function from $[p_K, p_0]$ into itself. By Tarsky's (1955) theorem there is a fixed point $p^* = \psi(p^*)$.

5. The market price p^* is attained in the market of firm i if all firms produce $x^* = \tilde{\chi}(p^*)$. On the other hand, if all of firm i's competitors produce x^*, then firm i adjusts production such that its best response market price is p^*, with best response quantity $\chi(p^*, x^*, .., x^*) = x^*$ because χ is strictly decreasing in p. Therefore a symmetric equilibrium exists where all firms produce x^* and market price is p^* in all markets.

3'. Continuation 2 (Fixed point in quantities): Given identical outputs $y \in [0, K]$ for all competitors, quantity best responses $\tilde{r} : [0, K] \to [0, K]$ are given by $\tilde{r}(y) = \chi\left(\tilde{P}(y), y, ..., y\right)$. Then $\tilde{r}$ is continuous but for upward jumps, since χ is continuous in all arguments, and decreasing in its first, while $P(x_{-i})$ is nonincreasing and therefore has no upward jumps, only downward jumps. Therefore $\tilde{r}$ has a fixed point (Milgrom and Roberts 1994b, cor. 1), which is an equilibrium output. ■

9.4.2 Stability and Multiple Symmetric Equilibria

We will now prove that instability of an equilibrium point corresponds to a slope larger than 1 of the fixed point map defining this equilibrium point, i.e. it cuts the diagonal from below. Note that since the fixed point map starts above the diagonal, multiple symmetric equilibria will exist if and only if it crosses the diagonal from below at least once. In particular, if the map jumps upwards over the diagonal (which cannot happen if profits are quasiconcave) then multiple symmetric equilibria will exist and all of them may be stable. Kolstad and Mathiesen (1987) give necessary and sufficient conditions for uniqueness of equilibrium with homogenous goods which boil down to $p' - c'' < 0$ and $\Pi_{11} + (n-1)\,\Pi_{12} < 0$ (stability). They do assume quasiconcavity of profits, and the above heuristic argument shows that this assumption is indispensable.

Remember the fixed point map used in the proof of theorem 1, $\psi(p) = \tilde{P}(\tilde{\chi}(p))$ on $p \in [p_K, p_0]$. From maximizing profits over prices we find that

$$\begin{aligned} \frac{d}{dy}\tilde{P}(y) &= (n-1)\left(-\tilde{\Pi}^{pj}/\tilde{\Pi}^{pp}\right), \\ \frac{d}{dp}\tilde{\chi}(p) &= \frac{1}{p^i + (n-1)\,p^j}, \end{aligned}$$

with

$$\begin{aligned} \tilde{\Pi}^{pp} &= 2\chi^p + (p - c')\,\chi^{pp} - c''(\chi)\,(\chi^p)^2 \\ &= \Pi^{ii}/\left(p^i\right)^2 \end{aligned}$$

where we used $\chi^{pp} = -p^{ii}/\left(p^i\right)^3$, and

$$\begin{aligned} \tilde{\Pi}^{pj} &= \chi^j + (p - c')\,\chi^{pj} - c''(\chi)\,\chi^p\chi^j \\ &\quad - \left(\Pi^{ii} - \tfrac{p^i}{p^j}\Pi^{ij}\right) p^j/\left(p^i\right)^2 \end{aligned}$$

if the first order condition $\chi + p\chi^p - c'(\chi)\,\chi^p = 0$ holds. Then the slope of the fixed-point map is

$$\begin{aligned} \tfrac{d}{dp}\psi(p) &= \tfrac{d}{dy}\tilde{P}(\tilde{\chi}(p))\,\tfrac{d}{dp}\tilde{\chi}(p) \\ &= (n-1)\,\frac{p^j\Pi^{ii} - p^i\Pi^{ij}}{\left(p^i + (n-1)\,p^j\right)\Pi^{ii}}, \end{aligned}$$

which is larger than 1, i.e. the fixed point map cuts the diagonal from below, if and only if

$$\Pi^{ii} + (n-1)\,\Pi^{ij} > 0 \;\Leftrightarrow\; \tfrac{\partial}{\partial x_j} r\,(x_{-i}) = -\Pi^{ij}/\Pi^{ii} > 1/\,(n-1)\,,$$

i.e. if the equilibrium is unstable according to Seade (1980).

