Housing Prices and Credit Constraints in Competitive Search*

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Abstract

This paper shows that, when utility is imperfectly transferable and the search process is competitive (or directed), wealthier buyers pay higher prices to speed up transactions. This result is established in a dynamic model of the housing market where households save both to smooth consumption and to build a down payment. "Block recursivity" is ensured by the existence of risk-neutral housing intermediaries. The calibrated version of our benchmark economy features greater indebtedness and higher housing prices in the long run compared to a Walrasian model, especially when the elasticity of new housing supply is low. We also show that the long-run effect of greater credit availability on housing prices depends crucially on whether or not rental and real estate housing stocks are segmented. Under full segmentation, price effects are much larger, with and without search frictions. But, even if there is no segmentation, these effects are substantial in our search model when supply elasticity is low, being larger than in the Walrasian version of the model. The last result is reversed with full segmentation, when search frictions dampen the price effect of the credit expansion.

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1 Introduction

We study a competitive search equilibrium model where risk-averse buyers who seek to purchase an indivisible good sort by their wealth. Specifically, wealthier buyers pay higher prices on average because this allows them to speed up transactions, while buyers with lower wealth tend to choose cheaper offers that feature longer queues. Intuitively, since the marginal utility of wealth is decreasing, wealthier buyers care relatively more about completing a transaction, while poorer ones are more concerned about paying lower prices. In turn, as is standard in these models, pricier goods take longer to sell. The sorting result is general, but we derive it in the context of the housing market. We know that wealthier buyers tend to purchase better homes. The added insight underlying the sorting result is that, conditional on the attributes of the homes they intend to buy (e.g. for a given quality, location and home size), buyers who are more wealthy pay higher prices in order to reduce trading delays. This behavior generates frictional price dispersion in equilibrium. In a similar fashion, wealthier travelers would opt for more expensive airlines or car rentals to avoid delays, whereas poorer buyers with similar traveling plans choose cheaper deals and typically face longer wait times. Likewise, wealthier customers prefer less crowed restaurants that are pricier, while poorer customers opting for similar food quality choose cheaper restaurants that are subject to delays (e.g. queues and slow service). These wealth effects are bound to be more important when demand is high, and the corresponding markets become more congested.

The housing market is arguably the most important application of our theory. A household's primary asset is usually its home (e.g. housing wealth accounts for about half of household net worth in the US). Since houses are big ticket items, wealth effects are likely to play a role in home purchasing decisions. Several studies find variations in house prices after controlling for house characteristics and location (e.g. Lisi and Iacobini, 2013; Guren, 2018; Kotova and Zhang, 2020). Indeed, our sorting result is consistent with empirical work in the real estate literature which finds that, after controlling for housing attributes and location, richer buyers tend to pay higher prices (see Elder et al., 1999; Qiu and Tu, 2018), and search for a shorter period of time on average (see Elder, Zumpano, and Baryla 1999, 2000). There is also widespread evidence of a positive relationship between the price of real estate property and its average time on the market (e.g. Merlo and Ortalo-Magné, 2004; de Wit and van der Klaauw, 2013).

The environment is a small open economy with long-lived households who consume a nondurable good and housing services. Individuals' bear uninsurable idiosyncratic earnings risk. Households may own or rent their homes, and may also differ in their liquid asset wealth. Owner-occupied housing is associated with a utility premium, but its illiquidity makes it ineffective at shielding consumption against permanent shocks. Yet there are always some homeowners who sell because of exogenous preference and moving shocks, and who will therefore need to buy a new house or become renters. Home purchases can be partially financed with non-defaultable mortgage loans, and houses serve as collateral for new loans (i.e., their owners can always remortgage). Households can accumulate a risk-free asset both to build a down payment and to smooth non-housing consumption. Buyers must search for a home in a decentralized market and the search process is competitive. We assume that the properties that are up for sale in this market are symmetric, so all price dispersion arising in equilibrium is purely frictional. With competitive search, agents may choose to trade at different prices, knowing that lower prices generate longer queues. The housing market is then described as segmented. Different market segments (or "submarkets") feature different prices for identical homes and thus different trading probabilities for buyers and sellers.[1] The construction of new housing is undertaken by competitive developers each period. We consider an economy where rental units can be converted into owner-occupied housing (e.g. as in Kaplan et al., 2020), as well as one where this conversion is not possible and rents are exogenous (as in Garriga and Hedlund, 2020).[2] We focus our analysis on stationary equilibria.

[1]The endogenous segmentation of different agent types across submarkets is a typical property of competitive search models, where different types trade off prices against trading probabilities at different rates (e.g. see Wright et al., 2021). Search theory has long been used to rationalize the existence of frictional price dispersion. Recent related work by Piazzesi et al. (2015) documents differential search patterns by buyers at the ZIP code level using data from California's website Trulia, and argues that these patterns can explain differences in the prices of houses with similar characteristics across ZIP codes. Their model assumes risk-neutral searchers and hence no wealth effects.

[2]The estimates in Greenwald and Guren (2021) indicate substantial market segmentation and Sommer et al. (2013) (among others) show that rents have been relatively flat over the last few decades, so the second economy is more in line with these findings.

The model is highly tractable because it is "block recursive", in the sense that the agents' value and policy functions depend on the distribution of households across individual states through a one-dimensional state variable that summarizes the relevant information about the terms of trade in the housing market. This is unlike random search models, where these functions depend on the entire household distribution (e.g. Molico, 2006).[3] Block recursivity arises because we assume that home buyers and homeowners, both of whom are risk averse, do not trade directly with each other in the search market (see also Hedlund, 2016a; Karahan and Rhee, 2019; Garriga and Hedlund, 2020). Instead, homeowners sell their homes in a Walrasian market to a housing intermediary, who then looks for potential buyers in the competitive search market. The prices in both markets are related because there is free entry into intermediation, so competitive intermediaries make zero profits in each segment of the search market.

To illustrate how the model works and what its quantitative properties are, we calibrate it to match selected statistics for the U.S. economy (which include measures of turnover). Our steady state exercises show that price dispersion, market congestion, and wealth accumulation are tightly linked. Take the case of a highly liquid market, where demand is high, average buying times are long. In this scenario, buyers who do not find a trading opportunity (a likely event for poor households) accumulate more assets and, in the next period, they target more expensive homes to increase their trading probability. As competition for these homes intensifies, wealthier buyers start to target homes that are even more expensive and borrow more. This competition, arising from sorting, propagates throughout the entire wealth distribution and produces frictional price dispersion in an economy where owner occupied housing units are homogeneous. This mechanism results in greater indebtedness in the long run compared to a Walrasian version of the model (where all buyers trade instantaneously at the same price). Moreover, if rental and real estate housing stocks are not segmented and credit is limited, it also generates higher housing prices. The less elastic the supply of new housing, the more important these differences are.

[3] This block-recursive structure is slightly more involved than that in Shi (2009) and Menzio and Shi (2010), where the agents' value and policy functions depend only on the exogenous state of the economy (e.g., aggregate productivity). In these labor search models, block recursivity arises from the combination of directed search and free entry of job vacancies created by risk-neutral firms under constant returns.

We also investigate the extent to which greater credit availability affects housing prices when the sorting mechanism described above is at play.[4] When rental units can be converted into owner-occupied housing and the elasticity of new housing supply is low, the effects are substantial in our search model and are larger than its Walrasian counterpart. On the other hand, when the rental and real estate housing stocks are segmented, price effects are much larger in both models. This intuitive result is in line with the results in Greenwald and Guren (2021), who model the real estate market as Walrasian. Interestingly, under market segmentation, search frictions dampen the effect of the credit expansion on housing prices, which is now higher in the Walrasian economy. This happens for two reasons: these frictions act as bottlenecks that delay home purchases when demand is high, and they also convexify tenure choices, making homeownership less responsive to changes in credit (in comparison with the Walrasian economy).

Whether or not markets are segmented, the interaction between greater credit availability and search and matching frictions leads to more buyers borrowing in a larger amount. Our results are in line with papers reporting evidence on the expansion of mortgage debt during the boom across income levels, as Foote et al. (2020), or Han et al. (2021), who find evidence that changes in down payment requirements can lead to substantial price effects in hot segments of the housing market (and argue that search frictions and competition among traders are key to rationalize their findings). Price dispersion in our quantitative economy is one order of magnitude smaller than that estimated, for instance, by Lisi and Iacobini (2013). This is partly due to the fact that price dispersion in our model only reflects the buyers' heterogeneous wealth effects (i.e., home sellers do not face search frictions). Nonetheless, the results of our analysis shed light into the different channels that affect price dispersion when credit conditions are eased. On the one hand, there are more poor buyers at the lower end of the price distribution (with a higher mass of agents concentrated there), which

[4] It is well-known that search models constitute a powerful mechanism for demand shocks to affect aggregates (e.g. see Díaz and Jerez, 2013; Ngai and Tenreyro, 2014; Head et al., 2014; Hedlund, 2016b; Garriga and Hedlund, 2020; Anenberg and Bayer, 2020; Ngai and Sheedy, 2020; Han et al., 2021). Yet most of the literature assumes that households are risk neutral and ignores their savings decisions. The recent quantitative studies by Hedlund (2016b)), Garriga and Hedlund (2020), and Eerola and Maattanen (2018) are notable exceptions which feature related amplification mechanisms in models where real estate and rental markets are fully segmented.

compresses the distribution. On the other hand, wealthier buyers borrow more in order to target pricier homes and speed up their transactions, which makes the distribution more disperse. Ultimately, the overall effect on price dispersion is determined by the underlying earnings risk (which determines households' saving decisions) and by the degree of market segmentation. In particular, when rental and real estate markets are segmented, a credit relaxation reduces price dispersion. This result is in line with recent evidence reported by Kotova and Zhang (2020).

This paper also makes a technical contribution to the directed search literature. The model's tractable structure allows us to derive several properties of the households' value and policy functions. We first show that the value functions exist and are differentiable along the optimal paths. This suffices to obtain the Euler equations. These results are not trivial since the model has some features that make it impossible to apply standard techniques of dynamic programming. First, the decision problem of a potential buyer is not jointly concave in the choice and state variables and, secondly, the buyer's value function cannot be assumed to be differentiable a priori. We thus develop new analytical tools to study the properties of the value and policy functions. These tools are of independent interest, as they can be applied to general non-concave and non-differentiable dynamic models that involve both discrete and continuous choices. Menzio et al. (2013a) circumvent the technical difficulties arising from the non-concavity by introducing lotteries in a related monetary search model. This makes the model tractable, but obviously not equivalent to the original problem since the optimal policy functions differ. In this paper, we do not need to introduce lotteries but work directly within the non-concave framework. We show that the households' value functions are concave on the range of assets that corresponds to participation and non-participation in the search market, respectively, provided the optimal consumption policy of households who rent are monotone in financial wealth. This is the case in all our quantitative experiments. To the best of our knowledge, these results are novel and provide a new benchmark for analyzing similar block-recursive search models with an endogenous asset distribution without the need of introducing lotteries.

Our last contribution is computational. Equilibria in related models are typically computed by discretizing household choices and using value function iteration to solve the household's problem. This is the procedure used by Hedlund (2016b), Chaumont and Shi (2022), and Eeckhout and Sepahsalari (2020), for instance. By contrast, our theoretical results allow us to apply the Endogenous Grid Method to the Euler equations of the households' problems, so we do not need to resort to discretization. This is particularly important to measure the effects of credit liberalization on price dispersion. Additionally, this procedure yields substantial gains in accuracy and computational time.

The paper is organized as follows. In Section 2 we describe the environment and the problems solved by households and intermediaries, and define a stationary equilibrium. We also show that, under free entry, the model's block recursive structure follows from the intermediaries' optimization problem, and use this result to derive the properties of the households' value and policy functions and the sorting result. Section 4 discusses the calibration, and some key comparative-statics results. Section 5 concludes. Proofs and computational details are relegated to the Appendix.

2 The model economy

In this section we present our model economy and define a stationary equilibrium.

2.1 Household preferences and endowments

Consider a location populated by a continuum of infinitely-lived households. Time is discrete. Households derive utility from a nondurable numeraire good and the service flow provided by a durable good which we refer to as *housing*. Their lifetime utility is $\sum_{t=0}^{\infty} E_0 \beta^t u(c_t, h_t)$, where c_t, $h_t \in \mathbf{R}_+$ are the respective amounts of the nondurable good and housing services consumed each period, and β is the discount factor. The function u is strictly increasing, strictly concave and $\mathcal{C}^2$, with $u_{ch} \geq 0$ and $\lim_{h \to 0} u(c, h) = -\infty$.

Each period households are endowed with an amount z of efficiency units of labor, which follows a stationary Markov process, denoted by Π_z, with finite support Z. Households supply labor inelastically. The wage per efficiency unit of labor is exogenous and denoted by w.[5]

Households can either rent or own a (single) home in order to obtain housing services. The owner-occupied housing stock consists of indivisible units of identical size, $\hbar$. On the other hand, rental units come in a continuum of sizes: $h \in [0, \hbar]$. This assumption is introduced because renters typically live in smaller homes than owners, and also to avoid the possibility that rents exceed labor income for low productivity households.

Each period homeowners face i.i.d. preference shocks, and can be in two individual states, $\mu \in \{0, 1\}$. An owner consumes $\hbar > 0$ housing services if $\mu = 1$, in which case she is matched with her home. Otherwise, she is mismatched and obtains zero housing services. The state μ follows a Markov process with transition probabilities $P(\mu' = 1 | \mu = 1) = 1 - \pi_\mu \in (0, 1)$ and $P(\mu' = 0 | \mu = 0) = 1$. In words, a matched owner becomes mismatched with probability π_μ each period. Also, $\mu = 0$ is an absorbing state; so mismatched households will find profitable to sell their home and move.[6] Households who rent a unit of size h enjoy $\omega\, h$ services, where $\omega \leq 1$. Thus we allow for a taste for ownership.

Additionally, households may be hit by an idiosyncratic migration shock that depends on their housing tenure status. Owners are hit by a migration shock with probability ξ_o, in which case they become unproductive in town. To leave town, they then have to sell their homes. In turn, renters migrate with probability ξ_r. We can think of these shocks as capturing the effect of migration flows, as well as the effect of the life cycle on housing demand.[7] We assume that households who leave move to a symmetric town in an unspecified rest of the world at no cost, and are replaced by new immigrants who do not own any housing. The details on these entry flows are specified in Section 2.4.4. The constant measure of households in the

[5] Alternatively, we could assume that the numeraire is produced with labor according to the linear technology $Y = w\, N_c$, where N_c is labor demand.

[6] We assume that mismatched owners sell their home before they buy a new one to simplify the model. Anenberg and Bayer (2020), Ngai and Sheedy (2020), and Moen et al. (2021) explicitly model the joint decision to buy and sell in environments with transferable utility.

[7] Because state $\mu = 0$ is an absorbing state, in the absence of migration shocks all renters have previously owned houses, so they hold a house's liquid value. Although this is not important for our theoretical results, it does matter for the calibration of the model and its ability to match some data counterparts.

town is normalized to one.

2.2 Housing construction

Housing construction is undertaken by competitive developers, using the nondurable good and new land available for construction that is owned by the government. These developers pay the rental price of land to the government. We proceed as Kaplan et al. (2020) and assume that every period new housing is built according to the production function

$$I_h = B\,N^{\alpha}L^{1-\alpha}, \tag{2.1}$$

where N is employment in the construction sector and L is new developed land. For simplicity, we assume that $L = 1$ every period. This new land is owned by the government, who taxes away the profits that developers may have in equilibrium. All tax and land rent revenue is used to fund government spending that does not affect agents. As we explain below, the new housing is either bundled into indivisible units of size $\hbar$ (at no cost) or it can be sold in divisible amounts. A developer solves the static problem

$$\begin{aligned} &\max_{I_h,N} \quad \overline{p}\,I_h - w\,N \\ &\text{s. t.} \qquad I_h = B\,N^{\alpha}L^{1-\alpha}, \end{aligned} \tag{2.2}$$

where $\bar{p}$ is the per-unit price of housing that developers charge and w is the wage. The solution to this problem, assuming that $L = 1$, yields a supply function

$$I_h = \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} B^{\frac{1}{1-\alpha}}\,\overline{p}^{\frac{\alpha}{1-\alpha}}. \tag{2.3}$$

As in Sommer and Sullivan (2018) and Kaplan et al. (2020), the key parameter in this production function is α. This parameter determines the price elasticity of new housing supply, which is given by $\alpha/(1-\alpha)$. Housing depreciates at rate $\delta \in (0,1)$. All agents who own housing are required to maintain their property, and maintenance costs exactly offset depreciation.

2.3 Market arrangements and real estate intermediation

Financial market arrangements are as in Díaz and Luengo-Prado (2008). Each period households can save by investing in a one-period risk-free asset with price $1/R \in \mathbf{R}_+$. Their home purchases can be partially financed with a non-defaultable mortgage loan. Specifically, a household can borrow up to a fraction $1-\zeta$ of the home's liquidation value, so it must save to meet the corresponding down payment. The mortgage is a loan in perpetuity with no associated costs if there is early repayment. Houses also serve as collateral for loans: homeowners can obtain a home equity loan for up to a fraction $1-\zeta$ of the home's value (i.e., they can always remortgage). Alternatively, mortgages in this model can be thought of as home equity lines of credit that can be renegotiated every period although they are non-defaultable contracts. Since households who rent do not own any collateral, they cannot borrow (see also Kaplan et al., 2020). For simplicity, we assume that there is no spread between borrowing and lending rates.

A key assumption of the model is that housing transactions are intermediated by risk-neutral agents who may freely enter the town. Specifically, these intermediaries purchase homes from mismatched owners and from developers in a Walrasian market. Then, they decide whether to search for potential home buyers or to rent their homes to non-owners. We are aware that this kind of intermediation is not common in reality, mainly because the high involved transaction costs (e.g. taxes). In the real world, most real estate agents are match-makers (rather than dealers). Yet this assumption is crucial to generate a block-recursive structure (see also Hedlund, 2016a; Garriga and Hedlund, 2020). In our model, buyers and sellers with different financial wealth participate in the real estate market each period. If buyers and sellers—both of whom are risk averse—were to trade directly with each other in the search market, the model would fail to be block recursive and would become intractable.

Intermediaries are infinitely-lived with discount factor $1/R$ and have deep pockets, so they do not require credit to finance their purchases. We assume that these agents always purchase housing units (bundles) of size $\hbar$ in the Walrasian market at price $\bar{p}$. They then decide whether to sell or rent these units. Recall that, whereas properties for sale are indivisible, rental units are divisible. The rental market is competitive, and the (per-unit) rental price is denoted by r_h.

It could be argued that the assumption that owners sell their homes in a Walrasian market makes the housing market very liquid. For instance, in Garriga and Hedlund (2020), owners who want to sell participate in a frictional market (which is also intermediated by risk neutral agents). However, this does not necessarily imply that owner-occupied housing is more liquid in our model, for two reasons. First, in Garriga and Hedlund (2020), owners have the option of defaulting on their mortgage (and being banned from the housing market for a stochastic number of periods), in which case their home is immediately liquidated by the bank in a Walrasian market. Second, whereas in our model owning does not entail a default risk, matched owners are not allowed to sell their homes. Thus they can not change their tenure status to smooth earnings risk. In other words, owning is risky in both models.

There are indirect taxes on real estate transactions. Households who sell pay taxes on the value of their home at the rate τ_s, whereas the buyers' tax rate is τ_b. Intermediaries do not pay taxes. Below we specify the timing of the model, and describe the market structure in detail. Each period is divided into three subperiods: *morning*, *afternoon*, and *night*.

