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Abstract

We show that asymmetric equilibria may exist in an *ex ante* symmetric step-by-step R&D race if product market competition is intense. The corresponding symmetric equilibrium is unstable and therefore less likely the one selected, while the asymmetric ones result in lower economic growth (reversing earlier results) but higher industry profits. Secondly, we show that the assumption of 'no leap-frogging' imbodied in the step-by-step structure of the model imposes no restrictions on the optimal strategies if and only if product market competition is small and imitation is easy. Both these results indicate that predictions or policy conclusions based on these models may have to be qualified.

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1 Introduction

We examine a model of technical progress that has been used as a building block in the 'Schumpeterian' or endogenous growth literature, e.g. in Aghion, Harris and Vickers (AHV, 1997). It models non-drastic innovation by strongly restricting the dynamics of competition: Innovations occur 'step-by-step', which means that a firm that has fallen behind must first catch up and equalize with the leading firm before overtaking it. In particular, 'leapfrogging' is ruled out by assumption. AHV analyze this model to give an example of a model of dynamic innovation competition where contrary to the results on drastic innovations "more intense product market competition and/or imitations may be growth-enhancing". They do this by comparing Bertrand and Cournot competition in the product market.

We elaborate on their model concerning two points: First, the authors only analyze symmetric equilibria and show that there is a unique one. Their analysis does not reveal whether there are asymmetric equilibria as well, and whether the symmetric one is stable. In fact, these two points are connected, since in the presence of asymmetric equilibria the sole symmetric one is necessarily unstable.[1] Stability of equilibrium is a desirable property for predictions or empirical applications, since the players will move away fast from unstable equilibria if there is the slightest amount of noise. We show that a pair of asymmetric equilibria arises under Bertrand competition in the output market if firms are very patient and imitation is difficult, while Cournot equilibrium is always stable. Where there is higher growth in the symmetric unstable Bertrand equilibrium than in the symmetric Cournot equilibrium, growth at the stable asymmetric equilibria is lower, reversing the conclusion that more competition in the product market leads to higher growth.

Second, we will discuss the assumption that firms cannot leapfrog the leaders and delineate the circumstances where firms would leapfrog

[1] This is essentially an index-theoretic result, as was applied to Cournot oligopoly by Kolstand and Mathiesen (1987).

in equilibrium if they had the possibility to do so. Leapfrogging in equilibrium would occur under Bertrand competition, but may not occur under Cournot competition. Our analysis provides hints that traditional Schumpeterian leapfrogging models of drastic innovation are more appropriate if market competition is high, while the step-by-step assumption is justified when competition is less intense.

2 The Model

We will first describe the setup of the AHV model. Two duopolists with constant marginal cost compete in a market for a homogeneous product either in quantities or prices (Cournot or Bertrand competition, respectively). Market demand has unit-elasticity, i.e. it is of the form $p = 1/Q$, where Q is total output. Therefore if firms' marginal costs are c_i and c_j, flow profits of firm i under Cournot competition are $\pi^i = 1/\left(1 + c_i/c_j\right)^2$, while with Bertrand competition they are $\pi^i = \max\left\{1 - c_i/c_j, 0\right\}$.

Both firms conduct research to make cost-reducing innovations. These innovations reduce unit costs by a fixed factor $\gamma > 1$, i.e. $c_i' = c_i/\gamma$, and arrive randomly and independently in continuous time with Poisson hazard rates determined by the research efforts of the two firms. A fundamental assumption of this model is that any firm can be maximally one step ahead: After a firm moves ahead it has to wait until the other one has caught up. This assumption can be justified by assuming that any further innovation would immediately disclose the last innovation to the other firm, so that in practice a new innovation does not change the cost gap. The possible states of competition are therefore $S = \{-1, 0, 1\}$, meaning that either firm i is behind, or firms are neck-to-neck, or firm i is in front.