9.4.3 Asymmetric equilibria

Proof. (Proposition 2)

Consider an asymmetric equilibrium $x = (x_1, \ldots, x_n)$, where w.l.o.g. $x_1 = y + \varepsilon > x_2 = y$. Then $x' = (x_2, x_1, x_3, \ldots, x_n)$ is also an equilibrium. For conciseness we now suppress the arguments $(x_3, \ldots, x_n)$. Equilibrium prices for firm 1 are $p = p\,(y+\varepsilon, y)$ and $p'\,(y, y+\varepsilon)$, with $p' > p$ by condition BS or $p' \geq p$ by condition BW.

First impose condition BS, leading to $p' > p$. Under condition AS, i.e. under the dual strict single-crossing property of firm i's profit $\tilde{\Pi}$ in (p_i, y) for all $i = 1..n$, all selections of best price responses are nonincreasing, i.e. $p' \leq p$ since $y + \varepsilon > y$, which is a contradiction to $p' > p$.

For the second statement impose condition BW, leading to $p' \geq p$. Since under condition ASD for all $i = 1..n$ the partial derivative $\partial\tilde{\Pi}/\partial p_i$ exists and is strictly decreasing in x_j $(j \neq i)$, price best responses are strictly decreasing in x_j (Theorem 2.8.5 in Topkis 1998). Therefore, since p is a best response at y and p' at $y + \varepsilon > y$, we must have $p' < p$, and again arrive at a contradiction. ■

9.4.4 Entry: Prices

Before giving the proof of theorem 3, we will shortly discuss why it is not possible to give a corresponding proof for the non-aggregative case. Let us pay attention to the dependence on the number of firms in the fixed point map (20) used in the proof of theorem 1:

$$\psi\,(p, n) = \tilde{P}\,(\tilde{\chi}\,(p, n)\,, n) \tag{21}$$

Then there are two opposing effects of an increase in n: First, best price responses are lower since $\tilde{P}(x,n)$ is nonincreasing in n; second, the market price p can only be sustained if all firms produce less, since $\tilde{\chi}(p,n)$ is decreasing in n. The first effect moves ψ downwards, while the second effect moves it upwards. Under aggregation the first effect is stronger, but this is hard to show here. In the following proof we avoid this difficulty by constructing a different fixed point map making strong use of the assumption of aggregation.

Proof. (Theorem 3)

1. Price best responses: Because competitors' outputs are aggregated by $f(x_{-i})$, price best response is a correspondence $P : F \to [p_\infty, p_0]$, $p = P(f(x_{-i}))$, where $p_\infty = p(K,K,...)$, $p_0 = p(0,0,...)$, and maximal and minimal selections exist. Under condition A these selections are nonincreasing in f.

2. Symmetric outputs (this is the hard part, where the aggregation is really used): If the first $n-1$ competitors of firm i are active and the others produce zero, let

$$\tilde{f}(x,n) = f(x,..,x,0,...),\ x \in [0,K],$$

with image $\phi(n) = \left[\tilde{f}(0,n), \tilde{f}(K,n)\right] = \left[\underline{f}, \bar{f}(n)\right]$, where $\underline{f} = f(0,...)$ and $\bar{f}(n) = f(K,...,K,0,...)$. Then $\tilde{f}$ is strictly increasing and continuous in x, and strictly increasing in n. Let $\Phi = \{(f,n) \in F \times \mathbb{N} | f \in \phi(n)\}$, then we can express every firm's output x by the value of the aggregator and the number of firms through a function $\tilde{x} : \Phi \to [0,K]$ such that $\tilde{x}$ is strictly increasing and continuous in f, and strictly decreasing in n.