2.3.1 Morning

At the end of period $t-1$, there are two types of households, depending on their tenure status: *owners* and *renters*. At the start of period t, the housing stock depreciates, and preference, migration and labor endowment shocks are realized. Then the Walrasian market opens. Supply in this market includes new construction and the depreciated homes of mismatched owners and intermediaries. As for demand, new intermediaries can freely enter the town to purchase housing bundles of size $\hbar$ at the market clearing price, $\overline{p}$. Also, in order to maintain their home, all agents who own housing must purchase the depreciated part of their homes in the Walrasian market.

2.3.2 Afternoon

During the afternoon, those households who sold their home in the morning and did not migrate, those who were renters in the previous period, and the newly arrived immigrants may search for a home to buy. We refer to these households as *potential buyers*. Matched owners make no economic decisions in this subperiod, so we refer to them as *non-traders*. Intermediaries decide whether to put their units up for sale during the afternoon or wait until the night when they can rent them to non-owners. A competitive search market operates in the afternoon, where intermediaries who seek to sell supply their indivisible units and buyers may search for a home at a negligible participation cost[8]. Intermediaries who are not able to sell their units in this market are not allowed to rent at night. Potential buyers may choose not to participate in the search market (e.g. if they have not accumulated enough assets to meet the corresponding down payment).

Purchasing a home may require borrowing subject to a collateral constraint. Specifically, buyers may borrow up to a fraction $1-\zeta$ of the home's value in the Walrasian market; i.e., their borrowing limit is $(1-\zeta)\,\overline{p}\,\hbar$. The implicit assumption (as in Kiyotaki and Moore, 1997) is that banks lend the amount they can recover in this market if they seized the house.

The competitive search process can be described as follows. As in Moen (1997), buyers and intermediaries can participate in different submarkets where they meet bilaterally and at random, and where each trader experiences at most one bilateral match. The matching probabilities in a given submarket depend on the associated buyer-seller ratio θ (or tightness). Specifically, an intermediary is matched to a buyer with probability $m_s(\theta)$, and a buyer is matched to an intermediary with probability $m_b(\theta) = m_s(\theta)/\theta$.[9] As is standard, $m_s(\theta)$ is strictly increasing, strictly concave and $\mathcal{C}^2$, with $m_s(0) = 0$ and $\lim_{\theta\to\infty} m_s(\theta) = 1$, and $m_b(\theta)$ is strictly decreasing and $\mathcal{C}^2$, with $\lim_{\theta\to 0} m_b(\theta) = 1$ and $\lim_{\theta\to\infty} m_b(\theta) = 0$. In words, the higher the buyer-seller ratio θ, the easier it is for intermediaries to contact buyers, and the harder it is for buyers to locate a home for sale (due to congestion externalities). As

[8]This rules out equilibria where some households participate in the frictional market (because doing so is costless) even though they do not plan to trade there.

[9]The underlying assumption is that the total number of bilateral trading meetings is determined by a matching function with constant returns to scale and that the Law of Large Numbers holds.

θ goes to infinity (zero) the intermediary's matching probability goes to one (zero), and the buyer's matching probability goes to zero (one). The elasticity $\eta(\theta) \equiv \frac{m_s'(\theta)\theta}{m_s(\theta)} \in [0,1]$ is assumed non increasing, and $\hat{m}_s(m_b) \equiv m_s(m_b^{-1}(\cdot))$ is such that $\ln \hat{m}_s$ is concave.[10] To model market participation, we introduce a "fictitious submarket" $\theta_0 \in \mathbf{R}_-$, and extend the functions m_b and m_s to $\Theta \equiv \mathbf{R}_+ \cup \{\theta_0\}$ by setting $m_b(\theta_0) = m_s(\theta_0) = 0$. Households who choose submarket θ_0 do not participate in the afternoon market.

To describe the price determination process in the competitive search market, we follow the price-taking approach in Jerez (2014). The idea is to think of houses traded in submarkets with different tightness levels $\theta \in \mathbf{R}_+$ as different commodities, which are characterized by different degrees of trading uncertainty. The prices of these differentiated commodities are described by a continuous function $p : \Theta \to \mathbf{R}_+$, with $p(\theta_0) = 0$. That is, $p(\theta)$ is the price per unit of space in a submarket with tightness $\theta \in \mathbf{R}_+$. Buyers and intermediaries choose the submarkets they enter taking $p(\theta)$ as given. The difference with the standard Walrasian equilibrium notion is that, in these submarkets, demand does not equal supply (as agents on both sides of the market face a positive rationing probability). The market clearing condition is then replaced by an aggregate consistency condition which requires that, given the agents' optimal decisions, the *equilibrium* buyer-seller ratio in submarket θ is indeed θ whenever this submarket attracts both buyers and intermediaries (see Section 2.4.4).

As shown in Jerez (2014), our equilibrium notion is equivalent to that of directed search. With directed search, each intermediary first posts (and commits to) to price offer p. Then, buyers seek the most attractive offers. In making these strategic decisions, all traders form common beliefs about the buyer-seller ratio $\theta(p)$ associated to each offer p (i.e., the mass of buyers seeking offer p over the mass of intermediaries posting p). In equilibrium, beliefs are rational. To see the connection with our price-taking equilibrium notion, think of a submarket θ as a market segment that is associated to a particular price offer p. Our equilibrium price functional $p(\theta)$ is the inverse of the schedule $\theta(p)$ describing the agents'

[10]Equivalently, $-\hat{m}_s{}'(m_b)/\hat{m}_s(m_b)$ is non decreasing. This assumption guarantees that the problem solved by potential buyers is concave and has a unique solution (see Sections B-D in the Appendix), and can be further relaxed (see Section E.1). See also Menzio and Shi (2010) where $\hat{m}_s$ is assumed concave (a slightly stronger assumption).

beliefs in a directed search equilibrium. In turn, our aggregate consistency condition is the equivalent of the corresponding rational expectations condition. As we shall see, in equilibrium $p(\theta)$ is decreasing, so prices are lower in more congested submarkets. This is equivalent to saying that lower price offers attract relatively more buyers under directed search. We choose the price-taking formulation because it makes the connection with the standard notion of recursive competitive equilibrium more direct and transparent.

2.3.3 Night

Households who bought a home in the afternoon are *owners* at night, just as the nontraders. The rest of the households are *renters*. In this subperiod, households receive their labor income, $w\,z$, and choose their nondurable consumption and the level of assets to be carried to the next period. For simplicity, we assume that the payments corresponding to home maintenance (purchased in the Walrasian morning market) are made at night.

2.4 Stationary equilibrium

In this section, we state the problems of the agents in the afternoon and night subperiods given the Walrasian price $\overline{p}$, the price schedule $p(\theta)$, and the rental price r_h (starting at night and going backwards).[11] By free entry, intermediaries make zero profits in each segment of the search market. This implies that the equilibrium price schedule—an infinite-dimensional object—is pinned down by the value of $\overline{p}$. On the other hand, equilibrium rents are determined as a function of $\bar{p}$ by an arbitrage condition that ensures that intermediaries who choose to rent also make zero profits. In sum, all the price information regarding the real estate and rental markets is summarized by $\bar{p}$. Hence, the problems solved by individual households do not depend directly on the distribution of households (over financial assets, income levels and tenure states). This distribution only affects the households' decisions through its effect on $\overline{p}$. At the end of the section, we state the law of motion of the distri-

[11] Recall that, during the morning, owners hit by a shock sell their home at price $\overline{p}$ to the intermediaries that enter the Walrasian market, while the rest of the households are inactive.

bution of households and that of the vacancy stock held by intermediaries as a function of the agents' optimal decisions. A stationary equilibrium is then defined.

2.4.1 Night value functions

Let $A = [\underline{a}, \infty)$ be the set in which the households' financial assets can take values, and denote the household's assets at the start of the night by $a \in A$. The set of individual states is then $X = A \times Z$. The afternoon value functions of potential buyers and non-traders are $W_b : X \to \mathbf{R}$ and $W_n : X \to \mathbf{R}$, respectively. The night value function of an *owner* is given by

$$\begin{aligned}
W_o(a, z) = \quad & \max_{c, a', \widetilde{a}} \left\{ u\left(c, \hbar\right) + \beta\, (1 - \pi)\, E_z\, W_n\left(a', z'\right) + \beta\, \pi\, E_z\, W_b\left(\widetilde{a}, z'\right) \right\} \\
\text{s.t.} \quad & c + \tfrac{1}{R}\, a' \leq w\, z + a - \delta\, \overline{p}\, \hbar, \\
& \widetilde{a} = a' + (1 - \tau_s)\, \overline{p}\, \hbar, \\
& a' \geq -(1 - \zeta)\, \overline{p}\, \hbar, \\
& c \geq 0,
\end{aligned} \tag{2.4}$$

where c and $\hbar$ are the amounts of the nondurable good and housing services consumed, and a' is the level of financial assets carried to the next period. Owners choose the values of c and a' to maximize their expected lifetime utility subject to a standard intertemporal budget constraint and also face a borrowing limit equal to $(1 - \zeta)\, \overline{p}\, \hbar$. As mentioned above, they can remortgage their home, in which case the price of reappraisal is the value of their home in the Walrasian morning market. Also, owners pay the maintenance cost $\delta\, \overline{p}\, \hbar$; i.e., they purchase the depreciated part of their bundle in the Walrasian market. Owners will sell their home at the start of the next period with probability $\pi = \xi_o + (1 - \xi_o)\, \pi_\mu$, which is probability of being hit by mismatch or migration shocks. These homes are sold at price $\overline{p}\, \hbar$ in the Walrasian market, and the transaction is subject to the corresponding indirect taxes. Note that the assumption that agents hit by a migration shock move to a symmetric location at no cost implies that the owners' continuation value is the same regardless of the kind of shock that hits them. Owners hit by the mismatch shock will be potential buyers in their current location, whereas owners hit by the migration shock will be potential buyers elsewhere. In both cases, their continuation value is $E_z\, W_b\,(\widetilde{a}, z')$, where $\widetilde{a} = a' + (1-\tau_s)\,\overline{p}\,\hbar$ is the household's financial wealth after the home is sold. We denote the owners' optimal decision policies by $g_o^c(a,z)$ and $g_o^a(a,z)$.

The night value function of a *renter* is defined in a similar way:

$$\begin{aligned} W_r(a,z) = \quad & \max_{c,h,a'} \Big\{ u\,(c, \omega\, h) + \beta\, E_z\, W_b\,(a', z') \Big\} \\ \text{s.t.} \quad & c + \tfrac{1}{R}\, a' \leq w\, z - r_h\, h + a, \\ & a' \geq 0,\ c \geq 0,\ 0 \leq h \leq \hbar, \end{aligned} \tag{2.5}$$

and $g_r^c(a,z)$, $g_r^h(a,z)$, and $g_r^a(a,z)$ denote the optimal policies. Differently from owners, renters choose their home size h, and are not allowed to borrow. While they face a migration shock, they do not change financial status when they migrate (and recall that moving does not entail any cost). Hence, renters only face labor uncertainty.

The value of an intermediary who rents at night is

$$J_r = -\kappa + r_h\,\hbar - \delta\,\overline{p}\,\hbar + \frac{1}{R}\,J. \tag{2.6}$$

Recall that these rental companies hold $\hbar$ units of housing which (differently from the units sold in the afternoon real estate market) are divisible. They pay the cost of posting their vacancy in the night rental market, κ, as well as the maintenance of their property, $\delta\,\overline{p}\,\hbar$. In the next period, they will decide whether to rent their property again at night or sell it in the morning or afternoon markets. That is, their continuation value is

$$J = \max\Big\{J_r, J_s, \overline{p}\,\hbar\Big\}, \tag{2.7}$$

where J_s is the value from participating in the afternoon market.

2.4.2 Afternoon value functions

Let $a \in A$ be the household's financial assets at noon. *Non-traders* are inactive during the afternoon, so their value function is given by

$$W_n(a, z) = W_o(a, z). \tag{2.8}$$

Potential buyers choose the submarkets they join taking as given the price schedule, $p(\theta)$, and the maximum loan they can obtain, $(1-\zeta)\,\overline{p}\,\hbar$. Their value function is given by

$$\begin{aligned} W_b(a, z) = \quad & \max_{\theta \in \Theta} \left\{ m_b(\theta)\ W_o\left(a - (1+\tau_b)\, p(\theta)\, \hbar, z\right) + \left(1 - m_b(\theta)\right)\ W_r(a, z) \right\} \\ \text{s. t.} \quad & a + (1-\zeta)\,\overline{p}\,\hbar \geq (1+\tau_b)\ p(\theta)\,\hbar \ \text{ if } \theta \in \mathbf{R}_+, \end{aligned} \tag{2.9}$$

and $g_b^\theta(a, z)$ denotes their optimal decision rule. The collateralized borrowing constraint in problem (2.9) ensures that buyers who join submarket $\theta \in \mathbf{R}_+$ have enough assets to pay for the corresponding down payment and the associated taxes. The maximum amount of credit a buyer gets is $(1-\zeta)\,\overline{p}\,\hbar$. Based on Kiyotaki and Moore (1997), we assume that financial intermediaries lend according to the liquidation value of the house, which is $\overline{p}\,\hbar$. With probability $m_b(\theta)$, these households buy a home at price $p(\theta)$ per unit of space, and enter the night with financial assets $a - (1+\tau_b)\, p(\theta)\,\hbar$.[12] With complementary probability, they do not trade and carry their assets a into the night, when they will be renters (just as those potential buyers who choose not to participate in the afternoon market).

Similarly, *realtors* choose the submarkets they join in order to maximize their expected lifetime value:

$$J_s = \max_{\theta \in \mathbf{R}_+} \left\{ m_s(\theta)\ p(\theta)\,\hbar + \left(1 - m_s(\theta)\right) \left(\frac{1}{R}\, J - \delta\,\overline{p}\,\hbar \right) \right\}. \tag{2.10}$$

Realtors who join submarket $\theta \in \mathbf{R}_+$ sell their bundle $\hbar$ with probability $m_s(\theta)$ and earn revenue $p(\theta)\,\hbar$, in which case they leave town. With complementary probability, they do not

[12] Households with a mortgage have negative assets at night and pay interests on that debt, as implied by the intertemporal budget constraint in (2.4).

trade. They must then pay the maintenance costs of their property and wait until the next period to decide (as the rental companies) whether to put their property up for sale in the morning or afternoon markets or rent it at night.

2.4.3 Block-recursivity and the afternoon price schedule

Note that intermediaries are indifferent between the three markets. Moreover, by free entry, their expected profits are zero, so their expected value in each market equals the price they pay for their dwellings in the Walrasian morning market:

$$J = J_r = J_s = \overline{p}\,\hbar. \tag{2.11}$$

Equation (2.6) and the zero-profit condition (2.11) pin down the equilibrium rental price as a function of the Walrasian morning price:

$$r_\hbar = \frac{\kappa}{\hbar} + (1 - 1/R + \delta)\,\overline{p}. \tag{2.12}$$

Combining (2.10) and (2.11) yields:

$$p(\theta) \leq \frac{(1 - 1/R + \delta)\,\overline{p}}{m_s(\theta)} + (1/R - \delta)\,\overline{p}, \text{ for all } \theta \in \mathbf{R}_+, \tag{2.13}$$

with strict equality if θ solves (2.10). In active submarkets, $p(\theta)$ is then given by the right-hand side of (2.13). In particular, $p(\theta)$ decreases with θ. Intuitively, since intermediaries make zero expected profits in all active submarkets, prices are lower in submarkets where the probability of completing a sale, $m_s(\theta)$, is higher. Prices in inactive submarkets instead imply weakly lower expected profits.

In fact, there is no loss of generality in assuming that intermediaries make zero expected profits in all submarkets, whether active or not. A standard feature of general equilibrium models with a continuum of commodities is that prices in inactive markets are indeterminate. Assuming that (2.13) holds with equality for all $\theta \in \mathbf{R}_+$ is equivalent to selecting the highest prices that support the equilibrium allocation. This price selection rule is equivalent to the restriction typically imposed on out-of-equilibrium beliefs in directed search models, known as the market utility property (see Jerez, 2014). With this selection rule, $p(\theta)$ is pinned down by $\overline{p}$. Given $\overline{p}$, households know the price schedule $p(\theta)$. As shown in Figure 1, $p(\theta)$ is strictly convex and $\mathcal{C}^2$ (since m_s is strictly concave and $\mathcal{C}^2$). It is also bounded below by $\overline{p}$. This lower bound is the price intermediaries would charge if the probability of completing a sale was one (to break even). Since trade is subject to rationing, no intermediary would trade at a price $p \leq \overline{p}$.

2.4.4 Stationary equilibrium definition

Before defining the equilibrium, we describe the law of motion of the distribution of households and the stock of vacancies held by realtors.

Let $\mathcal{X}$ denote the Borel σ-algebra on X. The distribution of non-traders and potential buyers across individual states at noon is described by the Borel measures, ψ_n and ψ_b, respectively. Likewise, ψ_o and ψ_r represent the distributions of owners and renters at night. Since the mass of households in town is one,

$$\int_{x\in X} d\psi_n + \int_{x\in X} d\psi_b = \int_{x\in X} d\psi_o + \int_{x\in X} d\psi_r = 1. \tag{2.14}$$

Define the transition function $Q_o : X \times \mathcal{X} \to [0,1]$ which gives the probability that an agent with state $x \in X$ who owns at night will be in state $x' \in X' \in \mathcal{X}$ in the next morning. Likewise, Q_r represents the transition function for renters. We use primes to denote the corresponding measures in the next period.

The laws of motions from the night to the following afternoon are

$$\psi_n'(X') = (1-\pi_\mu)(1-\xi_o)\int_{x\in X} Q_o(x,X')\,d\psi_o, \tag{2.15}$$

$$\begin{aligned}\psi_b'(X') = \;& (1-\xi_r)\int_{x\in X} Q_r(x,X')\,d\psi_r + \\ & \pi_\mu(1-\xi_o)\int_{x\in X} Q_o(x,X')\,d\psi_o + \psi_i(X'),\end{aligned} \tag{2.16}$$

for each $X' \in \mathcal{X}$. In (2.16), ψ_i is a measure on X representing the exogenous distribution of immigrants, which ensures that net migration flows are zero. Recall that owners who do not migrate and remain matched at the start of t are non-traders in the afternoon. This event has probability $(1-\pi_\mu)(1-\xi_o)$. On the other hand, renters who do not migrate, owners who do not migrate and become mismatched, and immigrants are potential buyers in the afternoon.