Let firm i's research efforts be $(x, y) \in \mathbb{R}_+^2$, and firm j's $(\bar{x}, \bar{y}) \in \mathbb{R}_+^2$. When firms are neck-to-neck their research efforts are x and $\bar{x}$, and when they fall behind the efforts are y and $\bar{y}$. Research is costly, with cost $c\left(z\right) = z^2/2$. Catching up is easier than moving ahead: The hazard rate of a firm that has fallen behind is $y + h$, where $h \geq 0$ measures the ease of imitation. Market profit flows are given by π_1, π_0, and π_{-1}, for a firm that is leading, neck-to-neck, or behind, respectively; $r > 0$ is the common discount factor. As in AHV, we will only be concerned with pure strategy equilibria in perfect Markov strategies, i.e. pure strategies that form a subgame-perfect Nash equilibrium of the dynamic game, and only depend on the state in $\{-1, 0, 1\}$ of the game.

Expected equilibrium payoffs are then characterized by the value functions

$$\begin{aligned} rV_1 &= \pi_1 - (\bar{y} + h)(V_1 - V_0), \\ rV_0 &= \pi_0 - c(x) + x(V_1 - V_0) - \bar{x}(V_0 - V_{-1}), \\ rV_{-1} &= \pi_{-1} - c(y) + (y + h)(V_0 - V_{-1}), \end{aligned} \tag{1}$$

where V_1, V_0, and V_{-1} are the values of being ahead, neck-to-neck, and behind, respectively. For example, V_0, the value of being neck-to-neck, is determined as flow profits π_0 minus costs of research $c(x)$, plus the expected gain from innovating $x(V_1 - V_0)$, minus the expected loss caused by an innovation of the other firm, $\bar{x}(V_0 - V_{-1})$. Given the strategy pair $(\bar{x}, \bar{y})$ of the other firm, the optimal strategies x and y of a firm that is neck-to-neck or behind, respectively, have to satisfy the first order necessary conditions

$$\begin{aligned} c'(x) &= (V_1 - V_0) \\ c'(y) &= (V_0 - V_{-1}). \end{aligned} \tag{2}$$

Taking pair-wise differences between the value functions (1) and inserting the first order conditions (2) we obtain the following system of equations characterizing the best responses $(x, y) \in b(\bar{x}, \bar{y})$, where $b : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ is the best response correspondence (let $P = \pi_1 - \pi_0$ and $C = \pi_1 - \pi_{-1}$ describe the "profit incentive" and "competitive threat" as in Beath et al. 1989, and $s = h + r$):

$$\begin{aligned} \tfrac{1}{2}x^2 + (\bar{y} + s)x - \bar{x}y &= P, \\ \tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 + (\bar{x} + s)y &= C - P. \end{aligned} \tag{3}$$

Corresponding equations characterize $\bar{x}$ and $\bar{y}$. Note that we explicitly allow asymmetric choices for all effort levels. For symmetric strategy pairs, $x = \bar{x}$ and $y = \bar{y}$, (3) are solved by AHV to yield the equilibrium strategies

$$\begin{aligned} x &= \sqrt{s^2 + 2P} - s, \\ y &= \sqrt{s^2 + x^2 + 2C} - \sqrt{s^2 + 2P}. \end{aligned} \tag{4}$$

They also show that the growth rate of the economy if there are many identical sectors is given by

$$g = 2x \frac{y + h}{y + h + 2x} \ln \gamma.$$

It can be shown that in asymmetric equilibria the average growth rate is given by

$$g = \frac{(x + \bar{x})(y + h)(\bar{y} + h)}{(y + h + \bar{x})(\bar{y} + h + x) - x\bar{x}} \ln \gamma.$$

3 Asymmetric Equilibria

In theory, (3) could be solved explicitly for (x, y) and $(\bar{x}, \bar{y})$, since it can be shown that these four equations can be 'reduced' to four independent polynomial equations of order four, which still have analytical solutions. Instead, we will use the inverse response map $(\bar{x}, \bar{y}) = b^{-1}(x, y)$, similar to Harris and Vickers (1987).