Consider market price at a given value of the aggregator, if all firms produce the same amount, even firm i (this is the basic trick):

$$\hat{p}(f,n) = p(\tilde{x}(f,n), f),$$

where $\hat{p} : F \times \mathbb{N} \to [p_\infty, p_0]$ is strictly decreasing and continuous in f, and strictly increasing in n. For fixed n, its image is $\hat{\pi}(n) = [p\left(K, \bar{f}(n)\right),$

$p\left(0,\underline{f}\right)]$, where the upper limit is fixed, and the lower limit is nonincreasing in n. Let $\hat{\Pi}=\{(p,n)\in[p_\infty,p_0]\times\mathbb{N}|p\in\hat{\pi}(n)\}$. Invert $\hat{p}$ with respect to f, to obtain a function $\hat{f}:\hat{\Pi}\to\mathbb{R}$ which is strictly decreasing (and continuous) in p and strictly increasing in n. The interpretation of $\hat{f}$ is: given price p and number of active firms n, value of aggregator if all n firms produce the same amount (even firm i), resulting in price p.

3. Symmetric equilibria: Let $\bar{P}:F\to[p_\infty,p_0]$ be a maximal or minimal selection of the price best response map P. Consider the family of maps $\psi_n:\hat{\pi}(n)\to[p_\infty,p_0]$ defined by

$$\psi_n(p)=\bar{P}\left(\hat{f}(p,n)\right),$$

then a maximum or minimum fixed point $p_{(n)}$ of this map is maximal or minimal equilibrium price. Under condition A, ψ_n is nonincreasing in n since $\bar{P}$ is nonincreasing, so that extremal equilibrium prices are nonincreasing in n (taking into account that $\pi(n)\subset\pi(n+1)$) by corollary 2.5.2 of Topkis (1998).

Under the (*non-dual*) weak single-crossing property of profits in (p_i,y), $\bar{P}$ is nondecreasing, and ψ_n is nondecreasing in n. Thus extremal equilibrium prices are *nondecreasing* in n if equilibria exist.

4. If $\partial\tilde{\Pi}/\partial p_i$ exists and is strictly decreasing in x_j, then by theorem 2.8.5 of Topkis (1998), which follows Amir (1996) and Edlin and Shannon (1998b) extremal price best responses $\bar{P}(x_{-i})$ are strictly decreasing while interior, i.e. positive. Since $\hat{f}$ is strictly increasing in n, the map $\psi_n(p)$ is strictly decreasing in n for each p, and therefore by corollary 2.5.2 of Topkis (1998) the extremal fixed points of ψ_n are strictly decreasing in n (Note that a subtle point of this proof is that we can only say something about the extremal fixed points and not about the others). ■

9.4.5 Entry: Interior equilibrium prices

Proof. (Corollary 4)

Denote the partial derivatives of inverse demand $p(x_i,f(x_{-i}))$ with respect to x_i and f as $p_1<0$ and $p_2<0$, of $f(x_{-i})$ with respect to x_j

$(j \neq i)$ as $f_j > 0$ (therefore $p^i = p_1$ and $p^j = p_2 f_j$), and of $g(x,n)$ with respect to x and n as $g_x = (n-1) f_j > 0$ and $g_n > 0$, respectively. The first-order necessary condition for an interior best response in terms of quantity, and the second derivatives of profits, are

$$\begin{aligned}
\Pi^i &= \tfrac{\partial}{\partial x_i}\Pi(x_i, x_{-i}) = p(x_i, f(x_{-i})) + x_i p_1(x_i, f(x_{-i})) - c'(x_i) = 0, \\
\Pi^{ii} &= \tfrac{\partial^2}{\partial x_i^2}\Pi(x_i, x_{-i}) = 2p_1 + x_i p_{11} - c'', \\
\Pi^{ij} &= \tfrac{\partial^2}{\partial x_i \partial x_j}\Pi(x_i, x_{-i}) = (p_2 + x_i p_{12}) f_j.
\end{aligned}$$

At a symmetric equilibrium with n firms,

$$\begin{aligned}
\tilde{\Pi}^i\left(x_{(n)}, n\right) = \Pi^i\left(x_{(n)}, .., x_{(n)}\right) = p\left(x_{(n)}, g\left(x_{(n)}, n\right)\right) \\
+ x_{(n)} p_1\left(x_{(n)}, g\left(x_{(n)}, n\right)\right) - c'\left(x_{(n)}\right) = 0,
\end{aligned}$$

we find the following second derivatives with respect to $x_{(n)}$ and n, respectively:

$$\begin{aligned}
\tilde{\Pi}^{ix} &= \tfrac{\partial}{\partial x_{(n)}}\tilde{\Pi}^i = \Pi^{ii} + \left(p_2 + x_{(n)} p_{12}\right) g_x = \Pi^{ii} + (n-1)\Pi^{ij}, \\
\tilde{\Pi}^{in} &= \tfrac{\partial}{\partial n}\tilde{\Pi}^i = \left(p_2 + x_{(n)} p_{12}\right) g_n = \Pi^{ij} g_n / f_j.
\end{aligned}$$