The laws of motion from the afternoon to the night are

$$\psi_o'(X') = \psi_n(X') + \int_{x\in X} \Pi_o(x, X')\, d\psi_b, \tag{2.17}$$

$$\psi_r'(X') = \int_{x\in X} \Pi_r(x, X')\, d\psi_b, \tag{2.18}$$

where the transition functions $\Pi_o : X \times \mathcal{X} \to [0,1]$ and $\Pi_r : X \times \mathcal{X} \to [0,1]$ give the probability that a potential buyer with state x at the start of the afternoon will be an owner or a renter with state in X' at night, respectively. These probabilities are related to the probability that the buyer purchases a home in the afternoon, which depends on the submarket θ she joins. A successful trade implies, not only a change in tenure status, but also a change in the financial assets (which again depends on θ). Specifically,

$$\Pi_o((a,z), X') = \begin{cases} m_b\left(g_b^\theta(a,z)\right), \text{ if } \left(a - (1+\tau_b)\, p\left(g_b^\theta(a,z)\right) \hbar, z\right) \in X', \\ 0, \text{ otherwise}, \end{cases} \tag{2.19}$$

$$\Pi_r((a,z), X') = \begin{cases} 1 - m_b\left(g_b^\theta(a,z)\right), \text{ if } (a,z) \in X', \\ 0, \text{ otherwise}. \end{cases} \tag{2.20}$$

Let H_o be the amount of housing owned by households at night, that is,

$$H_o = \hbar \int_X d\,\psi_o. \tag{2.21}$$

Let H_r be the supply of rental properties. The market clearing condition in the night rental market is

$$H_r = \int_X g_r^h(x)\, d\,\psi_r. \tag{2.22}$$

Finally, let V denote the amount of vacancies that realtors hold overnight. These are the units that remain unsold in the afternoon market. Recall that these units cannot be rented at night, and will join the pool of housing that can be traded in the Walrasian market in the next morning. Hence, $H_o + H_r + V$ is the total housing stock at night.

It is easy to show that, at the stationary equilibrium, the market clearing condition in the morning market can be written as

$$\delta\ (H_o + H_r + V) - I_h = 0, \tag{2.23}$$

where the left-hand side of (2.23) represents aggregate excess demand in this market.[13] In words, in equilibrium, the production of housing must equal the depreciation of the stock.

To pin down the equilibrium value of V, we use the consistency condition in the competitive search market. Let $\tilde{X} \subseteq X$ denote the set of states of potential buyers who participate in this market. That is, $x \in \tilde{X}$ if and only if $g_b^\theta(x) \neq \theta_0$. We can construct a measure ψ^s on $\tilde{X}$ such that

$$\psi^s(\Xi) = \int_\Xi \frac{1}{g_b^\theta(x)}\, d\,\psi_b, \tag{2.24}$$

for each Ξ in the Borel σ-algebra $\tilde{\mathcal{X}}$ defined on $\tilde{X}$. Recall that the consistency condition implies that $g_b^\theta(x)$ is the equilibrium buyer-seller ratio in the submarket where buyers with state x participate.[14] Hence, there ought to be $1/g_b^\theta(x)$ intermediaries per buyer there. Since

[13]Supply includes new construction and the depreciated homes of mismatched owners, rental companies and realtors. That is, supply is given by $I_h + (1-\delta)[(H_o - H_n) + H_r + V]$, where H_n denotes the number of owners who remain matched each morning. In turn, demand includes home maintenance by matched owners and home purchases by new intermediaries (which include the associated maintenance payments). Demand then equals $\delta H_n + H_r + \tilde{V}$, where $\tilde{V}$ denotes the number of homes for sale in the search market. Since $\tilde{V}$ equals sales in this market plus overnight vacancies, and sales equal $H_o - H_n$, it follows that $\tilde{V} = V + H_o - H_n$. Aggregate excess demand is then $\delta(H_o + H_r + V) - I_h$.

[14]In section 2.5.2, we show that all buyers with the same state x will participate in the same submarket in equilibrium.

$d\psi_b(x)$ is the density of buyers with state x, the number of intermediaries in this submarket must then be $\frac{1}{g_b^\theta(x)}\, d\psi_b(x)$. Therefore, for the consistency condition to hold, the number of intermediaries who are randomly matched to buyers with state $x \in \Xi$ is $\psi^s(\Xi)$ for each Ξ. The total number of intermediaries who do not trade in the afternoon is then

$$V = \int_{\tilde{X}} \left(1 - m_s\left(g^\theta(x)\right)\right)\, d\,\psi^s. \tag{2.25}$$

We are now ready to define a stationary equilibrium.

Definition 1. *A recursive stationary equilibrium for this economy, given the interest factor, R, the wage w, and the distribution of the immigrants, ψ_i, is a list of value functions and optimal decision policies for the households $\left\{W_o, W_r, W_n, W_b, g_o^c, g_o^a, g_r^c, g_r^h, g_r^a, g_b^\theta\right\}$, value functions for intermediaries, $\left\{J, J_s, J_r\right\}$, prices $(\overline{p}, p(\cdot), r_h)$, Borel measures $\left\{\psi_o, \psi_r, \psi_n, \psi_b, \psi^s\right\}$, and a tuple (I_h, V, H_o, H_r) such that:*

1. *$\left\{W_o, W_r, W_n, W_b, g_o^c, g_o^a, g_r^c, g_r^h, g_r^a, g_b^\theta\right\}$ solve the households' problems shown in* (2.4)–(2.9)*, given $(\overline{p}, p(\cdot), r_h)$.*

2. *The supply of new housing is given by* (2.3)*.*

3. *Realtors make zero expected profits in all submarkets, and rental companies make zero profits:* (2.10) *holds with equality for all $\theta \in \mathbf{R}_+$, and* (2.12) *holds.*

4. *The night rental market and the morning housing market clear, and the consistency condition is satisfied in the afternoon search market:* (2.22), (2.23), *and* (2.24) *hold.*

5. *The stationary probability measures $\left\{\psi_o, \psi_r, \psi_n, \psi_b\right\}$ satisfy* (2.15)–(2.18)*, and the overnight vacancy stock is stationary, so* (2.25) *holds.*

The non-standard condition in Definition 1 is the consistency condition in the afternoon market. By condition 3, realtors are indifferent between all submarkets $\theta \in \mathbf{R}_+$. Equation (2.24) in condition 4 says that the distribution of realtors across active submarkets is such that the *actual* buyer-seller ratios in these submarkets are equal to the ratios (or tightness levels) that households take as given when they make their optimal afternoon decisions. In turn, given this distribution, (2.23) and (2.25) ensure that the realtors' overnight vacancy stock equals their vacancy stock at the start of each period.

2.5 Some properties of the stationary equilibrium

Here we discuss some properties of the stationary equilibrium.

2.5.1 Properties of the value functions

The block-recursive structure of the model allows us to derive several properties of the households' value functions, which in turn support the characterization and computation of their policy functions. These derivations involve two main difficulties: (i) the decision problem of a potential buyer's is not concave, and (ii) the buyer's value function, W_b, cannot be assumed to be differentiable a priori. There are two sources of non-concavity in problem (2.9): the discrete decision of the buyer whether to participate or not, and the objective function not being jointly concave in the choice and state variables. The latter feature is due to the dependence of the matching probability on the market tightness, a variable that also affects the surplus of trade in the afternoon market. The product of these two terms is not concave in general, preventing the use of standard dynamic programming techniques, which start from the assumption that the objective function is jointly concave in the choice and state variables. We thus develop new analytical tools to study the properties of the value and policy functions. Appendixes A and B describe these tools, which are of independent interest, as they can be applied to general non-concave and non-differentiable dynamic models that involve both discrete and continuous choices.

In Appendix A we show that, given the price schedule in (2.13), the dynamic programming problems (2.4), (2.5), (2.8) and (2.9) admit continuous solutions W_o, W_r, W_n, and W_b, which are unique in a suitable class of functions (under quite general conditions). Also, W_o, W_r, and W_n are strictly increasing and W_b is non-decreasing. Whereas these functions need not be differentiable and concave in general, in Appendix B we show that they are differentiable along the optimal paths. This is all we need to establish the sufficiency of the Euler equations. Moreover, if we restrict to the range of assets of the households who participate in the afternoon market, W_o, W_r, and W_n are strictly concave and W_b is concave, provided the renters' consumption policy function $g_r^c(a, z)$ is non-decreasing on this range.[15] This implies that the household's optimal choices are unique.

2.5.2 Sorting and participation in the competitive search market

In this section, we exploit these results to characterize the equilibrium sorting pattern and establish the existence of a participation threshold asset level, $a_{part}(z)$, for each productivity state z. The proof of these results can be found in Appendix C.

The optimal decision rule of a buyer who participates in the afternoon market is

$$\begin{aligned} g_b^\theta(a, z) \in \quad & \underset{\theta \in \mathbf{R}_+}{\arg\max} \left\{ W_r(a, z) + m_b(\theta) \left[W_o(a - (1 + \tau_b) p(\theta) \hbar, z) - W_r(a, z) \right] \right\} \\ \text{s. t.} \quad & a + (1 - \zeta) \overline{p} \hbar \geq (1 + \tau_b) \, p(\theta) \hbar. \end{aligned} \tag{2.26}$$

For buyers with state (a, z), the ex-post gains from trading at price p are

$$S(a, z, p) = W_o(a - (1 + \tau_b) p \hbar, z) - W_r(a, z). \tag{2.27}$$

Hence, $g_b^\theta(a, z)$ maximizes the buyer's (ex-ante) expected gains, $m_b(\theta) S(a, z, p(\theta))$. These maximal expected gains are non-negative, since θ_0 is a feasible choice for all buyers. Figure 1 depicts the buyers' indifference curves on the space (θ, p) as a function of their state (a, z). Buyers prefer submarkets with low prices (which yield higher ex-post gains) and low congestion (which imply a higher trading probability). In the case of a buyer with state (a, z), an indifference curve is given by $m_b(\theta) S(a, z, p) = \overline{S}_{az}$ for some fixed value $\overline{S}_{az} \geq 0$. Thus $g_b^\theta(a, z)$ attains the highest value of $\overline{S}_{az}$ along the price schedule $p(\theta)$, subject to the borrowing constraint. To illustrate the role of financial wealth, Figure 1 depicts the optimal

[15] In particular, due to the endogenous participation decision, W_b is not concave on A, but it is concave on the range of assets that correspond to participation (those $a \in A$ with $W_b(a, z) > W_r(a, z)$).

choices of three buyers with identical labor productivity z and different financial assets. When the borrowing constraint does not bind, the buyer's indifference curve is tangent to the schedule $p(\theta)$. This is the case for buyers with assets a_1 or a_2, in the figure. Since the schedule $p(\theta)$ corresponds to the realtors' zero isoprofit curve on the space (θ, p), the indifference curve of an unconstrained buyer is tangent to this isoprofit curve. This is the standard characterization of a competitive search equilibrium in the absence of borrowing constraints (e.g. Moen, 1997; Acemoglu and Shimer, 1999). If the constraint binds, this tangency point is not feasible. This is the case of a buyer with lower assets, a_3. Constrained buyers join the submarket where homes are sold at the maximum price they can afford to pay given their financial wealth, the taxes involved in the transaction, and the borrowing limit.

Since $W_o(a, z)$ is differentiable with respect to a, so is the buyer's objective function. The first-order condition for problem (2.26) is

$$\frac{m_b'(\theta)\, S(a, z, p(\theta))}{\hbar\,(1+\tau_b)} - m_b(\theta)\, W_o'\,(a - (1+\tau_b)\, p(\theta)\, \hbar, z)\; p'(\theta) = \lambda(a, z)\, p'(\theta), \tag{2.28}$$

where W_o' is the derivative of W_o with respect to its first argument, and $\lambda(a, z)$ is the Lagrange multiplier of the constraint. If the constraint is slack, (2.28) simplifies to

$$\left(\frac{1}{1+\tau_b}\right)\left(\frac{1-\eta(\theta)}{\hbar\,\theta}\right)\left(\frac{S(a, z, p(\theta))}{W_o'\,(a - (1+\tau_b)\, p(\theta)\, \hbar, z)}\right) = -p'(\theta), \tag{2.29}$$

where $\eta(\theta)$ is the elasticity of $m_s(\theta)$. Equation (2.29) describes the tangency between the buyer's indifference curve and the price schedule. In particular, the left-hand side of (2.29) represents the buyer's marginal rate of substitution of θ for p. The last term in this expression gives the buyer's ex-post gains measured in units of consumption (rather than in utils):

$$\hat{S}(a, z, p) = \left(\frac{S(a, z, p)}{W_o'\,(a - (1+\tau_b)\, p\, \hbar, z)}\right), \tag{2.30}$$

since W_o' is the marginal utility of wealth of an owner at night. This term will be key for our sorting result, as it determines how the rate at which buyers trade off prices and congestion varies with their financial wealth.

Using the expression of the equilibrium price schedule in (2.13), the tangency condition (2.29) can be written as

$$\left(\frac{1}{1+\tau_b}\right)\hat{S}(a,z,p(\theta)) = \frac{\eta(\theta)}{1-\eta(\theta)}\left(p(\theta)\hbar - \overline{p}\hbar\left(\frac{1}{R}-\delta\right)\right), \tag{2.31}$$

where $p(\theta)\hbar - \overline{p}\hbar(1/R-\delta)$ are the realtor's ex-post gains in submarket θ. In the absence of taxation ($\tau_b = 0$), (2.31) generalizes the well-known Hosios (1990) condition for transferable-utility environments to our setting, where utility is imperfectly transferable. It says that a fraction $\eta(\theta)$ of the bilateral surplus is appropriated by the buyer and the rest goes to the realtor.

If the borrowing constraint binds, $g_b^\theta(a,z)$ satisfies

$$p\left(g_b^\theta(a,z)\right) = \frac{a+(1-\zeta)\,\overline{p}\,\hbar}{(1+\tau_b)\,\hbar} > \overline{p}. \tag{2.32}$$

Recall that the prices buyers pay exceed $\overline{p}$ (otherwise, realtors trading at these prices would make negative profits). Constrained buyers start the night with a negative asset position equal to $-(1-\zeta)\,\overline{p}\,\hbar$. As one would expect, for a given z, the multiplier $\lambda(a,z)$ decreases with a (see Lemma D.1 in the Appendix). There are then three possible cases. Either all buyers with productivity z are unconstrained, they are all constrained, or the constraint only binds below a threshold that depends on z.

Proposition 1 provides conditions under which the buyer's optimal choice is unique, so buyers in the same state join the same submarket in equilibrium. This is always the case for constrained buyers, whose unique optimal choice is characterized by (2.32). In turn, the problem of an unconstrained buyer has a unique solution provided $g_r^c(a,z)$ is non decreasing in a on the range of assets that correspond to participation. This guarantees that W_o is strictly concave with respect to a on this range, which implies that there is a single tangency point between the buyer's indifference curve and the schedule $p(\theta)$.[16]

[16] Since $\eta(\theta)$ is non-increasing, one cannot conclude from (2.29) that the buyer's marginal rate of substitution increases along an indifference curve as θ rises (as depicted in Figure 1). In the Appendix, we circumvent

Proposition 1. *A solution for problem* (2.26) *exists. Suppose that, for each* $z \in Z$, $g_r^c(a,z)$ *is non decreasing in* a *on the range of assets for which* $\theta_0 \notin g_b^\theta(a,z)$. *Then* $g_b^\theta(a,z)$ *is single-valued on this range.*

We now turn to the sorting result. In the case of constrained buyers, the result follows trivially from (2.32). These buyers pay the maximum price they can afford to pay, and this price increases with a (and does not depend on z). In other words, constrained buyers who are wealthier trade in less congested submarkets, where prices are higher.

Proposition 2. *For constrained buyers,* $g_b^\theta(a,z)$ *does not depend on* z, *and* $g_b^\theta(a,z) > g_b^\theta(a',z)$ *if* $a < a'$.

In the case of unconstrained buyers, prices depend on both a and z, and a similar sorting result holds provided wealthier buyers have steeper indifference curves than poorer buyers with identical productivity z. Under this single-crossing property, the former are willing to accept a larger price increase in order to increase their trading probability (while remaining indifferent). As depicted in Figure 1, for a given z, buyers who are wealthier choose lower values of θ, and pay higher prices. As noted above, the buyer's marginal rate of substitution at a given (θ, p) is proportional to $\hat{S}(a,z,p)$. Hence, the single crossing property holds when $\hat{S}(a,z,p)$ increases with a.

Proposition 3. *Suppose that the condition in Proposition 1 holds and* $S(a,z,p)$ *is non decreasing in* a *for each* $p \geq \overline{p}$ *and each* $z \in Z$. *Then* $g_b^\theta(a,z) > g_b^\theta(a',z)$ *if* $a < a'$.

The result in Proposition 1 is intuitive. If W_o is strictly concave in a, wealthier owners have lower marginal utilities of wealth at night. Hence, as long as the buyers' ex-post gains do not decrease with financial wealth, the gains measured in units of consumption, $\hat{S}(a,z,p)$, are higher for wealthier buyers (see (2.30)). Since these are the buyers who gain more when a transaction is completed, they care relatively more about reducing trading delays.

this issue by assuming that traders choose m_b rather than θ, since there is a one-to-one mapping between both variables. If $W_o(a,z)$ is strictly concave, the buyer's indifference curve has a strictly convex shape in the space (m_b, p), just as the intermediary's zero isoprofit curve, so both curves are tangent at most one point.

By contrast, poorer buyers care more about paying lower prices. Note that $S(a,z,p) = W_o(a-(1+\tau_b)\,p\,\hbar,z) - W_r(a,z)$ is non decreasing in a whenever the marginal utility of wealth at night is not lower when agents buy (rather than rent) a home. It is direct to check, using the Envelope Theorem, that a sufficient condition for this is that the purchase of a home always implies lower consumption at night; i.e., $g_o^c(a-(1+\tau_b)\overline{p}\,\hbar,z) \leq g_r^c(a,z)$. This will be the case if housing prices are sufficiently high. Indeed, this sufficient condition holds in all our quantitative exercises, where $S(a,z,p)$ always increases with a. In any case, the sorting result will still hold if $S(a,z,p)$ decreases with a at a lower rate than $W_o'(a-(1+\tau_b)\,p\,\hbar,z)$, so $\hat{S}(a,z,p)$ still increases with a.

Proposition 4. *Suppose that the condition in Proposition 1 holds. Also, given z, $\hat{S}(a,z,p)$ is increasing in a for each $p \geq \overline{p}$ and each $z \in Z$. Then $g_b^\theta(a,z) > g_b^\theta(a',z)$ if $a < a'$.*

Consider now the participation decision. Potential buyers with financial assets $a \leq (\tau_b+\zeta)\,\overline{p}\,\hbar$ do not participate, since they cannot afford the down payment and associated taxes in any submarket. For wealthier agents, the expected gains from participating in a given submarket are $m_b(\theta)\,S(a,z,p(\theta))$. If S increases with a, so do the agents' (maximal) gains from participation, as wealthier agents can afford to trade in more expensive submarkets than poorer ones (i.e., their feasible choice sets are larger). Take agents with productivity z. As long as their gains are strictly positive if $a \in A$ is sufficiently high, there is then a threshold $a_{part}(z) \in A$ such that agents with assets $a > a_{part}(z)$ strictly prefer to participate (because the associated gains are positive), those with assets $a_{part}(z)$ are indifferent between participating or not (because the gains are zero), and the rest do not participate (because the gains are negative). Thus $W_b(a,z) > W_r(a,z)$ for all $a > a_{part}(z)$, and $W_b(a,z) = W_r(a,z)$ for $a \leq a_{part}(z)$. These participation thresholds depend on the Walrasian price, so they change with aggregate conditions.

Proposition 5. *Suppose that the condition in Proposition 1 holds and $S(a,z,p)$ increases with a for each $p \geq \overline{p}$. If $W_b(a,z) > W_r(a,z)$ for some $a \in A$, there exists $a_{part}(z) \in A$ such that $g_b^\theta(a,z) \in \mathbf{R}_+$ if $a > a_{part}(z)$, $g_b^\theta(a,z) = \theta_0$ if $a < a_{part}(z)$, and $g_b^\theta(a_{part}(z),z) = \{\theta_0, \overline{\theta}_z\}$.*

In the above statement, $\overline{\theta}_z$ denotes the optimal submarket for buyers with state $(a_{part}(z), z)$ (who are indifferent between participating or not). The *marginal buyer type* in this economy is the one who participates in the cheapest active submarket, which is also the most congested one. This type is $(a_{part}(\underline{z}), \underline{z})$ where

$$\underline{z} = \arg\max_{z \in Z} \overline{\theta}_z. \tag{2.33}$$

These buyers face the lowest down payment requirement and the longest trading delays in the afternoon market.