3.1 The Inverse Best Response

We will move in two steps. First, as we show in appendix 6.1, note that the strategies used in any equilibrium are exactly the solutions to the fixed point equation $(x, y) \in b(b(x, y))$, while strategies in symmetric equilibria make up the subset for which also $(x, y) \in b(x, y)$ holds. Second, if b^{-1} is the inverse of the best response, $(x, y) \in b(b(x, y))$ if and only if $(x, y) \in b^{-1}(b^{1}(x, y))$, as shown in appendix 6.2. Therefore we can work with the inverse reaction map which in this case is more straightforward, and have the following: $(x, y) \in \mathbb{R}^2_{++}$ is a strategy played in some pure strategy Markov perfect equilibrium (symmetric or asymmetric) if and only if $(x,y) \in (b^{-1})^2(x,y)$. It is part of some *symmetric* equilibrium if and only if $(x,y) \in b^{-1}(x,y)$.

Best responses $(x,y) = b(\bar{x},\bar{y})$ in pure Markov strategies for the R&D-race are given by the equations (3). Under the natural assumptions that $\pi_1 > \pi_0$ (profits of a leader are strictly higher than those of a firm that is neck-to-neck), and $\pi_0 \geq \pi_{-1}$, for any best response $(x,y) \in \mathbb{R}^2_+$ to $(\bar{x},\bar{y}) \in \mathbb{R}^2_+$ we must have $x > 0$ and $y > 0$, i.e. $(x,y) \in \mathbb{R}^2_{++}$. The inverse response map $b^{-1} : \mathbb{R}^2_{++} \to \mathbb{R}^2$ is then given by solving (3) for $\bar{x}$ and $\bar{y}$,

$$\begin{aligned} \bar{x} &= \left(C - P + x^2/2 - y^2/2\right)/y - s, \\ \bar{y} &= \left(C - y^2/2 - ys\right)/x - s. \end{aligned} \tag{5}$$

Note that for large x or y the images $\bar{x}$ or $\bar{y}$ may be negative. This simply means that this (x,y) is not a best response to any feasible (i.e. non-negative) strategy of the other firm. Therefore, (x,y) is a best response to some feasible strategy by the other firm if and only if $b^{-1}(x,y) \geq 0$.

3.2 The Graphical Solution

The fixed point condition on pure Markov perfect equilibrium strategies, $(x,y) \in (b^{-1})^2(x,y)$, is still difficult to visualize, since $(b^{-1})^2$ is a map with a four-dimensional graph. The same is true for the fixed point condition on symmetric equilibria, $(x,y) \in b^{-1}(x,y)$. They can be 'solved' numerically, but this is little intuitive, and also there is no guarantee that all solutions are found. We therefore propose a graphical solution where the fixed points can be visualized in two-dimensional space. Let F be either one of the maps $(b^{-1})^2$ or b^{-1}, and let $F = (F_x, F_y)$, where $F_x, F_y : \mathbb{R}^2_+ \to \mathbb{R}_+$. That is, if $F(x,y) = (\bar{x},\bar{y})$, then $F_x(x,y) = \bar{x}$, and $F_y(x,y) = \bar{y}$. The fixed point condition $(x,y) = F(x,y)$ can then be expressed equivalently by the two conditions $x = F_x(x,y)$ and $y = F_y(x,y)$. These describe two curves in $\mathbb{R}^2_+$, and equilibrium strategies can be found at their intersections. Denote by A^1_x and A^1_y the curves pertaining to $F = b^{-1}$ (for symmetric equilibria), and by A^2_x and A^2_y the curves pertaining to $F = (b^{-1})^2$ (all equilibria).

Let us consider the cases of Cournot and Bertrand competition. Under Cournot competition and unit-elasticity demand the profit function is $\pi_i = 1/\left(1+\gamma^{-i}\right)^2$; with $s = 0.02$ and $\gamma = 2$ we obtain the following figure:

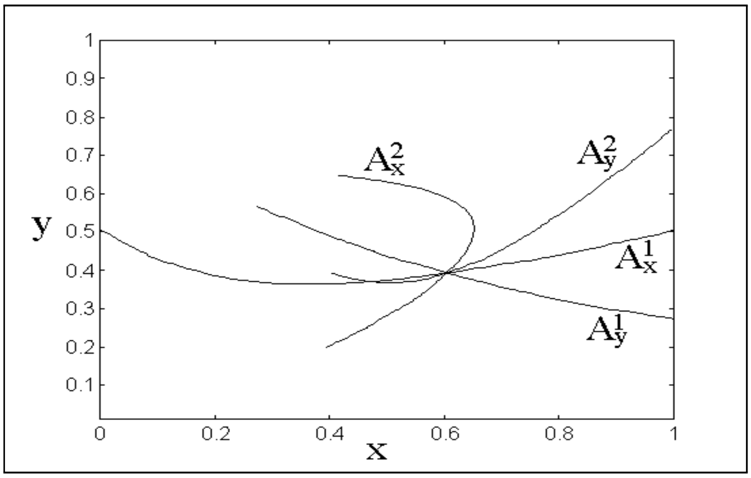

Figure 1: Equilibria under Cournot competition ($s = 0.02$, $\gamma = 2$).

We can see that A_x^2 and A_y^2 meet at the same point as A_x^1 and A_y^1, since A_x^1 and A_y^1 describe the symmetric equilibria. Since A_x^2 and A_y^2, and also A_x^1 and A_y^1, do not meet anywhere else, there is exactly one equilibrium, and it is symmetric (at $x \approx 0.604$ and $y \approx 0.392$, with average growth rate $g \approx 0.209$). This is the generic result under Cournot competition as we will argue below.

However, under Bertrand competition ($\pi_i = \max\{0, 1-\gamma^{-i}\}$), also with $s = 0.02$ and $\gamma = 2$, we obtain Figure 2:

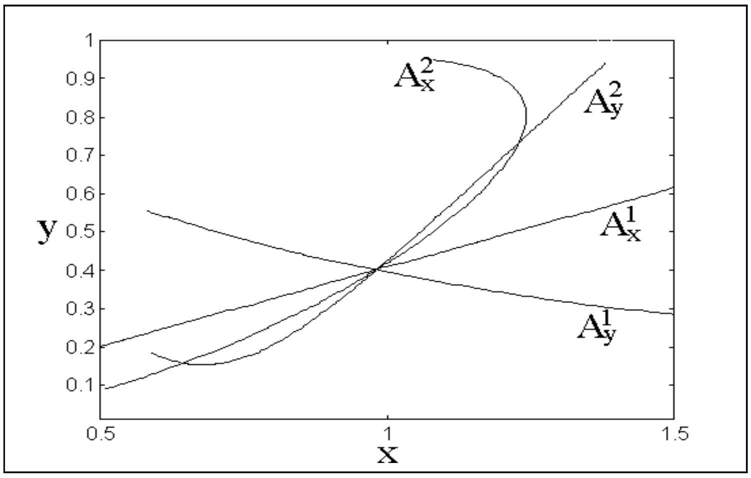

Figure 2: Equilibria under Bertrand competition ($s = 0.02$, $\gamma = 2$).

Here there is a pair of asymmetric equilibria (involving the strategy pairs $(x_1, y_1) \approx (1.229, 0.734)$ and $(x_2, y_2) \approx (0.643, 0.156)$, and growth rate $g \approx 0.140$) and one symmetric equilibrium, with $(x, y) \approx (0.980, 0.400)$ and growth rate $g \approx 0.235$. We can see that the existence of asymmetric equilibria depends on the relative slopes of the loci A_x^2 and A_y^2 around the symmetric equilibrium: The example in Figure 3 ($s = 0.15$, $\gamma = 2$) shows that asymmetric equilibria may exist if and only if the slope of A_y^2 is steeper than the slope of A_x^2 (in coordinates (x, y)):