Equilibrium quantities evolve with

$$\frac{dx_{(n)}}{dn} = -\frac{\tilde{\Pi}^{in}}{\tilde{\Pi}^{ix}} = -\frac{\Pi^{ij} g_n / f_j}{\Pi^{ii} + (n-1)\Pi^{ij}},$$

where by Seade's stability condition the denominator $\tilde{\Pi}^{ix}$ is negative. The total derivative of equilibrium prices is

$$\begin{aligned}
\frac{dp_{(n)}}{dn} &= \frac{d}{dn} p\left(x_{(n)}, g\left(x_{(n)}, n\right)\right) = (p_1 + p_2 g_x)\frac{dx_{(n)}}{dn} + p_2 g_n \\
&= \frac{g_n p_2}{\Pi^{ii} + (n-1)\Pi^{ij}}\left(\Pi^{ii} - \frac{p_1}{p_2 f_2}\Pi^{ij}\right).
\end{aligned}$$

Since by condition AD the second term on the right-hand side is non-positive, the sign of $dp_{(n)}/dn$ depends entirely on whether the equilibrium is stable: Prices are decreasing (increasing) if the equilibrium is stable (instable), i.e. $\Pi^{ii} + (n-1)\Pi^{ij} < (>)\, 0$. ■

9.4.6 Entry: Quantities

Proof. (Corollary 5)

From the proof of corollary 4 we already know that

$$\frac{dx_{(n)}}{dn} = -\frac{x_{(n)}\Pi^{ij}}{\Pi^{ii} + (n-1)\Pi^{ij}}.$$

If condition AD holds and equilibrium price is decreasing, then from (16) we can conclude that $\Pi^{ii} + (n-1)\Pi^{ij} < 0$. Therefore $dx_{(n)}/dn < (>)\, 0$ if $\Pi^{ij} < (>)\, 0$, i.e. if goods are strategic substitutes (complements). Similarly, we can find from the first order condition $\Pi^{i}\left(Q_{(n)}/n, (n-1)\, Q_{(n)}/n\right)$ $= 0$ that

$$\frac{dQ_{(n)}}{dn} = \frac{Q_{(n)}}{n}\frac{\Pi^{ii} - \Pi^{ij}}{\Pi^{ii} + (n-1)\Pi^{ij}},$$

i.e. $dQ_{(n)}/dn > 0$ if $\Pi^{ii} - \Pi^{ij} < 0$. Now from condition AD and condition BS, whose differential form is $p^{i} < p^{j} < 0$,

$$0 \geq \frac{p^{j}}{p^{i}}\Pi^{ii} - \Pi^{ij} > \Pi^{ii} - \Pi^{ij}$$

since at interior equilibria $\Pi^{ii} \leq 0$ and $0 \leq p^{j}/p^{i} < 1$. ■

9.4.7 Entry: Profits

Proof. (Corollary 6)

From the proof of theorem 3 it is easy to see that under condition A $f_{(n)} = \hat{f}\left(p_{(n)}, n\right)$ is strictly increasing in n, since $\hat{f}$ is strictly decreasing in $p_{(n)}$ and strictly increasing in n, and $p_{(n)}$ is nonincreasing in n.

Since goods are substitutes, profits $\Pi(x, f)$ are nonincreasing in f. As $f_{(n)}$ is increasing in n,

$$\Pi\left(x_{(n)}, f_{(n)}\right) \geq \Pi\left(x_{(n+1)}, f_{(n)}\right) \geq \Pi\left(x_{(n+1)}, f_{(n+1)}\right),$$

where $x_{(n)}$ and $x_{(n+1)}$ are the corresponding equilibrium outputs, and the first inequality expresses the fact that $x_{(n)}$ maximizes profits. If p is strictly decreasing in f then the second inequality is strict. ■

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