2.6 Computation of equilibrium

We now outline our strategy for computing the equilibrium. Appendix D provides specific details on the numerical algorithm.

2.6.1 The household's problem

As already noted, the Walrasian morning price, $\overline{p}$, determines the afternoon price schedule and the night rental price. Given these prices, households make three intertemporal decisions: the amount of financial assets for next period, whether or not to participate in the afternoon market, and their preferred submarket (conditional on participating). We allow households to choose both financial assets a' and a submarket θ in $\mathbf{R}_+$. In this way, we do not fix *ex-ante* the set of submarkets where agents can participate, Θ. We do this because the main action in our economy comes from agents trading off prices and trading probabilities. Discretizing and fixing the set Θ would bias the results and produce an artificially high or low equilibrium price dispersion. Instead, we compute the policy functions using the households' Euler equations, without resorting to discretization of Θ.

A difficulty in the computation is that the participation decision is endogenous, so households solve a non-concave problem. Thus we build on Fella (2014) in order to compute the household's optimal choice. The solution method proposed by Fella (2014) involves using the

Endogenous Grid Method (EGM hereafter) to find the local maximum and a Value Function Iteration step to verify whether the point is not only a local but also a global maximum. We discuss the main computational issues below.

In order to solve the buyer's afternoon problem, we need to know her gains in each submarket and the marginal utility of trading at a particular price. That is, we need to know the value functions of owners and renters. Proposition 1 ensures that the first-order conditions of problem (2.26) are sufficient. Consider now the night stage, when households decide the amount of financial assets for next period. As already noted, the buyer's value function is not globally concave. We know, however, that this function is concave on the range of assets that corresponds to participation ($a > a_{part}(z)$) and non-participation ($0 < a < a_{part}(z)$), respectively. We apply the EGM to each range. Solving this part of the problem requires an additional step of Value Function Iteration, comparing the local maximum if the agent does not participate in the afternoon market next period and if she does. The support of each range is endogenous (as the participation thresholds) and depends on $\overline{p}$.

2.6.2 Equilibrium in the morning market

The fact that we do not fix the set Θ ex-ante implies that we cannot use Monte Carlo simulations to find the stationary distribution of agents. The reason is that any change in the distribution of financial assets implies a change in the distribution of active submarkets. We instead compute directly the stationary distributions shown in (2.15)–(2.18).

The equilibrium price $\overline{p}$ clears the Walrasian morning market. Figure 2 represents the excess demand in this market; that is, the difference between the depreciated stock and new housing. Construction, I_h, is continuous and increasing in $\overline{p}$. In our quantitative experiments, the depreciated stock, $\delta\ (H_o + H_r + V)$, decreases smoothly with $\overline{p}$, as shown in the Figure. Intuitively, as $\overline{p}$ rises, the afternoon price schedule shifts upwards, and fewer households want to own (H_o falls). Since rents also rise, households rent smaller units (H_r also falls). Finally, there are fewer vacancies for sale in the afternoon because demand is lower in the search market (V falls). We use a standard iterative tatonnement-type algorithm to find the equilibrium value of $\overline{p}$. As shown in the Figure, if we start by assuming that the Walrasian price is equal to $\overline{p}_0$, supply exceeds demand, so the price must be adjusted downwards. In particular, the lower price at which demand meets this supply (I_h^{S0}) is $\overline{p}_0'$, and is below the equilibrium price. Hence, the equilibrium price lies in $[\overline{p}_0', \overline{p}_0]$. Our next guess is a weighted average of $\overline{p}_0'$ and $\overline{p}_0$. A similar argument applies if demand exceeds supply at the guessed prices. See Section D.4 in the Appendix.

3 Parameterization

The model period is a month. Since a property's average time on the market (TOM) is always below a quarter (as we will see), we do not want to amplify the role of search and matching frictions by imposing a lower frequency. Some model parameters are chosen externally. The remaining parameters are chosen to minimize the distance between a selection of moments of the stationary distribution and their data counterparts.

3.1 Functional forms

As in Díaz and Luengo-Prado (2010), we use the additively-separable felicity function

$$u\left(c,h\right) = \frac{c^{1-\sigma}}{1-\sigma} + \phi\,\frac{h^{1-\sigma}}{1-\sigma}. \tag{3.1}$$

The risk aversion parameter is set equal to $\sigma = 2$. Recall that rental units of size $h \in [0, \hbar]$ yield $\omega\, h$ housing services. Thus, for renters, $u\left(c,h\right) = (c^{1-\sigma})/(1-\sigma) + \phi\ \left((\omega\, h)^{1-\sigma}\right)/(1-\sigma)$.

Matching probabilities in the search market are as in Menzio and Shi (2011):

$$m_s(\theta) = \left(1 + \theta^{-\gamma}\right)^{\frac{-1}{\gamma}}, \quad m_b(\theta) = m_s(\theta)/\theta, \tag{3.2}$$

with $\gamma > 0$. Unlike the standard urn-ball matching process, this process has an extra degree of freedom in that γ governs the elasticity of $m_b(\theta)$ with respect to θ. The parameter γ

also determines the severity of search and matching frictions. As γ increases, frictions are reduced. Since our computation method requires a one-to-one mapping between θ and m_b, we cannot use the standard (truncated) Cobb-Douglas matching function (which implies $m_b = 1$ for θ sufficiently low). Note that (2.13) can be written as

$$m_s(\theta) = \frac{(1 - 1/R + \delta)\,\overline{p}}{p(\theta) - (1/R - \delta)\,\overline{p}}. \tag{3.3}$$

This expression shows that the probability of selling a home in submarket θ is a function of the ratio $p(\theta)/\overline{p}$. This relation is independent of the matching process we use, but the functional form of the matching process does affect the tightness level and the probability of buying. This insight will be very useful in the computation of the equilibrium (see Section D.1).

3.2 Externally chosen parameters

As in Díaz and Luengo-Prado (2010), we set the annualized real interest rate at 3%. We set $\tau_s = 6\%$ and $\tau_s = 2.5\%$, following Díaz and Luengo-Prado (2008). The depreciation rate of housing is 1.50% in annual terms, as in Sommer and Sullivan (2018). We follow Kaplan et al. (2020) and set α so that the price elasticity of new housing supply, $\alpha/(1-\alpha)$, is equal to 1.5, which is the median value across MSAs estimated by Saiz (2010).

The process for labor productivity is chosen in two steps. First, we calibrate an AR(1) process:

$$\ln \widehat{z}_t = \rho \ln \widehat{z}_{t-1} + \epsilon_t, \tag{3.4}$$

so its annualized version has the properties of the permanent component of labor earnings estimated by Storesletten et al. (2004). Hence, $\rho = 0.952^{1/12} = 0.9959$ and $\sigma_\epsilon = 0.17/\left(\sqrt{\sum_{i=1}^{12} \rho^{2\,(i-1)}}\right) = 0.0502$. The Rouwenhorst method is then used to discretize $\ln z_t$ into a 3-state Markov chain, $\Pi_{\widehat{z}}$. Next, we add a transitory state which can be thought of as an unemployment state. This state plays a similar role to the catastrophic state of Díaz and

Luengo-Prado (2008), who show that agents prefer renting to owing when they face more transitory risk. We proceed as Broer et al. (2021) and assume that, when hit by this shock, the agent's productivity drops to 40% of their lowest previously calibrated productivity state. This implies that z takes values in the set $Z = \{0.88, 1.00, 2.19, 4.81\}$. The probability of the transitory state is always $\varphi = 5\%$, which is roughly the average unemployment rate in the US. The probability of exiting unemployment to any other state is equal to the associated stationary probability implied by (3.4). The Markov process on Z is shown in Table 1.

The probability of becoming mismatched is set so that owners move every 9 years on average, as the National Association of Realtors (NAR) reports. Similarly to Head et al. (2014), we have assumed that households move across locations and target the annual frequency of owners and renters moving across counties in the US, which is about 3.2 and 12 percent, respectively, according to the Census Bureau. These three targets combined are used to calibrate the probabilities of the mismatch and migration shocks, π_μ, ξ_o, and ξ_r. The value of the wage per efficient unit of labor is set equal to $w = 1000$. Also, we set $\hbar = w\, mean(z)$.

The rental price r_h and the Walrasian housing price $\overline{p}$ are linked by the non-arbitrage condition (2.12). We calibrate κ so that the price-to-rent ratio (in annual terms), measured as $\overline{p}/r_h$, is 12.5%, as in Sommer and Sullivan (2018). This gives a value of κ equal to 20% of the monthly wage w.

We have assumed that immigrants own no residential assets. Since we do not have a sensible way to calibrate the distribution of their financial assets, we assume that they all enter the location with zero assets.

3.3 Parameters jointly calibrated

The rest of the parameters, β, ϕ, ω, γ, ζ, and B, are chosen jointly to minimize the distance between a number of selected equilibrium moments and their data counterparts. The data moments are chosen from the Survey of Consumer Finances (SCF). We have taken various waves from the SCF, from 1989 to 2007, and have selected the sample of households with positive earnings. We proceed as Budria et al. (2002) to compute household earnings. For each wave, we compute the same statistics and we take the mean across waves. In this paper, we refer to the homeownership rate as the fraction of households who own their home. In the data, we take it to be the fraction of households who own residential real estate, which is 69.15%. The average median wealth-to-earnings ratio for renters is 0.22. Matching housing wealth ratios requires that we take a stand regarding the value of housing. In the SCF, households are asked about the market value of their property. The counterpart of that value in our economy is the liquidation value of the house, $\overline{p}$. This is why we value housing at price $\overline{p}$ when we measure housing wealth. The median housing-wealth-to-earnings ratio for homeowners is 2.57. To have a sense of the size of mortgage debt, we calculate the median ratio for homeowners whose financial wealth (financial network minus real estate debt) is negative and call it the median loan-to-value ratio. The average of this ratio across waves is -0.43. Finally, we follow Kaplan et al. (2020) and target an average house size ratio of 1.5 between owners and renters. Table 2 summarizes the calibration of our benchmark economy. Interestingly, the calibrated down payment is 26%, which is very close to the number used by Favilukis et al. (2017), 25%, and slightly higher than that in Sommer and Sullivan (2018).

3.4 Alternative economies

We also consider alternative search economies that differ in the specification of the production of housing and the rental market, as shown in Table 3. All economies are calibrated so that they lead to the same steady state whenever the down payment is $\zeta = 0.26$.

The *low elasticity* economy is one where the elasticity of new housing supply is 0.6, the lowest value estimated by Saiz (2010) for the US MSAs areas.[17] This implies a reduction in the value for α from 0.47 to 0.37. The TFP parameter, B, of the housing production function is recalibrated so that this economy generates the same equilibrium as our benchmark economy when $\zeta = 0.26$. Additionally, we consider a *very low elasticity* economy where we further reduce the supply elasticity to 0.1. This is consistent with the estimates in

[17]Specifically, it corresponds to Miami, FL.

Baum-Snow and Han (2019) for US urban neighborhoods.

In addition, to assess the importance of the absence of segmentation of rental and owner-occupied housing stocks in the model, we consider also the opposite extreme case with full market segmentation. Specifically, we assume that rental and owner-occupied units are different objects and that rents are exogenous (so they are unaffected by changes in housing demand). We fix the rental price equal to its calibrated value in the benchmark economy, and assume that rental units are elastically supplied at this price (and, for simplicity, do not depreciate). We then recalibrate the TFP parameter, B, so that housing production only replaces depreciated owner-occupied housing and vacancies: $I_h = \delta\left(H_o + V\right)$. This is done for the three supply elasticities considered above.

4 Quantitative results

In this section, we present the results of our quantitative experiments. We first describe our benchmark economy, and then explore the effects of relaxing credit conditions in the alternative economies we consider.

4.1 The benchmark economy

Table 4 shows selected statistics of our benchmark economy. Let us focus on the untargeted moments (pointed with *). The share of owners who hold debt in equilibrium is 68.15%, whereas the number reported in Sommer and Sullivan (2018) is 65%. In the SCF, though, the mean of working age households with negative financial assets across 1989-2007 waves is 42%. Median rental expenditures are 19.37% in the steady state are, whereas in the data they are about 25%, according to Sommer and Sullivan (2018). We have calibrated the matching function parameter to match the median time to buy and let the model determine average time on the market (TOM). The National Association of Realtors reports a TOM between 4 and 17 weeks. Average TOM is 9.89 weeks in the steady state, which is about the mean estimate of the National Association of Realtors. A remark is in order. In reality, TOM refers to the average time between listing and sale of a property. Thus, although households sell their property without delays in the morning market, we think that the appropriate model counterpart of this statistic is the average time it takes an intermediary to sell the property in the afternoon market.

In our economy there are vacancies overnight; these are the units that could not be sold in the afternoon and are not occupied. According to the American Housing Survey, the ratio of the stock of year-round vacant units for sale to the total stock of owner occupied units (plus those vacant units) is 1.59% every quarter for the period 1965:1-2010:4. We take this as the data counterpart of our overnight vacancy rate, computed as vacant units, V, over $V + H_o$, where H_o is the stock of owner-occupied housing. The implied rate in our benchmark model is 1.35%, which is pretty close to the data. In our economy there is a difference between the stock of houses for sale in the afternoon and the stock that remains unsold over night (and will be up for sale again the next period). We also report the rate of vacancies for sale in the afternoon, which is 2.27%. The greater the difference between these two rates, the more liquid the afternoon market is.

Our model generates frictional housing price dispersion, as illustrated in Figure 3. Panel 3(a) plots the afternoon policy function, $g^{\theta}(a, z)$. In line with our theoretical results, given the productivity state z, there is sorting by financial wealth, meaning that buyers with higher wealth trade in less congested market segments. As shown in panel 3(b), both the probability of buying and the price buyers pay rise with wealth. However, there is no sorting by labor earnings in general. In particular, buyers in state 1 trade in less congested submarkets and pay higher prices than buyers in state 2 who have identical financial wealth. The reason is that state 1 is a transitory state with a much lower persistence than state 2. In fact, agents in state 1 are more likely to enter states 3 and 4 than agents in state 2. Our results suggest that there can be sorting by earnings only if earnings shocks are sufficiently persistent. This non-monotonicity is also reflected in the participation thresholds, $a_{part}(z)$, as shown in Figure 3.

Price dispersion, measured as the coefficient of variation of the price distribution in the search market, is small in the steady state, 0.14%. For instance, Lisi and Iacobini (2013) estimate the coefficient of variation of house prices to be about 2% in the data. Kotova and Zhang (2020) find a much larger dispersion, about 15%, for a selection of US counties. There are three factors compressing the distribution of prices in the model. First, owning is risky as owners cannot sell at will; recall that we are allowing to sell, on average, every 9 years. Since earnings risk operates at a shorter frequency, agents do not want to be caught with too much debt. This discourages households with high financial wealth from searching in submarkets with higher prices (compressing the upper tail of the afternoon price distribution). Additionally, the parameter γ of the matching process affects price dispersion because it governs the trade off between prices and congestion. This parameter is chosen to match the observed median time to buy. The calibrated value makes the probability of buying "highly concave" in the price, and this tends to reduce price dispersion. Finally, it is important to note that, in our model economy, price dispersion only reflects the buyers' heterogeneous wealth effects (across the wealth distribution), since sellers in the search market are risk neutral.

4.2 The role of search and matching frictions

Search and matching frictions act as bottlenecks: home buyers would like to trade instantaneously, but they cannot. Prices in the afternoon market then reflect both how buyers value housing services, as well as how they value the speed of the transaction. To understand how these bottlenecks affect the economy, we consider a version of the model without search and matching frictions, where home buyers trade directly with mismatched owners and developers in the morning market. In equilibrium, all buyers then trade at price $\overline{p}$ with probability one. In this economy, there are no realtors and the frictional afternoon market shuts down. Again, households can either buy an indivisible home that yields services $\hbar$ or rent out one that yields $\omega\, h$, where $h \leq \hbar$. The buyer's value function is then

$$\begin{aligned} W_b(a,z) = \quad & \max_{m_b \in \{0,1\}} \left\{ m_b\, W_o\left(a - (1+\tau_b)\,\overline{p}\,\hbar, z\right) + (1-m_b)\; W_r\left(a,z\right) \right\} \\ \text{s. t.} \quad & a \geq (\zeta + \tau_b)\,\overline{p}\,\hbar. \end{aligned} \tag{4.1}$$

The effect of eliminating search and matching frictions in our environment is described in columns 4 to 6 of Table 4 for each of the supply elasticities we consider. Column 3 (Walras^1) describes the equilibrium that results when these frictions are eliminated in our benchmark economy, where the price elasticity of new housing supply is 1.5. The other two columns (Walras^2 and Walras^3) present the corresponding effect in the low and very low elasticity economies.

While we do not recalibrate the Walrasian economy (in column 4), note that the targeted moments remain similar, except for the lower homeownership rate (66.47%) and for time to buy (which is now zero). The Walrasian price is 2% lower than in our benchmark economy. This is so because of demand factors. First, fewer agents now turn to owning. This may seem surprising, but it is explained by the fact that search and matching frictions partially convexify the binary tenure decision in problem (4.1); this makes owning relatively more attractive in the benchmark economy for our risk-averse households. Furthermore, in the Walrasian economy, there are no intermediaries who demand owner-occupied housing. As a result of the drop in total housing demand, there is less construction every period, and the Walrasian price is lower. The existence of search and matching frictions is also important to understand the share of indebted owners and the magnitude of their debt. In the Walrasian economy, 40.32% of owners hold debt, as opposed to 68.15% in the benchmark economy and 65% in the data. The median debt is also lower. Recall that our economy has been calibrated to match the median LTV ratio, but the ratio of indebted owners is determined by the model. This ratio is tightly linked to the existence of equilibrium price dispersion. Since home buyers not only compete to obtain housing services but also to speed up transactions, they borrow to afford a higher price. The existence of search and matching frictions also affects the distribution of financial assets. While there are more renters in the Walrasian economy, their median wealth-to-earnings ratio is smaller (0.20 versus 0.22 in the benchmark economy). This is again due to the fact that, with search frictions, potential buyers accumulate more assets in order to afford higher prices and reduce trading delays. This competitive effect is nonexistent in the Walrasian model, as the buyers trade instantaneously at the same price, which is also why the participation rate falls from 7.68 to 2.33%.

Naturally, the lower the housing supply elasticity, the larger the reduction in the Walrasian price generated by the elimination of bottlenecks. This price is 7.3% lower in the *low elasticity* economy and 16% lower in the *very low elasticity* economy, compared to the benchmark economy. It is interesting to note, however, that homeownership rates bounce back up as owner-occupied housing becomes cheaper. In summary, search and matching frictions increase the overall demand of housing, and imply that more households borrow, and this translates into higher housing prices. Notice that this is so even though the model period is a month and homeowners move once every 9 years on average. That is, we are imposing relatively mild search and matching frictions, which in turn are consistent with monthly average TOM and buying times.

4.3 The long-run effect of credit expansions

Here we conduct a series of experiments in which we lower the down payment from 26% to 5%. This is similar to the exercise in Favilukis et al. (2017), and is performed in our benchmark economy and in the alternative economies we consider. The results are shown in columns 3 to 8 of Table 5.