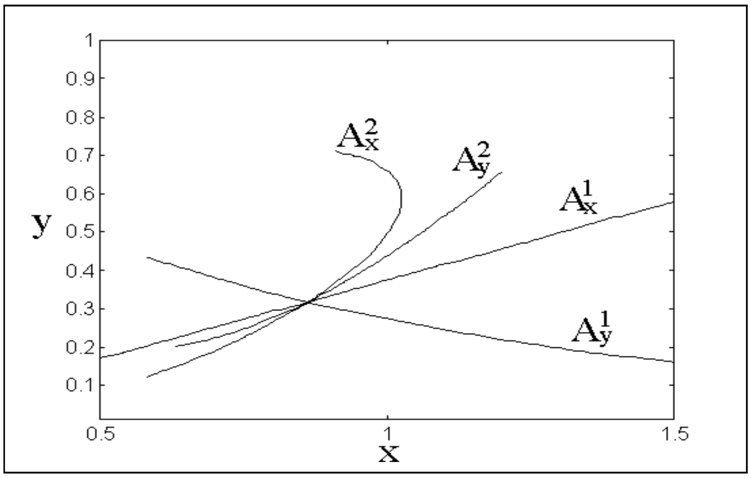

Figure 3: Equilibria under Bertrand competition ($s = 0.15$, $\gamma = 2$).

We show in appendix 6.3 that the symmetric equilibrium with Cournot competition is always stable, while the symmetric equilibrium under Bertrand competition is unstable if and only if $(1 - \gamma^{-1}) / (r + h)^2 > K^* \approx 65.172$, i.e. if the discount rate r and the ease of imitation h are sufficiently small. A large innovation size γ makes instability more likely, but is in general not sufficient to cause instability.

3.3 Welfare Properties of Asymmetric Equilibria

The existence of asymmetric equilibria under Bertrand competition has a straightforward interpretation: More relative advantage for the leader, either because the product market is more competitive (Bertrand instead of Cournot competition), or because cost reduction through innovations are bigger (γ higher), or because imitation is more difficult (lower h), or because players are patient and care about long-term advantage (lower r), may result in endogenous asymmetry. *Ex ante* identical firms choose different strategies because 'the market is too small', and one of them emerges as a 'natural leader', whereas the other becomes a 'natural follower'. Beliefs about each others' strategies then reinforce the asymmetry even though there are times when both firms are neck-to-neck and have the *same* cost of production, because they follow different investment strategies.

As argued above, considering the symmetric equilibrium as the "legitimate solution" or even prediction of the game is questionable if it is unstable. Here the most reasonable prediction would be that the market ends up in one of the asymmetric equilibria, even if this is subject to the equilibrium selection problem. The comparison of payoffs and growth rates between symmetric and asymmetric equilibria is also of interest: It turns out that the (neck-to-neck) payoffs for the Bertrand example in Figure 2 are 28.39 and 1.525, whereas in the symmetric equilibrium they are 8.813 for each firm. Therefore, in the asymmetric equilibrium the 'follower' is much worse off, but *joint payoffs* are higher than in the symmetric equilibrium. We were not able to prove this analytically, but suspect that this may hold generally: Since for the disadvantaged firm it is rational to hold back its efforts, there will be less dissipation of monopoly rents than in a symmetric duopoly.

On the other hand, average growth rates are higher in symmetric equilibria. The reason is that even though one of the firms credibly exerts very high research efforts when firms are neck-to-neck and when it is behind, since the other firm invests less in research in both cases, in equilibrium there will be fewer innovations, which lowers the growth rate. The decrease in growth rates, as the above example shows, can be so strong that conclusions about the relation between the intensity of competition in the product market and economic growth can be reversed.

This divergence between industry payoffs and economic growth, together with the instability of the symmetric equilibrium, may make asymmetry a welfare or even policy issue: The presence of a too aggressive firm (even if the competitors are ex ante on equal footing) may effectively slow down growth, while keeping industry profits at a higher level. Therefore, higher competition in the product market will only be able to raise growth if the equilibrium if firms behave similarly in equilibrium.

4 To leapfrog or not to leapfrog

AHV assume that cost-reducing innovations are of a fixed size γ, and that a firm that has fallen behind first has to catch up with the leader (make an innovation of size γ) instead of leapfrogging him (making an innovation of size γ^2). In this section we will discuss the equilibrium outcomes if leapfrogging is possible.

We will analyze whether in the present model 'no leapfrogging' is an optimal choice if leapfrogging to the leader's position is possible. In this case the assumption of 'no leapfrogging' imposes no restriction on the equilibrium strategies.