4.3.1 The role of search and matching frictions

The effect of a credit expansion in our benchmark economy is described in column 3, whereas column 4 shows the effect in its Walrasian counterpart. The enormous credit expansion makes owning less risky, as households can borrow more to smooth earnings risk. A word of caution is needed here. As discussed in Kaplan et al. (2020), the way in which mortgages are modeled matters for a credit expansion to impact prices significantly. We have assumed that a reduction in the down payment allows both new and existing owners to increase their borrowing. In reality, this reduction affects mainly new mortgages, unless many owners refinance. We believe that this distinction is important when studying the transitional dynamics of housing prices, but it matters less when studying long-run effects. Also, Foote et al. (2020) document that a large part of the growth in mortgage debt during the housing boom can be attributed to income rich households who were refinancing their mortgages.

In the benchmark economy, the supply of new housing is quite elastic, and construction increases by 5.87% as demand rises in the face of the credit expansion. This implies a mild increase of 3.88% in the Walrasian price. The reason is that rental companies are supplying their units in the Walrasian market to meet the higher demand of owner occupied housing. These units are repackaged at no cost, and supplied as vacant homes in the frictional market. As a result, there is a stark 24.38% increase in supply in this market relative to the benchmark. This swift conversion of rental units into owner-occupied housing then explains why the homeownership rate increases sharply from 69.17 to 88.29%, while the increase in the Walrasian price is moderate. These aggregate effects imply some interesting distributional changes. The median loan-to-value ratio rises by around 40% (from 46.74 to 66.15%), and the fraction of indebted owners increases from 68.15 to 80.24%. The credit expansion increases not only because there are new home borrowers, but because everyone borrows more.

Since there are more owners, renters concentrate among the poor, which is why their median wealth-to-earnings ratio falls from 0.22 to 0.10. The standard deviation of prices in the search market increases by 13.56 %, and as a percentage of the mean price, it rises from 0.14 to 0.15, relative to the benchmark economy. This mild change in price dispersion is due to various countervailing forces. On the one hand, there are more poor buyers at the lower end of the price distribution (with a higher mass of agents concentrated there), which compresses the distribution. This is why the buyers' median wealth-to-earnings ratio drops from 0.86 to 0.34%, and so does the ratio of this median to the mean. On the other hand, wealthier buyers borrow more in order to target pricier homes and speed up their transactions, which makes the distribution more disperse. As a result of these two opposing effects, the standard deviation rises, but as a percentage of the mean price, it only rises slightly. Note that the participation rate in the frictional market rises sharply from 7.68 to 27.04%. Overall, the increase in both demand and supply in this market translates into a slight rise in median time to buy and a small reduction in average TOM and overnight vacancies.

The corresponding effects in the Walrasian counterpart of the benchmark economy are shown in column 4. A few differences stand out. First, the increase in the Walrasian price is slightly smaller (3.63%), implying that search frictions contribute in 0.25 percentage points of the total price increase. This is because TOM is zero in the Walrasian economy, where the rise in construction that is necessary to meet demand is smaller (5.49%). Search and matching frictions thus amplify the demand increase triggered by a credit expansion, although the quantitative effect is small in this case. Second, the Walrasian economy features an even larger increase in the homeownership rate, which rises to 92.64%, as households can buy a home instantaneously. Third, while search and matching frictions act as bottlenecks, they also lead to more borrowing, both in terms of the median loan-to-value ratio and the share of indebted owners. Intuitively, in response to increasing demand, competition aimed at reducing trading delays becomes more intense in our benchmark, resulting in more borrowing.

In summary, a credit expansion in our benchmark economy makes homeownership more attractive, as it gives insurance against earnings risk through borrowing. This rises the demand of owner-occupied housing, and induces rental companies to sell their properties to intermediaries, who also demand new construction to satisfy demand. As construction rises, so do vacancies for sale. In spite of this, the overnight vacancy rate falls due to the sharp increase in ownership. Search and matching frictions add 0.25 modest percentage points to the Walrasian price.

4.3.2 The interaction of search and matching frictions and new housing supply elasticity

We now investigate the importance of the price elasticity of new housing supply in determining the magnitude of the amplification effects arising from competitive search. To this aim, we quantify the effect of a credit expansion in the *low elasticity* and the *very low elasticity* economies in columns 6 to 8 of Table 5. The qualitative effect is the same—search and matching frictions imply greater increases in the Walrasian price—but the magnitude is larger when the supply elasticity is lower. In the *low elasticity* and *very low elasticity* economies, search and matching frictions add 1.27 and 2.11 percentage points to the price with respect to their Walrasian counterparts, respectively. Likewise, the higher borrowing induced by these frictions is a robust feature of the model, the effect being larger when the supply elasticity is lower. Notice also that, in these economies, the standard deviation of prices in the search market rises by 29.37 and 39.45%, respectively, relative to the benchmark. As already noted, there are two forces affecting price dispersion: more poor households search in cheaper market segments, while wealthier households borrow more to enter more expensive submarkets and speed up their transactions. Both forces almost compensate, and the coefficient of variation rises to 0.16 and 0.17%, respectively.

4.3.3 The role of market segmentation

In our benchmark economy, rental units can be converted into owner-occupied housing (and vice versa) at no cost within one month. This amounts to assuming that the housing supply in the Walrasian morning market has a flat segment (where it is infinitely elastic). Given that this is a rather extreme assumption, we also explore the opposite scenario: an economy where rental units are different objects and rents are exogenous (so they are unaffected by changes in housing demand). We fix the rental price equal to its calibrated value in the benchmark economy and conduct the same exercises as in Section 4.3.2 in order to explore the interaction of search frictions and credit constraints in this scenario. The results are shown in Table 6.

Consider first the implications of market segmentation in our benchmark economy, where the elasticity of new housing supply is 1.5 (columns 3 and 4). As one would expect, the increase in the Walrasian price is much larger when the rental and real estate markets are completely segmented. In fact, it is more than three times larger, 12.68% (versus 3.88% in the case of no segmentation). Since there is no possibility of converting rental units into owner-occupied ones, the increase in demand has to be met with construction, which increases by 19.62% (as opposed to 5.87% in the benchmark). The standard deviation of prices in the search market rises by 35.25%, relative to the benchmark economy, so it is more strongly influenced by wealthier households bidding for higher prices than by the presence of many new poorer buyers in the search market. However, since prices are also rising more sharply, the coefficient of variation only rises from 0.14 to 0.17.

Eliminating search frictions leads to an even larger price increase, 15.95%. This may come as a surprise, as we argue that search and matching frictions act as bottlenecks, so agents are willing to pay more to avoid them. There are two insights that we learn from this exercise. First, as we have argued in Section 4.2, search and matching frictions "convexify" the tenure decision so it makes the demand not as elastic as in the case without frictions. That is, the demand of owner occupied housing is more sensitive to credit conditions in the absence of search and matching frictions. If we add market segmentation we get a higher price increase. Additionally, search and matching frictions act as bottle necks. The second insight is that these bottlenecks also imply that less construction is needed to meet the increase in demand in this scenario. Note that median time to buy increases significantly after the credit relaxation when markets are segmented, so it is harder for buyers to trade. These stronger bottlenecks tend to slow down sales growth in the search market and, thus, the intermediaries demand in the Walrasian market, relative to the scenario without segmentation. In summary, while market segmentation significantly amplifies the effect of a credit expansion on housing prices, search and matching frictions tend to dampen that amplification effect in this case.

We also report for the sake of comparability, the effect of a credit expansion in the alternative search economies with lower housing supply elasticities in columns 5 and 6 of Table 6. As we can see, when the elasticity is very low, the price effect is much larger (22.77%), while there is almost no change in the homeownership rate (which rises to 70.95%). Additionally, it is important to note that the impact on price dispersion is now different, due to the interaction between market segmentation and the reduction in the supply elasticity. First of all, while the standard deviation of prices raises (to 22.94 and 22.76%, respectively), the increase is now smaller than in the scenario with no market segmentation. Since price increases are much stronger now, this in turn implies that the coefficient of variation does not rise in this case. In fact, it falls to 0.13% when the elasticity is very low. Kotova and Zhang (2020) report that price dispersion fell as prices rose during the housing boom that preceded the Great Recession. In the light of the results in Greenwald and Guren (2021), who provide evidence of substantial market segmentation, this is consistent with our sorting mechanism under market segmentation, provided the housing supply elasticity is low.

5 Final comments

This paper investigates how the interaction between search and matching frictions and risk-aversion affects the long-run level and dispersion of house prices when the search process is competitive. We also study how this interaction shapes the response of housing prices to a relaxation of credit constraints. We do so in an economy populated by households who live forever and face idiosyncratic uninsurable earnings risk. There is also a meaningful tenure choice: owner-occupied housing is associated with a utility premium, but its illiquidity makes it ineffective at shielding consumption against permanent shocks.

We show theoretically that, when search is competitive, wealthier households are willing to pay a higher price to speed up transactions. Hence, the equilibrium features frictional price dispersion. Our quantitative experiments show that search and matching frictions have a double positive effect on housing demand in the model. First, they act as bottlenecks, which is why buyers are willing to pay more in order to the speed up transactions. Second, they tend to convexify the tenure choice, making homeownership more attractive for risk-averse agents. In the long run, this double effect results in higher house prices and higher debt levels than in an economy without these frictions. These differences are more pronounced when the elasticity of new housing supply is low.

In our benchmark economy, search frictions amplify the long run effect of a credit relaxation with respect to a Walrasian economy. When borrowing constraints are relaxed, households who would not have searched for a home before now do so, whereas wealthier households purchase more expensive homes to avoid queues. The overall effect is a rise in the average price and a sharp increase in the amount of borrowing and the number of borrowers. Price dispersion increases due to the fact that wealthier households are willing to pay more to avoid congestion. Economies with low housing elasticity experience larger increases in housing prices.

We also uncover some interesting interactions of search and matching frictions and market segmentation. As pointed out by Kaplan et al. (2020) or Greenwald and Guren (2021), among others, segmentation between rental and owner occupied housing amplifies the effect of a credit relaxation. This is due to the fact that more construction is necessary to meet the increase in housing demand in this scenario. This is also the case in our framework. Naturally, the lower the elasticity of new housing supply, the stronger the amplification effects are. However, our results indicate that search and matching frictions act as a buffer in this case because they obstruct the direct channel through which credit affect prices: the change in the tenure decision. Interestingly, unlike in the case of no market segmentation, price dispersion may even fall when credit is eased if the new housing supply elasticity is small. This last result offers an additional margin of analysis that may be helpful when assessing the importance of market segmentation in the data.

We have made some simplifying assumptions to establish our results. We have abstracted from the property ladder. Ortalo-Magné and Rady (2006) show that credit relaxation allows households to invest in better, larger homes, pushing prices up. We leave it for future research to examine how the existence of a property ladder affects housing price dispersion. We have also abstracted from the life cycle. This is important as many buyers do not have previous real estate wealth to purchase a new home. According to the National Association of Realtors, around 30 percent of all buyers are first-time buyers. Therefore, credit conditions matter more to them than to repeat buyers. We have assumed that new agents enter the economy each period with zero assets, which somehow resembles the life cycle effect.

In our model, homeowners face no default risk and may sell their homes instantaneously when they become mismatched. Hedlund (2016a 2016b) argues that the joint interaction between tighter credit standards, default risk, and decreasing liquidity is important during a housing bust (see also Head et al. (2019)). A quantitative study of the housing market based on our theory is likely to incorporate several of these additional features.

Finally, the paper focuses on steady states. Studying the transitional dynamics of our model is not trivial. Out of the steady state, the Walrasian price, $\bar{p}_t$, at which intermediaries purchase homes—which is the key state variable of the model—equals their expected return in the competitive search market. Since intermediaries carry unsold inventories over time, their expected return depends not only on the prices that prevail in the search market in that period, but also in subsequent periods. This means that $\bar{p}_t$ depends on $\{\bar{p}_{t+l}\}_{l=1}^{\infty}$, since all price information is summarized by the Walrasian price. In particular, if intermediaries expect higher prices in the future, their current expected return and thus $\bar{p}_t$ will increase (shifting the price schedule that buyers face in the search market upwards). In the Walrasian version of the model we have studied, this effect is absent because there are no unsold inventories over night. The effect is also absent in Hedlund (2016b) and Garriga and Hedlund (2020), for the same reason. In Hedlund (2016b), the intermediation sector is modeled as a large real estate firm which consists of a continuum of agents. Each period the firm decides how to distribute these agents across submarkets, and also decides total construction. In taking these decisions, the firm faces no uncertainty. By the law of large numbers, it can pool the rationing risks faced by the individual agents that form the firm. This ensures that there are no unsold inventories in the intermediation sector at the end of a period. Taking price expectations seriously seems important, and an extension of our model along these lines would be extremely interesting. The problem amounts to finding a sequence of prices, which is much easier than finding sequences of higher dimensional objects. We leave all these interesting extensions for future work.

A Properties of the value functions

Let a denote the household's assets in a given subperiod (either night or afternoon). Denote $X = A \times Z$, where $A = [\underline{a}, \infty)$ and $Z = \{z_1, \ldots, z_n\}$ is a finite set of exogenous shocks, $0 < z_1 < z_2 < \cdots < z_n$. Let $C(X)$ be the space of continuous functions $f : X \to \mathbf{R}$, where we consider the usual topology on A and the discrete topology on Z. Define the two-dimensional Bellman operator T acting on $C(X) \times C(X)$ by $T = (T_o, T_r)$, where

$$\begin{aligned} T_o(f_o, f_r)(a, z) \quad &= \quad \max_{c,a'} \Big\{ u\left(c, \hbar\right) + \beta\ (1-\pi)\ E_z\, f_o\left(a', z'\right) \\ &\qquad\qquad + \beta\pi\, E_z\, T_b(f_o, f_r)\left(a' + (1-\tau_s)\, \overline{p}\, \hbar, z'\right) \Big\} \qquad \text{(A.1)} \\ &\qquad \text{s.t.} \quad c + \tfrac{1}{R}\, a' \le w\, z + a - \delta \overline{p} \hbar, \\ &\qquad\qquad\; a' \ge -(1-\zeta)\, \overline{p} \hbar, \quad c \ge 0 \end{aligned}$$

$$\begin{aligned} T_r(f_o, f_r)(a, z) \quad &= \quad \max_{c,h,a'} \Big\{ u\left(c, \omega\, h\right) + \beta\, E_z\, T_b(f_o, f_r)\left(a', z'\right) \Big\} \\ &\qquad \text{s.t.} \quad c + \tfrac{1}{R}\, a' \le w\, z + a - r_h\, h, \\ &\qquad\qquad\; a' \ge 0,\ c \ge 0,\ 0 \le h \le \hbar \end{aligned} \qquad \text{(A.2)}$$

and where $T_b(f_o, f_r)(a, z) =$

$$\max \left\{ \max_{\theta \in D(a)} \Big\{ m_b\left(\theta\right)\ f_o\left(a - (1+\tau_b)\, p(\theta) \hbar, z\right) + \left(1 - m_b\left(\theta\right)\right)\ f_r\left(a, z\right) \Big\}, f_r(a, z) \right\}. \qquad \text{(A.3)}$$

The feasible correspondence D of the inner maximization problem in (A.3) is defined by

$$D(a) = \{\theta \in \mathbf{R}_+ \,:\, a - (1+\tau_b)\, p(\theta)\, \hbar + (1-\zeta)\, \overline{p}\, \hbar \ge 0\} \quad \text{for } a \in A. \qquad \text{(A.4)}$$

If $D(a) = \emptyset$, we attach the value $-\infty$ to participation, and thus $T_b(f_o, f_r)(a, z) = f_r(a, z)$ in this case. Also, since

$$p(\theta) = \frac{(1 - \frac{1}{R} + \delta)\overline{p}}{m_s\left(\theta\right)} + \left(\frac{1}{R} - \delta\right)\, \overline{p} \quad \text{for all } \theta \in \mathbf{R}_+, \qquad \text{(A.5)}$$

$\lim_{\theta \to \infty} p(\theta) = \overline{p}$. Since p is decreasing, $D(a) \neq \emptyset$ if and only if $a > (\tau_b + \zeta)\, \overline{p}\, \hbar$. Since p is continuous in $\mathbf{R}_{++}$, D has closed sections. However, $D(a)$ is not compact. To circumvent this problem and be able to apply Bergé's Maximum Theorem, we assume that agents choose m_b rather than θ, which is allowed since m_b is strictly monotone. Let

$$\hat{p}(m_b) = \frac{(1 - \frac{1}{R} + \delta)\, \overline{p}}{\hat{m}_s\left(m_b\right)} + \left(\frac{1}{R} - \delta\right)\, \overline{p} \quad \text{for } m_b \in (0, 1), \qquad \text{(A.6)}$$

and $\hat{p}(0) = \overline{p}$. The function $\hat{p}$ is continuous in $[0, 1)$, since it is the composition of two continuous functions when $0 < m_b < 1$ and, for $m_b = 0$, $\lim_{m_b \to 0^+} \hat{p}(m_b) = \lim_{\theta \to \infty} p(\theta) = \overline{p}$. Also, since $\hat{m}_s$ is strictly decreasing and $-\hat{m}_s{}'/\hat{m}_s$ is non decreasing, $\hat{p}$ is strictly increasing and strictly convex. Finally, $\lim_{m_b \to 1^-} \hat{p}(m_b) = \lim_{\theta \to 0^+} p(\theta) = \infty$. By choosing m_b as the new decision variable, the feasible correspondence D becomes $\overline{D}$, defined by

$$\overline{D}(a) = \{m_b \in [0,1) \,:\, a - (1+\tau_b)\,\hat{p}(m_b)\,\hbar + (1-\zeta)\,\overline{p}\,\hbar \geq 0\}. \tag{A.7}$$

The sections of $\overline{D}$ are nonempty and compact for $a - (1+\tau_b)\,\hat{p}(m_b)\hbar + (1-\zeta)\,\overline{p}\hbar > 0$. In fact, when nonempty, $\overline{D}(a)$ is the bounded and closed interval $\left[0, \hat{p}^{-1}\left(\frac{a/\hbar + (1-\zeta)\,\overline{p}}{1+\tau_b}\right)\right]$. Problem (A.3) thus transforms into $T_b(f_o, f_r)(a,z) =$

$$\max\left\{\max_{m_b \in \overline{D}(a)} \Big\{m_b\, f_o\left(a - (1+\tau_b)\,\hat{p}(m_b)\,\hbar, z\right) + (1-m_b)\; f_r\left(a,z\right)\Big\}, f_r(a,z)\right\}. \tag{A.8}$$

In what follows, we assume that the minimum rental-unit size is $\varepsilon > 0$, so $0 < \epsilon \leq h \leq \hbar$. This is an innocuous assumption since the utility of renting 0 units in $-\infty$. Also, we assume that the poorest and less productive owner can sustain a strictly positive level of consumption at the borrowing limit, $w\,z_1 + \underline{a} > \delta\,\overline{p}\,\hbar + (1-\zeta)\,\overline{p}\,\hbar/R > 0$. In the same way, the poorest and less productive renter can sustain a strictly positive level of consumption when renting maximum-sized units, $w\,z_1 + \underline{a} > r_h\,\hbar$. Since u is non decreasing both with respect to c and h, this assumption assures that a positive level of consumption is always possible for both owners and renters, so that their utility functions remain bounded from below:

$$\begin{aligned} u(c,\hbar) &\geq u_o := u\left(w\,z_1 + \underline{a} + \delta\,\overline{p}\,\hbar + (1-\zeta)\,\overline{p}/R, \hbar\right) > -\infty \\ u(c,\omega h) &\geq u_r := u(w\,z_1 - r_h\,h + \underline{a}, \omega\,\varepsilon) > -\infty, \end{aligned} \tag{A.9}$$

for all $c > 0$, $\varepsilon < h \leq h_r$.