We will assume that 'no leapfrogging' is an equilibrium, with value functions as in (1) given by

$$\begin{aligned} r\bar{W} &= \pi_1 - (y_j + h)\left(\bar{W} - \bar{V}\right), \\ r\bar{V} &= \pi_0 - c\left(x_i\right) + x_i\left(\bar{W} - \bar{V}\right) - x_j\left(\bar{V} - \bar{U}\right), \end{aligned}$$

$$r\bar{U} = \pi_{-1} - c(y_i) + (y_j + h)\left(\bar{V} - \bar{U}\right),$$

and first order conditions for optimal effort levels given as in (2) by

$$\begin{aligned} c'(x_i) &= \left(\bar{W} - \bar{V}\right), \\ c'(y_i) &= \left(\bar{V} - \bar{U}\right). \end{aligned}$$

A follower who is deliberating to leap-frog faces the following value of leapfrogging (assuming that afterwards the equilibrium without leapfrogging is played):

$$rU_l = \pi_{-1} - c(z) + (z + h_l)\left(\bar{W} - U_l\right),$$

where we assume that imitation is more difficult than for just catching up: $0 \leq h_l < h$. Research effort is $z \geq 0$, and at the optimum is characterized by the usual first order condition $c'(z) = \left(\bar{W} - U_l\right)$. Using this first order conditions and solving[2] for U_l leads to (assuming quadratic cost of research as above)

$$\begin{aligned} \bar{U} &= \tfrac{1}{r}\left(\pi_{-1} + \tfrac{1}{2}y^2 + hy\right), \\ U_l &= \bar{W} - \sqrt{(h_l + r)^2 + 2\left(r\bar{W} - \pi_{-1}\right)} + (h_l + r). \end{aligned}$$

The follower prefers catching up over leapfrogging if $\bar{U} \geq U_l$. After some manipulations using the first order conditions, and because in equilibrium $\bar{W} = x + y + \bar{U}$, this condition can be written as

$$x^2 + 2xy + 2xh_l + 2yh_l \leq 2yh.$$

This can only hold if, in equilibrium, effort levels are very small and h_l is small enough. A necessary condition for preferring catching up over leapfrogging is $x^2 + 2xy \leq 2yh$, which is independent of the value of h_l. It can be shown that this condition is more likely to be satisfied if innovation size γ is small or discount rate/ease of imitation s is large (and therefore x and y are small), and an example for Cournot competition where catching up is preferred to leapfrogging is the equilibrium under

[2] The second root of the equation for U_l is excluded by $\bar{W} - U_l \geq 0$.

the parameters $\gamma = 1.1$, $h = 0.2$, and $r = 0.01$. On the other hand, under Bertrand competition this condition never holds for any $\gamma > 1$, and therefore firms prefer leapfrogging even if imitation is difficult, $h_l = 0$, and therefore the 'equilibrium' with catching up is never an equilibrium under Bertrand competition if we allow firms to leapfrog.

To sum up, if we relax the assumption that firms *cannot* leapfrog, then no leapfrogging arises as an *equilibrium outcome* only when the intensity of competition in the product market is small (Cournot competition), and innovation size is small, or the discount rate and ease of imitation are large.

5 Conclusion

For a simple model of step-by-step innovation competition we have shown that the unique symmetric equilibrium may be unstable if product market competition is high, innovations large, and discount rate and ease of innovation small. In this case asymmetric equilibria may exist as well, possibly altering the empirical predictions and welfare properties of this model.

We have also shown that the assumption that firms cannot leapfrog each other does restrict equilibrium strategies in the sense that firms would prefer to directly leapfrog each other unless market competition is low, innovations are small, and discount rate and ease of innovation are large.

6 Appendix

6.1 Symmetric and Asymmetric Equilibria

In a game with 2 players, strategy spaces S_i $(i = 1, 2)$ and given pure strategy best response maps $b_i : S_j \to S_i$ $(j \neq i)$, the pure strategy equilibria $s = (s_1, s_2) \in S_1 \times S_2$ are characterized by the fixed point equations $s_i \in b_i(s_j)$ $(j \neq i)$. This leads to the necessary condition on an equilibrium strategy s_i

$$s_i \in b_i(b_j(s_i)) \quad (j \neq i), \tag{6}$$

which does not depend on the strategy of firm j. For identical best response maps $b_1 = b_2 = b$ on $S_1 = S_2$, this condition becomes $s_i \in b(b(s_i)) = b^2(s_i)$ $(i = 1, 2)$. Any s_i with $s_i \in b^2(s_i)$ is *part* of some pure strategy Nash equilibrium $(s_i, \hat{s}_j)$ since there is $\hat{s}_j \in b(s_i)$ with $s_i \in b(\hat{s}_j)$, but of course not all combinations of (s_1, s_2) with $s_i \in b^2(s_i)$ $(i = 1, 2)$ are Nash equilibria (unless the game is zero-sum).

Also, s_i with $s_i \in b(s_i)$ are part of *symmetric* pure strategy equilibria (s_1, s_2) with $s_1 = s_2$. It is obvious that

$$\{s_i \in S_i | s_i \in b(s_i)\} \subset \left\{s_i \in S_i | s_i \in b^2(s_i)\right\},$$

i.e. symmetric equilibria are trivial 'asymmetric' equilibria. Strategies appearing as part of (non-trivial) *asymmetric* pure strategy equilibria are therefore given by

$$\left\{s_i \in S_i | s_i \in b^2(s_i) \wedge s_i \notin b(s_i)\right\}.$$

These conditions are illustrated in the context of the following static game: There are two players with strategy sets $S_1 = S_2 = [0, 1]$, and payoffs are $s_i (s_j - 1)^2 - s_i^2/2$ $(i = 1, 2; i \neq j)$. Best responses are found to be single-valued with $b(s_j) = (s_j - 1)^2$, and the pure strategies in equilibrium are given as the solutions (over $[0, 1]$) of the following equations

$$\begin{aligned} \text{symmetric} &: \quad s_i = (s_i - 1)^2 \Rightarrow s_i = \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}, \\ \text{(a)symmetric} &: \quad s_i = \left((s_i - 1)^2 - 1\right)^2 \Rightarrow s_i \in \left\{0, \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}, 1\right\}. \end{aligned}$$

The pure strategy Nash equilibria are therefore $(0, 1)$, $(1, 0)$, and $(\frac{3}{2} - \frac{1}{2}\sqrt{5}, \frac{3}{2} - \frac{1}{2}\sqrt{5})$.

6.2 The inverse best-response map

Above we argued that to find the set of all pure equilibrium strategies we can work with the inverse response map instead of the response map itself. To this end we prove the following lemma:

Lemma 1 *The sets of fixed points of* $b^2 : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ *and of* $\left(b^{-1}\right)^2 : \mathbb{R}^2_{++} \to \mathbb{R}^2$ *are identical:*

$$\left\{(x,y) \,|\, (x,y) \in b^2(x,y)\right\} = \left\{(x,y) \,|\, (x,y) \in \left(b^{-1}\right)^2(x,y)\right\}.$$

Proof. Extend b^{-1} to the whole of $\mathbb{R}^2$ by defining $b^{-1}(x,y) = \emptyset$ if $(x,y) \in \mathbb{R}^2 \backslash \mathbb{R}^2_{++}$, and define $b^{-1}(\emptyset) = \emptyset$. Let (x,y) be such that $(x,y) \in b^2(x,y)$; then there is $(\bar{x},\bar{y}) \in \mathbb{R}^2_{++}$ such that $(\bar{x},\bar{y}) \in b(x,y)$, $(x,y) \in b(\bar{x},\bar{y})$, and therefore $(x,y) \in \mathbb{R}^2_{++}$. Then it is obvious that $(\bar{x},\bar{y}) \in b^{-1}(x,y)$ and $(x,y) \in b^{-1}(\bar{x},\bar{y})$, therefore $(x,y) \in \left(b^{-1}\right)^2(x,y)$.