Let $\underline{u} = \min\{u_o, u_r\}$. Theorem 1 below uses (A.9) to deal with the utility functions postulated in the calibration and numerical exercises, but allows for unbounded from above utilities (e.g., logarithmic). In this latter case, we need to control for their rate of growth on the feasible correspondence, as well as for the size of the discount factor β to guarantee that the dynamic programming equations define a contraction operator. To this end, consider the sequence $\{a_0, a_1, \ldots, a_j, \ldots\}$, defined by

$$a_j = \left(\frac{R\,w\,z_n + R\,\delta\,\overline{p}\,\hbar}{R-1} + \underline{a}\right) R^j - \frac{R\,w\,z_n + R\,\delta\,\overline{p}\,\hbar}{R-1}, \quad j = 0,1,2,\ldots, \tag{A.10}$$

and recall that $z_n = \max Z$. Note that $\underline{a} \leq a_j \leq a_{j+1}$, $a_j \to \infty$ as $j \to \infty$, and $a_0 = \underline{a}$. Let

$$\begin{aligned} u_j^o &= \max_{a \in [\underline{a}, a_j]} \left| u\left(w\,z_n + a + \frac{(1-\zeta)\overline{p}\,\hbar}{R}, \hbar\right)\right|, \\ u_j^r &= \max_{a \in [\underline{a}, a_j]} \left|u\left(w\,z_n - r_h + a, \omega\varepsilon\right)\right|, \end{aligned}$$

and $u_j = \max\{u_j^o, u_j^r\}$. Note that both u_j^o and u_j^r are well defined because u is continuous and by (A.9). Define

$$v_j := \sum_{i=j}^{\infty} \beta^{i-j} u_j, \quad \text{for } j = 0, 1, 2, \ldots. \tag{A.11}$$

The following theorem establishes the existence of a unique solution to the Bellman equation in a suitable class of functions. The result covers both the bounded and unbounded-from-below cases under the hypotheses discussed above.

Theorem 1. *Suppose that*

$$\overline{u} := \lim_{j\to\infty} \frac{u_{j+1}}{u_j} < \frac{1}{\beta}. \tag{A.12}$$

Then, the dynamic programming equations (A.1), (A.2) *and* (A.3) *admit unique continuous solutions* W_o*,* W_r *and* W_b*, respectively, in the class of functions* $\mathcal{F}$ *defined by*

$$\mathcal{F} = \Bigg\{ f \in C(X) \,:\, f(a,z) \geq \frac{\underline{u}}{1-\beta}, \quad \textit{for all } a \in A,\ z \in Z,\ \textit{and} \max_{a\in[\underline{a},a_j]} f(a) \leq v_j,\ \textit{for all } j = 0, 1, \ldots \Bigg\}. \tag{A.13}$$

Moreover, both W_o *and* W_r *are strictly increasing and* W_b *is non decreasing.*

Proof. Let $(f_o, f_r) \in \mathcal{F}\times\mathcal{F}$. If $a \leq (1+\tau_b)\,\overline{p}\,\hbar - (1-\zeta)\,\overline{p}\,\hbar$, the optimal choice in the afternoon market is θ_0, and so $T_b(f_o, f_r)(a,z) = f_r(a,z)$, which is continuous. When $a > (1+\tau_b)\,\overline{p}\,\hbar - (1-\zeta)\,\overline{p}\,\hbar$, the function $(a, z, m_b) \mapsto m_b\, f_o\,(a - (1+\tau_b)\,\hat{p}(m_b)\hbar, z) + (1-m_b)\ f_r\,(a,z)$ is continuous and the correspondence $\overline{D}$ defined in (A.7) is nonempty valued, compact valued, and continuous. Hence, by the Theorem of the Maximum, the value function

$$\max_{m_b\in\overline{D}(a)} \Big\{ m_b\, f_o\,(a - (1+\tau_b)\,\hat{p}(m_b)\hbar, z) + (1-m_b)\ f_r\,(a,z) \Big\} \tag{A.14}$$

is continuous. Since $T_b(f_o, f_r)$ is defined as the maximum between this value function and f_r, it is also continuous. It follows that the functions defining the right-hand side of $T_o(f_o, f_r)$ and $T_r(f_o, f_r)$ given in (A.1) and (A.2), respectively, are continuous. Moreover, the feasible correspondence is nonempty valued, continuous and compact valued in both cases. Hence, by the Theorem of the Maximum, both $T_o(f_o, f_r)$ and $T_r(f_o, f_r)$ are continuous. Let us see that $T_i(\mathcal{F}\times\mathcal{F}) \subseteq \mathcal{F}$, for $i = o, r, b$. Let $(f_o, f_r) \in \mathcal{F}\times\mathcal{F}$. By the definition of T_b as the maximum of a convex combination of f_o and f_r, it is clear that $T_b(f_o, f_r) \geq \frac{\underline{u}}{1-\beta}$. Plugging this inequality into (A.1) and (A.2), we obtain

$$T_o(f_o, f_r)(a,z) \geq \max_{c,a'} u(c,\hbar) + \beta\frac{\underline{u}}{1-\beta} \geq \underline{u} + \beta\frac{\underline{u}}{1-\beta} = \frac{\underline{u}}{1-\beta}, \tag{A.15}$$

and

$$T_r(f_o,f_r)(a,z) \geq \max_{c,h,a'} u(c,\omega h) + \beta \frac{\underline{u}}{1-\beta} \geq \underline{u} + \beta \frac{\underline{u}}{1-\beta} = \frac{\underline{u}}{1-\beta}, \tag{A.16}$$

respectively. On the other hand,

$$T_b(f_o,f_r)(a,z) \leq m_b\, f_o\,(a-(1+\tau_b)\,\hat{p}(m_b)\hbar, z) + (1-m_b)\ f_r\,(a,z) \leq \\ m_b v_j + (1-m_b) v_j = v_j \tag{A.17}$$

and $T_b(f_o,f_r)(a,z) \leq f_r(a,z) \leq v_j$, for all $a \in [\underline{a}, a_j]$, for all $j = 0,1,\ldots$. Hence, given that for any $a \in [\underline{a}, a_j]$, $\overline{D}(a) \subseteq [\underline{a}, a_{j+1}]$ by the definition of v_j given in (A.12), we have

$$T_o(f_o,f_r)(a,z) \leq u_j + \beta v_{j+1} = v_j, \quad \text{for all } a \in [\underline{a}, a_j]. \tag{A.18}$$

By a similar computation, $T_o(f_o,f_r)(a,z) \leq v_j$ for all $a \in [\underline{a}, a_j]$. It thus follows that $T_i(\mathcal{F}\times\mathcal{F}) \subseteq \mathcal{F}$, for all $i = o,r,b$. Consider now $C(X)$ with the topology generated by the countable family of seminorms $\|f\|_j = \max_{a\in[\underline{a},a_j],z\in Z} |f(a,z)|$, for all $j = 0,1,\ldots$. This family is separated ($\|f\|_j = 0$ for all j implies that f is the null function). Since the compact intervals $[\underline{a}, a_j]$ form an increasing family that covers A and they have nonempty interiors, and the space Z is finite, the space $C(X)$ is complete with this topology (see Rincón-Zapatero and Rodríguez-Palmero, 2003). Consider the product space $\mathcal{F}\times\mathcal{F}$ with the seminorms $\|(f_o,f_r)\|_j = \max\{\|f_o\|_j, \|f_r\|_j\}$, for $j = 0,1,\ldots$ and $(f_o,f_r) \in \mathcal{F}\times\mathcal{F}$. It is clear that $\mathcal{F}\times\mathcal{F}$ is complete with this topology, thus closed. Consider the series $\sum_{j=0}^{\infty} c^{-j} u_j$, with $c > \overline{u}$, where $\overline{u}$ was defined in (A.12). By the ratio test and by (A.12),

$$\lim_{j\to\infty} \frac{c^{-(j+1)} u_{j+1}}{c^{-j} u_j} = \frac{\overline{u}}{c} < 1, \tag{A.19}$$

so the series converges. Moreover, since $\beta\,\overline{u} < 1$, it is possible to choose $c > \overline{u}$ with $\beta\, c < 1$. Following Theorem 4 in Rincón-Zapatero and Rodríguez-Palmero (2003), $T = (T_o, T_r)$ is a local contraction on $\mathcal{F}\times\mathcal{F}$, so T admits a unique fixed point in $\mathcal{F}\times\mathcal{F}$, that is, there are unique $W_o \in \mathcal{F}$, $W_r \in \mathcal{F}$ such that $T_o(W_o,W_r) = W_o$ and $T_r(W_o,W_r) = W_r$. Also, $T_b(W_o,W_r) = W_b$ is the buyer's value function.

To prove that W_o and W_r are increasing in a, let $z \in Z$ be fixed and let $a_1 < a_2$. Then $\overline{D}(a_1) \subseteq \overline{D}(a_2)$, since $\hat{p}$, as the composition of two decreasing functions, is increasing. Let $(f_o,f_r) \in \mathcal{F}\times\mathcal{F}$, where both f_o and f_r are non decreasing. Then $m_b\, f_o\,(a-(1+\tau_b)\,\hat{p}(m_b)\hbar, z) + (1-m_b)\ f_r\,(a,z)$ is non decreasing in a, since $0 \leq m_b < 1$. Hence,

$$\begin{aligned} &\max_{m_b\in\overline{D}(a_1)} \left\{ m_b\, f_o\,(a_1-(1+\tau_b)\,\hat{p}(m_b)\hbar, z) + (1-m_b)\ f_r\,(a_1,z) \right\} \\ &\qquad \leq \max_{m_b\in\overline{D}(a_1)} \left\{ m_b\, f_o\,(a_2-(1+\tau_b)\,\hat{p}(m_b)\hbar, z) + (1-m_b)\ f_r\,(a_2,z) \right\} \\ &\qquad \leq \max_{m_b\in\overline{D}(a_2)} \left\{ m_b\, f_o\,(a_2-(1+\tau_b)\,\hat{p}(m_b)\hbar, z) + (1-m_b)\ f_r\,(a_2,z) \right\}. \end{aligned}$$

It follows that $T_b(f_o, f_r)$ is continuous and, being the maximum of two non-decreasing functions, it is also non decreasing. Plugging this result into the definitions of T_o and T_r, we get, by the same reasoning, that both $T_b(f_o, f_r)$ and $T_r(f_o, f_r)$ are non decreasing, since the feasible correspondence of both problems is increasing in a. Actually, both $T_b(f_o, f_r)$ and $T_r(f_o, f_r)$ are strictly increasing, since the utility functions are increasing. Finally, the subset of non-decreasing functions of $\mathcal{F}$ is closed, so the fixed points W_o, W_r and W_b are non decreasing. However, in the case of W_o and W_r, they are increasing by the previous argument, as they satisfy $T_o(W_o, W_r) = W_o$ and $T_b(W_o, W_r) = W_b$, respectively. □

The general theorem above applies to the utility functions used in the calibration of the model in Section 3.

Corollary 2. *The conclusions of Theorem 1 hold under the same hypotheses when*

$$u(c,h) = \frac{c^{1-\sigma}}{1-\sigma} + \phi \frac{h^{1-\sigma}}{1-\sigma}, \quad \phi > 0,$$

for any $\sigma \geq 1$, or when $\sigma < 1$ but $R^{1-\sigma}\beta < 1$.

Note that $\sigma = 1$ corresponds to $u(c,h) = \ln c + \phi \ln h$.

Proof. We only need to show that (A.12) holds. Note that $u(\cdot, h)$ is increasing in cases 1 and 2. When $\sigma > 1$, u is negative and bounded above. The sequence $\{u_j\}$ defined just above Theorem 1, being increasing and bounded is convergent, thus $\overline{u} = 1 < \frac{1}{\beta}$. When $\sigma < 1$, u is positive but unbounded from above. Given the definition of a_j made n the proof of Theorem 1, it is immediate to see that

$$\overline{u} = \lim_{j\to\infty} \frac{u^o_{j+1}}{u^o_j} = \lim_{j\to\infty} \frac{\phi\left(w\, z_n + a_{j+1} + \frac{(1-\zeta)\overline{p}}{R}\right)^{1-\sigma} + v(\hbar)}{\phi\left(w\, z_n + a_j + \frac{(1-\zeta)\overline{p}}{R}\right)^{1-\sigma} + v(\hbar)} = R^{1-\sigma}, \tag{A.20}$$

hence $R^{1-\sigma} < \frac{1}{\beta}$ assures that the hypothesis of Theorem 1 are fulfilled. In the logarithmic case, where $\sigma = 1$, u^o_j is bounded by $\left|\log\left(w + a_j + \frac{(1-\zeta)\overline{p}}{R}\right)\right| + \phi|\log\hbar|$ for large enough j. The ratio

$$\frac{|\log\left(w + a_{j+1} + \frac{(1-\zeta)\overline{p}}{R}\right)| + \phi|\log\hbar|}{|\log\left(w + a_j + \frac{(1-\zeta)\overline{p}}{R}\right)| + \phi|\log\hbar|} \tag{A.21}$$

tends to 1 as $j \to \infty$, so (A.12) is satisfied. A similar computation holds for u^r_j. □

B Differentiability, Euler equations and concavity

In this section we prove differentiability of the value functions along the optimal paths, obtain rigorously the Euler equations and prove concavity of the value functions in the participation region. Our results are based on a generalization of the Envelope Theorem that we develop in Theorem 3, and on the approach recently introduced in Rincón-Zapatero (2019) for dealing with non-concave stochastic dynamic programming problems. Theorem 3 characterizes the so-called Fréchet differentials of the value function, which is a rather weak concept of differentiability. This is specially well suited for studying the household's problem, where, aside from non concavity, it is not legitimate to assume differentiability of the buyer's value function in the definition of (A.1) and (A.2). This is the main reason for which other approaches to prove differentiability of the value function in a non-concave framework (as those explored in Dechert and Nishimura (1983), Milgrom and Segal (2002), or Clausen and Strub (2016)) do not apply to our setting (Menzio et al. (2013b) in a related model enumerate another reasons that also apply to our model). Thanks to the results that we introduce in this section, we do not need to introduce lotteries but work directly within the original non-concave framework. We prove rigorously that the Euler equations still hold as necessary conditions of optimality, so they can be used to compute the optimal policies. We establish a link between the concavity of the value functions and the monotonicity of the optimal consumption policies.

We introduce the concepts of Fréchet super– and subdifferentials of a function (F-superdifferential and F-subdifferential, henceforth) to simplify the presentation and the proofs that follow. For a continuous function $f : \Omega \subseteq \mathbf{R}^n \to \mathbf{R}$, where Ω is an open set, the vector $p \in \mathbf{R}^n$ belongs to the F-superdifferential of f at $x_0 \in \Omega$, $D^+f(x_0)$, if and only if there exists a continuous function $\varphi : \Omega \longrightarrow \mathbf{R}$ which is differentiable at x_0 with $D\varphi(x_0) = p$, $f(x_0) = \varphi(x_0)$ and $f - \varphi$ has a local maximum at x_0. Similarly, $p \in \mathbf{R}^n$ belongs to the F-subdifferential of f at $x_0 \in \Omega$, $D^-f(x_0)$, if and only if there exists a continuous function $\varphi : \Omega \longrightarrow \mathbf{R}$ which is differentiable at x_0 with $D\varphi(x_0) = p$, $f(x_0) = \varphi(x_0)$ and $f - \varphi$ has a local minimum at x_0. $D^+f(x_0)$ and $D^-f(x_0)$ are closed convex (and possible empty) subsets of $\mathbf{R}^n$. Yet, if f is differentiable at x_0, then both $D^+f(x_0)$ and $D^-f(x_0)$ are nonempty and $D^+f(x_0) = D^-f(x_0) = \{Df(x_0)\}$. Reciprocally, if for a function f, both $D^+f(x_0)$ and $D^-f(x_0)$ are nonempty, then f is differentiable at x_0 and $D^+f(x_0) = D^-f(x_0) = \{Df(x_0)\}$, where Df denotes the derivative of f. Given two continuous functions f_1 and f_2, two non-negative numbers λ_1 and λ_2 and $p_i \in D^+f_i(x)$, for $i = 1, 2$, $\lambda_1 p_1 + \lambda_2 p_2 \in D^+(\lambda_1 f_1 + \lambda_2 f_2)(a)$. A similar proposition holds for D^-. Another property that we will use is that, whenever x_0 is a local maximum of f in Ω, $0 \in D^+f(x_0)$. Finally, $D^+f(x_0) \neq \emptyset$ if the function f is concave. See, for instance, Bardi and Capuzzo-Dolcetta (1997) for these and for other properties of the F-super– and subdifferentials of a function.

The next theorem characterizes the F-differentials of the value function

$$f(x) = \max_{y \in \Gamma(x)} F(x, y),$$

where $F : X \times Y \to \mathbf{R}$ is continuous, with $X, Y \subseteq \mathbf{R}^n$, and where Γ is a correspondence from X to Y is nonempty, compact valued and continuous. The result is well known in the case in which the correspondence Γ is constant (i.e., when $\Gamma(x) = Y$ for all $x \in X$), but for the general case it is a generalization of the Benveniste–Scheinkman envelope argument which applies to non-concave problems.

Theorem 3. *Consider the problem described above, $f(x) = \max_{y \in \Gamma(x)} F(x,y)$. Let x_0 be an interior point of X and $y_0 \in \Gamma(x_0)$ satisfying:*

(i) $f(x_0) = F(x_0, y_0)$, and

(ii) there is a ball $B(x_0, \varepsilon)$ in X with center x_0 and radius $\varepsilon > 0$, such that for all $x \in B(x_0, \varepsilon)$, $y_0 \in \Gamma(x)$.

Then $D_x^- F(x_0, y_0) \subseteq D^- f(x_0)$ and $D^+ f(x_0) \subseteq D_x^+ F(x_0, y_0)$, where $D_x^{\pm} F(x_0, y_0)$ denotes the F–upper/lower differential of the function $x \mapsto F(x, y_0)$.