For the converse, let $(x,y) \in \left(b^{-1}\right)^2(x,y)$. Then there is $(\bar{x},\bar{y}) \in b^{-1}(x,y) \cap \mathbb{R}^2_{++}$ such that $(x,y) \in b^{-1}(\bar{x},\bar{y})$, otherwise $\left(b^{-1}\right)^2(x,y)$ would be empty. Therefore $(x,y) \in b(\bar{x},\bar{y})$ and $(\bar{x},\bar{y}) \in b(x,y)$, i.e. $(x,y) \in b^2(x,y)$. ■

6.3 Stability

In this appendix we will give a precise definition of what we mean by "stability" of an equilibrium in this game where each player has two strategies instead of only one, and derive a necessary and sufficient condition for stability of the symmetric equilibria of this model.

Let Db and Db^{-1} be the (2x2) Jacobians of the best response and inverse best response maps, respectively. Then we say that an equilibrium is stable if the absolute values of the (real parts of the) eigenvalues of Db at this equilibrium are strictly smaller than 1, or equivalently, if the absolute values of the (real parts of the) eigenvalues of Db^{-1} at this equilibrium are strictly larger than 1. This definition is the natural generalization to higher-dimensional strategies of the usual notion of stability for two-player games with one-dimensional strategies as e.g. in Seade (1980).

We will make use of the inverse response map, with

$$Db^{-1}(x,y) = \begin{pmatrix} \frac{x}{y} & -\frac{1}{2}\frac{y^2+2C-2P+x^2}{y^2} \\ \frac{1}{2}\frac{2sy-2C+y^2}{x^2} & -\frac{s+y}{x} \end{pmatrix},$$

and eigenvalues

$$\begin{aligned} e_1 &= \frac{x^2-sy-y^2+\sqrt{(x^4+y^2x^2+s^2y^2-4Csy+4C^2+4Psy-4PC+2Py^2+2x^2C)}}{2xy} \\ e_2 &= \frac{x^2-sy-y^2-\sqrt{(x^4+y^2x^2+s^2y^2-4Csy+4C^2+4Psy-4PC+2Py^2+2x^2C)}}{2xy}. \end{aligned}$$

After substituting the symmetric equilibrium solution (4), the larger eigenvalue e_1 is always positive and larger than 1, while the smaller eigenvalue is negative and may fall in the interval $[-1,0]$, which happens if and only if $C/s^2 = C/(r+h)^2$ is larger than the positive root K (if it exists) of the equation

$$\begin{aligned} & 16(\lambda+1)^2(5\lambda-4)^2K^4 + \left(192-352\lambda^3-944\lambda^2-2992\lambda\right)K^3 \quad (7) \\ - & \left(568\lambda^2+2888\lambda+1404\right)K^2 - (1252+712\lambda)K - 267 = 0, \end{aligned}$$

where $\lambda = P/C$. With Bertrand competition, $\lambda = 1$ because $C = P = 1-\gamma^{-1}$, and the positive solution of (7) is $K \approx 65.172$. Therefore the symmetric equilibrium with Bertrand competition is unstable if and only if $(1-\gamma^{-1})/(r+h)^2 > K \approx 65.172$. By continuity of the reaction maps, and given that

On the other hand, e_2 is always smaller than -1 if $0 \le \lambda \le 4/5$, for which (7) does not have a positive solution. With Cournot competition, $C = \frac{\gamma-1}{1+\gamma}$ and $P = \frac{1}{4}\frac{(3\gamma+1)(\gamma-1)}{(1+\gamma)^2}$, and $P < \frac{4}{5}C$ is true for all $\gamma > 1$. Therefore the symmetric equilibrium with Cournot competition is always stable.

References

[1] Aghion, Philippe, Harris, Christopher, and Vickers, John (1997). "Competition and growth with step-by-step innovation: An example". *European Economic Review*, 41:771–82.

[2] Beath, John, Katsoulacos, Yannis, and Ulph, David (1989). "Strategic R&D Policy". *The Economic Journal*, 99 (Conference):74–83.

[3] Harris, Christopher and Vickers, John (1987). "Racing with Uncertainty". *Review of Economic Studies*, 54:1–21.

[4] Kolstad, Charles D. and Mathiesen, Lars (1987). "Necessary and Sufficient Conditions for Uniqueness of a Cournot Equilibrium". *Review of Economic Studies*, 54:681–90.

[5] Seade, Jesus K. (1980). "The Stability of Cournot revisited". *Journal of Economic Theory*, 23:15–27.