Proof. By Bergé's Theorem, f is continuous and the optimal policy correspondence is nonempty. Assumptions (i) and (ii) ensure that the function $x \mapsto f(x) - F(x, y_0)$ is well defined on the ball $B(x_0, \varepsilon)$ and attains a local minimum at x_0. If $D_x^- F(x_0, y_0)$ is empty, there is nothing to prove. Suppose that it is nonempty. Let φ be continuous in $B(x_0, \varepsilon)$ and differentiable at x_0 such that $F(x, y_0) - \varphi(x)$ has a local minimum at x_0 and $F(x_0, y_0) = \varphi(x_0)$. Then $f(x) - \varphi(x) \geq F(x, y_0) - \varphi(x) \geq 0$ and $f(x_0) - \varphi(x_0) = F(x_0, y_0) - \varphi(x_0) = 0$ by (i). Thus x_0 is a local minimum of $f - \varphi$, and so $D\varphi(x_0) \in D^- f(x_0)$. Now, if $D^+ f(x_0) = \emptyset$ then $D^+ f(x_0) \subseteq D_x^+ F(x_0, y_0)$, trivially. If $D^+ f(x_0) \neq \emptyset$, let φ be continuous in $B(x_0, \varepsilon)$ such that $D\varphi(x_0) \in D^+ f(x_0)$ and $f - \varphi$ has a local maximum at x_0, with $(f - \varphi)(x_0) = 0$. Then $F(x, y_0) - \varphi(x) \leq f(x) - \varphi(x) \leq 0 = F(x_0, y_0) - \varphi(x_0)$, for all $x \in B(x_0, \varepsilon)$. Hence, x_0 is a maximum of $x \mapsto F(x, y_0) - \varphi(x)$, and so $D\varphi(x_0) \in D_x^+ F(x_0, y_0)$. □

Remark 4. *The theorem is a generalization of the classical Envelope Theorem of dynamic programming, since when the value function f is concave, $D^+ f(x_0) \neq \emptyset$. If F is differentiable with respect to a then $D_x^- F(x_0, y_0) \neq \emptyset$, and hence $D^- f(x_0) \neq \emptyset$. Both Fréchet differentials of f are then non empty and thus f is differentiable. Note that $D_x^- F(x_0, y_0) \neq \emptyset$ is much weaker than the assumption of differentiability of F. On the other hand, (ii) is satisfied when (x_0, y_0) is an interior point of the graph of Γ, although it may be fulfilled more generally, as we will show in our housing model.*

We will apply the above theorem to show the validity of the Euler equations in our model, which is a non-trivial issue due to the lack of concavity. Although the household problem we study is stochastic, the theorem adapts easily since the set of shocks is finite. The properties of differentiability and concavity of the functions involved in our model have to be understood once $z \in Z$ is fixed. In particular, we will use the same notation $D^{\pm} f(x, z)$ for the upper or lower differential of the mapping $x \mapsto f(x, z)$, where z is fixed, for a function f that depends on the variables (x, z). Also, we will use the notation $f'(x, z)$ for the derivative of f with respect to x with preference over the more involved $D_x f(x, z)$ or $\frac{\partial f}{\partial x}(x, z)$, since z plays the role of an exogenous parameter.

After this preliminary exposition, we turn to our specific problem, given by (A.1)–(A.3). In the results that follow, we will assume that there are selections of g_o^a, g_r^a, g_r^h and g_b^θ such that g_o^a and g_r^a are interior, and

$$0 \leq g^\theta(a, z) < p^{-1} \left(\frac{a + (1 - \zeta)\bar{p}\hbar}{(1 + \tau_b}) \hbar \right), \tag{B.1}$$

for all $a \in A$. We do not assume uniqueness of the optimal policies. From (A.2), the renter's consumption and housing choices, when interior, are related by the optimality condition

$$r_h u_c(g_r^c(a,z), \omega g_r^h(a,z)) = \omega u_h(g_r^c(a,z), \omega g_r^h(a,z)).$$

Thus, since u is concave, the assumption $u_{ch} > 0$ guarantees that g_r^c and g_r^h have the same monotonicity properties with respect to a. We will assume that u is of class C^2 and that $u_{ch} > 0$ holds.

Our strategy for proving that the value functions are differentiable at the optimal policies, consists of showing that both the F-subdifferential and the F-superdifferential of the continuation value functions $E_z W_o$ and $E_z W_b$ are nonempty. This is key to show the validity of the Euler equations and to link concavity of W_o and W_r with the renter's and owner's optimal consumption being non-decreasing. The Euler equations are used in the computation part of the model combined with endogenous grid method (see Section D.2.2) and concavity allows us to prove differentiability of the value functions, which is used to derive the sorting result and to characterize the participation thresholds in the competitive search market (see Section 2.5.2). All this program is made possible thanks to Theorem 3, complemented with the results obtained in Rincón-Zapatero (2019). However, this approach does no apply directly to the Bellman equations satisfied by W_o, W_r and W_b, due to their complex structure, so we need to elaborate a bit more.

Lemmas 5, 6 and 7 below deal with the F-differentials of the value functions, Propositions 8 and 9 establish the Euler equations for renters and owners, and differentiability of $E_z W_b$ and $E_z W_o$, respectively. Concavity of W_r and W_o is proved in Propositions 11 and 12. Differentiability of the value functions W_r and W_o at the optimal policies is proved in Corollary 13.

Lemma 5. *Let $a_0 > \underline{a}$ and $z \in Z$. Then*

(i) $u_c(g_o^c(a_0,z), \hbar) \in D^- W_o(a_0,z)$, *and*

(ii) $u_c(g_r^c(a_0,z), \omega g_r^h(a_0,z)) \in D^- W_r(a_0,z)$.

Proof. For $a_0 > \underline{a}$ and $z \in Z$, $W_o(a_0,z)$ and $W_r(a_0,z)$ satisfy the Bellman equations (A.1) and (A.2), respectively. Since both $g_o^a(a_0,z)$ and $g_r^a(a_0,z)$ are interior and the feasible correspondence is a closed interval, there is an open interval I, centered at a_0, such that both $g_o^a(a_0,z)$ and $g_r^a(a_0,z)$ belong to $D(a)$ for all $a \in I$. Thus (i) and (ii) in Theorem 3 hold. To prove statement (i) in the lemma, consider the function F defined by

$$\begin{aligned} F(a, g_o^a(a_0,z), z) = u\left(wz + a - \delta \overline{p}\hbar - g_o^a(a_0,z)/R, \hbar\right) + \beta\left(1-\pi\right) E_z W_o\left(g_o^a(a_0,z), z'\right) \\ + \beta\pi\, E_z W_b\left(g_o^a(a_0,z) + (1-\tau_s)\overline{p}\hbar, z'\right), \end{aligned}$$

which is differentiable with respect to a, with derivative $u_c(g_o^c(a_0,z), \hbar)$ at $a = a_0$, since the second and third summands in the definition of F are constant. Note that $W_o(a_0,z) = F(a_0, g_o^a(a_0,z), z)$ and $W_o(a,z) \geq F(a, g_o^a(a_0,z), z)$. Thus Theorem 3 implies $u_c(g_o^c(a_0,z), \hbar) \in$

$D^-W_o(a_0,z)$. In order to prove statement (ii), let now the function F be defined by

$$F(a, g_r^a(a_0,z), z) = u\left(wz + a - r_h h - g_r^a(a_0,z)/R, \omega g_r^h(a_0,z)\right) + \beta\, E_z W_b\left(g_r^a(a_0,z), z'\right),$$

which is differentiable with respect to a, with derivative $u_c(g_r^c(a_0,z), \omega g_r^h(a_0,z))$ at $a = a_0$. Note that $W_r(a_0,z) = F(a_0, g_r^a(a_0,z), z)$ and $W_r(a,z) \geq F(a, g_r^a(a_0,z), z)$. Hence, Theorem 3 implies $u_c(g_r^c(a_0,z), \omega g_r^h(a_0,z)) \in D^-W_r(a_0,z)$. □

To prove that D^-W_b is nonempty is a bit more involved. We rewrite the problem of a potential buyer in an equivalent form. Let us define $a_{\min} = (1+\tau_b)\,\overline{p}\hbar - (1-\zeta)\,\overline{p}\hbar = (\zeta+\tau_b)\overline{p}\hbar$. This is the threshold value of a above which $D(a)$, as defined in (A.4), is nonempty. Remember the definition of $a_{\text{part}}(z) > a_{\min}$ as the maximum $a > a_{\min}$ such that $g^\theta(a,z) = \theta_0$ (if it exists.) Let, for $z \in Z$, the function

$$W(a, m_b, z) = \begin{cases} W_r(a,z), & \text{if } a \leq a_{\min},\ m_b \in [0,1], \\ m_b\left(W_o(a - (1+\tau_b)\hat{p}(m_b)\hbar, z) - W_r(a,z)\right) + W_r(a,z), & \text{if } a > a_{\min},\ m_b \in \overline{D}(a), \end{cases} \tag{B.2}$$

where $\hat{p}(m_b)$ in (A.6) and $\overline{D}(a)$ in (A.7). Let $\widetilde{D}(a) = \{0\}$ for $a \leq a_{\min}$, and $\widetilde{D}(a) = \overline{D}(a)$ for $a > a_{\min}$. The correspondence $\widetilde{D}$ is nonempty, compact valued and continuous. Formally, we are identifying the choice θ_0 in the original problem with $m_b = 0$. Given this, it is clear that the original problem is equivalent to the following new formulation: $\max W(a, m_b, z)$ subject to $m_b \in \widetilde{D}(a)$. Note that W is piecewise continuous and, when restricted to the graph of $\widetilde{D}$, it is continuous. To see this, let $(a_n, (m_b)_n)$ be a sequence converging to $(a_{\min}, m_b)$ along the graph of $\widetilde{D}$, where $m_b \in [0,1]$, then for $a_n > a_{\min}$, $(m_b)_n = \widehat{p}^{-1}(a_n) \to \widehat{p}^{-1}(a_{\min}) = 0$, and for $a_n < a_{\min}$, $(m_b)_n = 0$. Hence,

$$W(a_n, (m_b)_n, z) \to 0 \cdot (W_o(0,z) - W_r(a_{\min}), z) + W_r(a_{\min}, z) = W_r(a_{\min}, z) = W(a_{\min}, 0, z),$$

as $n \to \infty$. Since $m_b = 0$ is feasible for any a and $g_b^\theta(a,z) = 0$ in the region $a \leq a_{\text{part}}(z)$, $W_b(a,z) = W_r(a,z)$ in this region.

Lemma 6. *Let $a_0 > \underline{a}$ and $z \in Z$. Then $D^-W_b(a_0,z) = D^-W_r(a_0,z)$, for $a_0 < a_{\text{part}}(z)$, and*

$$m_b\left(g_b^\theta(a_0,z)\right) p_o + \left(1 - m_b\left(g_b^\theta(a_0,z)\right)\right) p_r \in D^-W_b(a_0,z), \quad \textit{for } a_0 > a_{\text{part}}(z), \tag{B.3}$$

where $p_o = u_c\left(g_o^c(a_0 - (1+\tau_b)\,p(g_b^\theta(a_0,z)), z), \hbar\right)$ and $p_r = u_c(g_r^c(a_0,z), \omega g_r^h(a_0,z))$.

Proof. For $\underline{a} < a < a_{\text{part}}(z)$, $W_b(a,z) = W_r(a,z)$, so (i) is trivial. Let now $a_0 > a_{\text{part}}(z)$. Since g_b^θ is interior, the optimal $g^{m_b}(a_0,z)$ is interior. Thus the function of a

$$F(a, g^{m_b}(a_0,z), z) = g^{m_b}(a_0,z) W_o(a - (1+\tau_b)\hat{p}(g^{m_b}(a_0,z))\hbar, z) + (1 - g^{m_b}(a_0,z)) W_r(a,z)) \tag{B.4}$$

is well defined in a suitable interval centered at a_0. Although we can not assert that F is differentiable with respect to a, as we have not proved yet differentiability of W_o and W_r, we can prove that[18] $D_a^- F(a_0, g^{m_b}(a_0, z), z) \neq \emptyset$. To see this, take

$$p_o \in D^- W_o\left(a_0 - (1+\tau_b)\,\hat{p}(g^{m_b}(a_0, z)\hbar), z\right) \quad \text{and} \quad p_r \in D^- W_r(a_0, z),$$

which exist by Lemma 5. By the property of convexity of the differentials mentioned just above Theorem 3, $g^{m_b}(a_0, z)\, p_o + (1 - g^{m_b}(a_0, z))\, p_r \in D_a^- F(a_0, g^{m_b}(a_0, z), z)$, or, equivalently,

$$m_b\left(g_b^\theta(a_0, z)\right)\, p_o + \left(1 - m_b\left(g_b^\theta(a_0, z)\right)\right)\, p_r \in D_a^- F(a_0, g_b^\theta(a_0, z), z), \tag{B.5}$$

with p_o and p_r as described in the statement of the lemma. Since $D_a^- F(a_0, g_b^\theta(a_0, z), z) \subseteq D^- W_b(a_0, z)$ by Theorem 3, the result in the lemma holds. □

The fact that the lower F–subdifferential of the value function is nonempty is not enough to get differentiability, since the value functions need not be concave and hence the F-superdifferential could be empty. Below we follow the path initiated in Rincón-Zapatero (2019) to prove differentiability in the absence of concavity, which uses the optimality condition in the Bellman equation, where the value function appears both at the left and the right of the equality defining the functional equation. This will provide us with conditions for the nonemptyness of the F-superdifferential of the value functions at the optimal policies.

The following results deal with the F-superdifferentials of the value functions. Actually, due to the stochastic nature of the problem, what is characterized is the F-superdifferentials of the expected value functions (at the optimal policies). In consequence, what can be asserted with full generality is the differentiability of the expected value functions, and not the value functions itself. This was pointed out for the first time in Rincón-Zapatero (2019).

Lemma 7. *Let* $a_0 > \underline{a}$. *Then* $\frac{1}{\beta R} u_c(g_r^c(a_0, z), \omega g_r^h(a_0, z)) \in D^+ E_z W_b(g_r^a(a_0, z), z')$.

Proof. Consider the Bellman equation (A.2) and the function of a' given by

$$F(a_0, a', z) := u(wz + a_0 - r_h h - a'/R, \omega g_r^h(a_0, z)) + \beta E_z W_b(a', z'). \tag{B.6}$$

Since $g_r^a(a_0, z)$ is an interior maximizer to the Bellman equation (A.2), $0 \in D_{a'}^+ F(a_0, g_r^a(a_0, z), z)$. But, since u is differentiable, $D_{a'}^+ F = \{-u_c/R\} + \beta D^+ E_z W_b$, where we have omitted the arguments. Hence, $\frac{1}{\beta R} u_c(g_r^c(a_0, z), \omega g_r^h(a_0, z)) \in D^+ E_z W_b(g_r^a(a_0, z), z')$. □

Our next result shows that $E_z W_b$ is differentiable at the renter's optimal policy, and establishes the validity of the renter's Euler equation.

[18] This is one of the advantages of working with F-sub or superdifferentials, and a sample of the usefulness of Theorem 3 and how it relax the classical assumption of differentiability. Note that at this stage nothing is known about the differentiability of $E_z W_o$ and $E_z Wb$, and consequently about the auxiliary function F; without resorting to the a weaker concept of differentiability as the Fréchet–differentials, we could not move forward.

Proposition 8. *Let $a > \underline{a}$, $z \in Z$. Then $E_z W_b$ is differentiable at $a' = g_r^a(a,z) \neq a_{\text{part}}(z)$, with derivative*

$$[E_z W_b]'(a', z') = \frac{1}{\beta R} u_c(g_r^c(a', z), \omega g_r^h(a', z))$$

and the Euler equation

$$-\frac{1}{\beta R} u_c(g_r^c(a,z), \omega g_r^h(a,z)) + E_z u_c(g_r^c(a', z'), \omega g_r^h(a', z')) = 0,$$

for $a' = g_r^a(a,z) < a_{\text{part}}(z)$ and

$$-\frac{1}{\beta R} u_c(g_r^c(a,z), \omega g_r^h(a,z)) + E_z \Bigg(m_b\left(g_b^\theta(a', z')\right) u_c\left(g_o^c(a' - (1+\tau_b)\, p(g_b^\theta(a', z'))), \hbar\right) \\ + \left(1 - m_b\left(g_b^\theta(a', z')\right)\right) u_c(g_r^c(a', z'), \omega g_r^h(a', z')) \Bigg) = 0,$$

for $a' = g_r^a(a,z) > a_{\text{part}}(z)$, holds.

Proof. Let $a > \underline{a}$ and $z \in Z$ such that $a' = g_r^a(a,z) \leq a_{\text{part}}(z)$. By Lemma 5 and Lemma 6, $u_c(g_r^c(a_0), \omega g_r^h(a_0, z)) \in D^- W_b(a_0, z)$. By the properties of the differentials listed above,

$$E_z u_c(g_r^c(a', z'), \omega g_r^h(a', z')) \in E_z D^- W_b(a', z'). \tag{B.7}$$

Similarly, if $g_r^a(a,z) > a_{\text{part}}(z)$, we have

$$E_z \Bigg(m_b\left(g_b^\theta(a', z')\right) u_c\left(g_o^c(a' - (1+\tau_b)\, p(g_b^\theta(a', z'))), \hbar\right) \\ + \left(1 - m_b\left(g_b^\theta(a', z')\right)\right) u_c(g_r^c(a', z'), \omega g_r^h(a', z')) \Bigg) \tag{B.8}$$

belongs to $D^- E_z W_b(a', z')$, where $a' = g_r^a(a,z)$. By Lemma 7, the F-superdifferential $D^+ E_z W_b(g_r^a(a,z), z')$ is nonempty, for all $a > \underline{a}$. Hence, $E_z W_b(\cdot, z)$ is differentiable at $g_r^a(a,z)$, $D^- E_z W_b(g_r^a(a,z), z') = D^+ E_z W_b(g_r^a(a,z), z')$, and these two sets are singletons. By Lemma 7, the unique element of $D^+ E_z W_b(g_r^a(a,z), z')$ is $\frac{1}{\beta R} u_c(g_r^c(a,z), \omega g_r^h(a,z))$, which has to be the unique element of $D^- E_z W_b(g_r^a(a,z), z')$ given in (B.7) and (B.8) above, obtaining in this way the renter's Euler Equation and the expression for the derivative stated in the lemma. □

Differentiability of $E_z W_b$ proved above will be used to prove differentiability of $E_z W_o$ and to obtain the owner's Euler equation.

Proposition 9. *Let $a > \underline{a}$, $z \in Z$. Then $E_z W_o$ is differentiable at $a' = g_o^a(a,z)$, with*

$a' \neq a_{\text{part}}(z) - (1-\tau_b)\overline{p}\hbar$, *with derivative*

$$[E_z W_o]'(a', z') = u_c(g_o^c(a', z), \hbar)$$

and the Euler equation

$$\beta R(1-\pi)\, u_c(g_r^c(a', z), \hbar) - u_c(g_o^c(a, z), \hbar) + \beta R \pi \; [E_z W_b]'(a' + (1-\tau_s)\overline{p}\hbar, z') = 0.$$

holds.

Proof. From (A.1), the function of a'

$$F(a, a', z) = u\left(wz + a - \delta \overline{p}\hbar - a'/R, \hbar\right) + \beta\,(1-\pi)\, E_z W_o\left(a', z'\right) + \beta\pi\, E_z W_b\left(a' + (1-\tau_s)\overline{p}\hbar, z'\right)$$

satisfies $0 \in D^+_{a'} F(a, g_o^a(a, z), z)$. Since $E_z W_b$ is differentiable by Proposition 8, we have by the properties of the differentials[19]

$$-\frac{1}{R} u_c(g_o^c(a, z), \hbar) + \beta\pi\; [E_z W_b]'(a' + (1-\tau_s)\overline{p}\hbar, z') \in -\beta\,(1-\pi) D^+ E_z W_o(a', z'),$$

showing that $D^+ E_z W_o$ is nonempty at $a' = g_o^a(a, z)$. This, combined with Lemma 5, and a reasoning similar to the proof of Proposition 8 above, imply that $E_z W_o$ is differentiable at $g_o^a(a, z)$ and that the derivative is given by the unique element in $D^- E_z W_o$, that is $[E_z W_o]'(a', z) = u_c(g_o^c(a', z), \hbar)$. Finally, the equality $D^- E_z W_o(a', z') = D^+ E_z W_o(a', z')$, gives the owner's Euler equation. □

A more explicit expression of the Euler equation id obtained after replacing $[E_z W_b]'(a' + (1-\tau_s)\overline{p}\hbar, z')$ by the value obtained in (B.8) above.

We now study concavity. Given an exogenous shock z, concavity of the value functions with respect to the endogenous variable a is proved in intervals where the renter's optimal consumption policy is non decreasing (to be precise, a suitable selection of g_r^c). We first establish concavity of $E_z W_b$.

Lemma 10. *Let $z \in Z$. Let I' be a subinterval of the image of $g_r^a(\cdot, z)$ such that $a_{\text{part}}(z) \notin I'$. Then $E_z W_b$ is concave in I' if and only if $g_r^c(\cdot, z)$ is nondecreasing in the inverse image of I', $(g_r^a)^{-1}(I', z) = \{a > \underline{a} \,:\, g_r^a(a, z) \in I'\}$.*

Proof. Let $a_i' \in I'$ and let $a_i > \underline{a}$ such that $a_i' = g_r^a(a_i, z)$, for $i = 1, 2$. Without loss of generality, suppose that $a_1' < a_2'$. By the Mean Value Theorem

$$E_z W_b(a_2', z') - E_z W_b(a_1', z') = [E_z W_b]'(a_z', z')(a_1' - a_2') = \frac{1}{\beta R} u_c(g_r^c(a_z', z), \omega g_r^h(a_z', z))(a_2' - a_1'), \tag{B.9}$$

[19] We are implicitly assuming that there is some asset value $\widetilde{a} \in A$ such that $g_o^a(a, z) + (1-\tau_s)\overline{p}\hbar = g_r^a(\widetilde{a}, z)$. Since g_r^a is an upper semicontinuous and unbounded correspondence, this must hold.

where $a'_1 < a'_z < a'_2$. If g^c_r is non decreasing with respect to a, then $g^c_r(a'_1, z) \leq g^c_r(a'_z, z)$ and $g^h_r(a'_1, z) \leq g^h_r(a'_z, z)$; since u is concave, u_c is decreasing, thus

$$u_c(g^c_r(a'_z, z), \omega g^h_r(a'_z, z)) \leq u_c(g^c_r(a'_1, z), \omega g^h_r(a'_1, z))$$

Plugging this inequality into (B.9), we have

$$E_zW_b(a'_2, z') - E_zW_b(a'_1, z') \leq \frac{1}{\beta R} u_c(g^c_r(a'_1, z), \omega g^h_r(a'_1, z))(a'_2 - a'_1) = [E_zW_b]'(a'_1, z')(a'_2 - a'_1),$$

proving that E_zW_b is concave in I'. Obviously, the reasoning above is reversible, that is, if E_zW_b is concave in I', then g^c_r is non decreasing with respect to a. □

Proposition 11. *Let $z \in Z$. Let I be an interval of A such that $g^a_r(I, z) = \{a' \in A : a' = g^a_r(a, z), a \in A\}$ is an interval and $a_{\text{part}}(z) \notin g^a_r(I, z)$. Then W_r is strictly concave in I if and only if $g^c_r(\cdot, z)$ is non decreasing in I.*

Proof. Let $a_1, a_2 \in I$ and let $\lambda_1, \lambda_2 \in [0, 1]$. Since $\lambda_1 a_1 + \lambda_2 a_2 \in I$ and $g^a_r(I, z)$ is an interval, $\lambda_1 g^a_r(a_1, z) + \lambda_2 g^a_r(a_2, z) \in g^a_r(I, z)$. Hence, $(\lambda_1 a_1 + \lambda_2 a_2, \lambda_1 g^a_r(a_1, z) + \lambda_2 g^a_r(a_2, z))$ belongs to the graph of the buyer's feasible correspondence, since it is convex. Moreover, W_b is concave on $g^a_r(I, z)$ by Lemma 10, from which is follows that W_r is concave. Let, to simplify the notation, $(g^i_r)^\lambda = \lambda_1 g^i_r(a_1, z) + \lambda_2 g^i_r(a_2, z)$, for $i = c, a, h$. Then

$$\begin{aligned} W_r(\lambda_1 a_1 + \lambda_2 a_2, z) &\leq u((g^c_r)^\lambda), \omega(g^h_r)^\lambda)) + \beta W_b((g^a_r)^\lambda), z) \\ &\leq \lambda_1 u(g^a_r(a_1, z), \omega g^h_r(a_1, z))) + \lambda_2 u(g^a_r(a_2, z), \omega g^h_r(a_2, z))) \\ &\quad + \beta\lambda_1 W_b(g^a_r(a_1), z) + \beta\lambda_1 W_b(g^a_r(a_1), z) \\ &= \lambda_1 W_r(a_1, z) + \lambda_2 W_r(a_2, z), \end{aligned}$$

where we have used (A.2), that u is concave and that E_zW_b is concave in the image of g^a_r. Hence, W_r is concave in I. Strict concavity of W_r follows from strict concavity of u. □

Proposition 12. *Let $z \in Z$. Let I be an interval of A such that both $g^a_o(I, z)$ and $g^a_r(I, z)$ are intervals, $a_{\text{part}}(z) \notin g^a_r(I, z)$ and $\{(1 - \tau_s)\overline{p}\hbar\} + g^a_o(I, z) \subseteq g^a_r(I, z)$. Then W_o is strictly concave on I if and only if $g^c_r(\cdot, z)$ is non decreasing in I.*

Proof. We use the fact that the restriction of the operator T_o to the set $\mathcal{F}$ defined in (A.13) is a contraction. This restricted operator is defined in the obvious way. Suppose that $g^c_r(\cdot, z)$ is non decreasing in I. First, fix the buyer's value function W_b which, given the hypotheses of the proposition and Lemma 10, is concave on $g^a_r(I, z)$. The restricted operator is then

$$T^b_o(f_o)(a, z) = \max_{c, a'} \left\{ U^b(c, \hbar, z) + \beta(1 - \pi)E_z f_o(a', z) \right\}, \tag{B.10}$$

where $U^b(c, a', z) = u(c, \hbar) + \beta(1 - \alpha)E_zW_b(a' + (1 - \tau_b)\overline{p}\hbar, z)$ is strictly concave with respect to (c, a'). Hence, if $f_o \in F$ is concave in a, then $T^b_o f_o$ is concave in a and hence the limit of the iterating sequence $(T^b_o)^n$, W_o, is concave in a. Once this is proved, the dynamic programming equation (B.10) implies that W_o is in fact strictly concave, since U^b is strictly concave. □

Corollary 13. *Let $z \in Z$. Suppose that $I \subseteq (\underline{a}, \infty)$ is an open interval where the assumptions of Propositions 11 and 12 hold and where $g_r^c(\cdot, z)$ is non decreasing. Then the value functions W_r and W_o are differentiable on $I \cap g_r^c(A, z)$.*

Proof. The functions W_r and W_o are (strictly) concave on I, and the F–superdifferential of a concave function is nonempty. Since the F–subdifferential of W_r and W_o are nonempty at g_r^c by Lemma 5, the result follows from the properties of the differentials. □

C Proofs of Propositions 1, and 3 to 5

The characterization results in Section 2.5.2 of the main text follow from the properties of the value functions established in Sections A and B. Potential buyers solve problem (A.8), or, equivalently, the problem described right after Lemma 5. Under the conditions of Theorem 1, an optimal solution to this problem exists, by the Theorem of the Maximum. Since the price function $\hat{p}$ in (A.6) is strictly increasing and strictly convex, the concavity result in Propositions 11 and 12 imply that, conditional on participating in the afternoon market, the optimal solution is unique under the assumptions in Proposition 1.

Proof of Proposition 1. Consider the buyer's problem (A.8) formulated in terms of the decision variable m_b and the price function $\hat{p}(m_b)$. By the properties of $\hat{p}$ and the monotonicity and concavity of W_o, the function $m_b \mapsto W_o(a - (1 + \tau_b)\hat{p}(m_b)\hbar, z)$ is differentiable, decreasing and strictly concave. Let us denote this function by $\hat{W}_o$. Note that the function $m_b \mapsto m_b \hat{W}_o'(m_b)$ is decreasing, since $0 < m_b^1 < m_b^2$ implies $m_b^2 \hat{W}_o'(m_b^2) < m_b^1 \hat{W}_o'(m_b^2)$ and $m_b^1 W_o'(m_b^2) < m_b^1 W_o'(m_b^1)$, thus

$$m_b^2 \hat{W}_o'(m_b^2) < m_b^1 \hat{W}_o'(m_b^2) < m_b^1 W_o'(m_b^1).$$

If follows that $(m_b \hat{W}_o(m_b))' = \hat{W}_o(m_b) + m_b \hat{W}_o'(m_b)$ is decreasing, thus $m_b \hat{W}_o(m_b)$ is strictly concave. In consequence, the optimal m_b, and hence the optimal g_b^θ, is unique, for each z. Hence, by the Theorem of the Maximum, the policy function is continuous. □

Proposition 3 follows from the properties of W_o in Theorem 1, and Propositions 9 and 12.

Proof of Proposition 3. Since W_o is differentiable (Proposition 9), the optimal choice of a buyer with state (a, z) who participates in the afternoon market satisfies the first-order condition:

$$\begin{aligned} &W_o\left(a - (1 + \tau_b)\hbar\hat{p}(m_b), z\right) - W_r\left(a, z\right) - m_b(1 + \tau_b)\hbar\hat{p}'(m_b)W_o'\left(a - (1 + \tau_b)\hbar\hat{p}(m_b), z\right) \\ &= \hat{\lambda}(a, z)(1 + \tau_b)\hbar\hat{p}'(m_b), \end{aligned} \tag{C.1}$$

where $\hat{\lambda}(a, z)$ is the Lagrange multiplier of the borrowing constraint in (A.7). If $\hat{\lambda}(a, z) = 0$, (C.1) can be written as:

$$\left(\frac{1}{1+\tau_b}\right)\left(\frac{W_o\left(a-(1+\tau_b)\hbar p,z\right)-W_r\left(a,z\right)}{m_b\hbar W_o'\left(a-(1+\tau_b)\hbar p,z\right)}\right)=\hat{p}'(m_b). \tag{C.2}$$

The term in the left-hand side of (C.2) is the buyer's marginal rate of substitution of p for m_b. Buyers prefer high values of m_b and low values of p. Also, W_o is increasing and concave, given z (by Theorem 1 and Proposition 12). Hence, (C.2) has a unique solution (in line with Proposition 1), as the buyer's marginal rate of substitution falls as m_b and p increase along an indifference curve. In addition, if $(W_o\left(a-(1+\tau_b)\hbar p,z\right)-W_r\left(a,z\right))$ is non-decreasing in a for each z and each $p\geq\bar{p}$, the fact that W_o is strictly concave in a implies that the buyer's marginal rate of substitution increases with a. Hence, so does the optimal choice of m_b. More generally, this result holds if the second term in the left-hand side of (C.2) increases with a for each z and each $p\geq\bar{p}$. □

When the borrowing constraint binds for some buyers and is slack for other buyers with identical productivity z, the existence of a threshold $a_c(z)$ below which the constraint binds follows directly from the following result, which uses the differentiability of W_o and W_r and the strict monotonicity of W_r.

Lemma 14. *For a given $z\in Z$, if $a<a'$ and $\hat{\lambda}(a,z)$, $\hat{\lambda}(a',z)>0$ then $\hat{\lambda}(a',z)<\hat{\lambda}(a,z)$.*

Proof. If $\hat{\lambda}(a,z)>0$, the price paid by the buyer is $\frac{a/\hbar+(1-\zeta)\overline{p}}{(1+\tau_b)}$, and assets at night are $-(1-\zeta)\overline{p}\hbar$. Thus (C.1) implies

$$\begin{aligned}\hat{\lambda}(a,z)&=\frac{W_o\left(-(1-\zeta)\overline{p}\hbar,z\right)-W_r(a,z)}{(1+\tau_b)\hbar\hat{p}'(m_b)}-m_bW_o'\left(-(1-\zeta)\overline{p}\hbar,z\right)\\&=\frac{W_o\left(-(1-\zeta)\overline{p}\hbar,z\right)-W_r(a,z)}{(1+\tau_b)\hbar\hat{p}'(m_b)}-m_bu_c\left(g_o^c\left(-(1-\zeta)\overline{p}\hbar,z\right),\hbar\right),\end{aligned} \tag{C.3}$$

where the last equality follows from the Envelope Theorem. On the other hand, since $\hat{p}(m_b)$ is given by (A.6), m_b satisfies

$$\frac{(1-1/R+\delta)\overline{p}}{\hat{m}_s(m_b)}+(1/R-\delta)\overline{p}=\frac{a/\hbar+(1-\zeta)\overline{p}}{(1+\tau_b)}. \tag{C.4}$$

As a increases to a', m_b increases, since $\hat{m}_s$ is strictly decreasing. So does $\hat{p}'(m_b)$, since $\hat{p}$ is strictly increasing and strictly convex. Since W_r is strictly increasing by Theorem 1, (C.3) then implies $\hat{\lambda}(a',z)<\hat{\lambda}(a,z)$. □

Proposition 5 follows from the continuity and differentiability of W_b and W_r, and Proposition 1. The proof is based on the original problem in (2.9), where potential buyers choose θ.

Proof of Proposition 5. Let $\tilde{W}_b(a,z)$ denote the value of problem (2.26), and let $\tilde{g}_b^{\theta}(a,z)$ be the associated policy function. Then

$$W_b(a,z) = \max\{\tilde{W}_b(a,z), W_r(a,z)\}, \tag{C.5}$$

and $g_b^\theta(a,z) = \tilde{g}_b^\theta(a,z)$ if $W_b(a,z) = \tilde{W}_b(a,z) > W_r(a,z)$. Fix an arbitrary $z \in Z$. Since θ_0 is the only feasible choice for a potential buyer when $a \le a_{min} = (\tau_b + \zeta)\,\overline{p}\,\hbar$, on this range $W_b(a) = W_r(a)$. Suppose $a > a_{min}$, so the constraint set of problem (2.26) is nonempty. Applying the Envelope theorem to the Lagrangian of this problem yields

$$\tilde{W}_b'(a,z) - W_r'(a,z) = m_b\left(\tilde{g}_b^\theta(a,z)\right)\left(W_o'\left(a - (1+\tau_b)\hbar p(\tilde{g}_b^\theta(a,z))\right) - W_r'(a,z)\right) + \lambda(a,z). \tag{C.6}$$

The right-hand side of (C.6) is strictly positive because $m_b(\theta) > 0$ for all $\theta \in \mathbf{R}_+$, the term in brackets is strictly positive by assumption, and $\lambda(a,z) \ge 0$. Thus $\tilde{W}_b(a,z) - W_r(a,z)$ is strictly increasing in a for $a > a_{min}$. By assumption, $W_b(a,z) = \tilde{W}_b(a,z) > W_r(a,z)$ for some a. Since $\tilde{W}_b$ and W_r are continuous, there then exists $a_{part}(z)$ such that $\tilde{W}_b(a,z) > W_r(a,z)$ for all $a > a_{part}(z)$ and $\tilde{W}_b(a_{part}(z),z) = W_r(a_{part}(z),z)$. Given that $p(g_b^\theta(a)) > \overline{p}$ for $a > a_{part}(z)$, $p(\theta)$ is continuous, and so is $g_b^\theta(a)$ on this range (by Proposition 2.26), it follows that $p(\lim_{a\to a_{part}(z)^+} g_b^\theta(a_{part}(z))) > \overline{p}$. Hence, $a_{part}(z) > a_{min}$ and, by continuity, $p(g_b^\theta(a)) > \overline{p}$ for any $a < a_{part}(z)$ sufficiently close to a_{part}. Since $\tilde{W}_b(a,z) - W_r(a,z)$ is strictly increasing on this range, $W_r(a,z) > \tilde{W}_b(a,z)$ and so $g_b^\theta(a,z) = \{\theta_0\}$ for any $a < a_{part}(z)$. □

D Computation

In order to compute a stationary equilibrium, it is best to rewrite the problems of potential buyers and intermediaries as follows. Instead of choosing m_b taking $\hat{p}(m_b)$ as given, they choose p taking as given the inverse of the function $\hat{p}(m_b)$, which we denote by $m_b(p)$. For this, it is crucial that $m_b(\theta)$ is a function rather than a correspondence. In particular, we cannot use the standard "truncated" Cobb-Douglas matching function.

D.1 The matching function and the equilibrium price schedule

Given the Walrasian price $\overline{p}$, equation (A.6) determines m_s as a function of p:

$$m_s(p) = \frac{(1 - 1/R + \delta)\,\overline{p}}{p - (1/R - \delta)\,\overline{p}}. \tag{D.1}$$

This function is strictly decreasing and strictly convex with $m_s(\overline{p}) = 1$ and $\lim_{p\to\infty} m_s(p) = 0$, and does not depend on the choice of the matching technology. We take $m_s(\theta) = (1 + \theta^{-\gamma})^{\frac{-1}{\gamma}}$

with $\gamma > 0$, and $m_b(\theta) = m_s(\theta)/\theta$. Thus $\widehat{m}_s(m_b) = (1 - m_b^\gamma)^{1/\gamma}$, and we can write

$$m_b(p) = (1 - m_s(p)^\gamma)^{1/\gamma}, \tag{D.2}$$

$$\theta(p) = \frac{m_s(p)}{(1 - m_s(p)^\gamma)^{1/\gamma}}. \tag{D.3}$$

Here, $\theta(p)$ is the inverse of $p(\theta)$, so it is strictly decreasing and strictly convex with $\lim_{p\to\infty} \theta(p) = 0$ and $\lim_{p\to\overline{p}} \theta(p) = \infty$. Also, $m_b(p)$ is strictly increasing with $m_b(\overline{p}) = 0$ and $\lim_{p\to\infty} m_b(p) = 1$. As shown in Appendix A, $m_b(p)$ is strictly concave provided $-\widehat{m_s}'(m_b)/\hat{m}_s(m_b)$ is non decreasing. This last assumption can be further relaxed. For instance, for the value of γ used in our calibration to match the value of median TTB in the data (and, in fact, for any $\gamma < 1$), the assumption only holds for values of m_b above some threshold. Yet we only require that it holds for the range of values of m_b which correspond the submarkets that are active in equilibrium (since eliminating inactive submarkets does not change the problem of a potential buyer). One can easily verify that it suffices to check that the slope of $-\widehat{m}_s'(m_b)/\widehat{m}_s(m_b)$ is positive for the lowest value of m_b observed in equilibrium (which corresponds to the optimal choice of a marginal buyer). If so, $m_b(p)$ is strictly concave on the range of prices at which agents trade in equilibrium, and the results in Propositions 1, and 3 to 5 again hold.

D.2 The household's problem

Here we describe in detail the algorithm to solve the household's problem.

D.2.1 The optimal choice of potential buyers

In order to extend the method in Fella (2014) to our framework, we proceed in two steps. The problem of a potential buyer with state (a, z), where $a > a_{part}(z)$, can be written as

$$\begin{array}{ll} W_b(a,z) = & \max_p \left\{ W_r(a,z) + m_b(p) \left[W_o\left(a - (1+\tau_b)\, p\, \hbar\right) - W_r(a,z) \right] \right\} \\ \text{s. t.} & \overline{p}\, \hbar \le p\, \hbar \le \frac{a + (1-\zeta)\, \overline{p}\, \hbar}{(1+\tau_b)}, \end{array} \tag{D.4}$$

with associated policy function $g^p(a, z)$. Since $m_b(\overline{p}) = 0$, the constraint $p \geq \overline{p}$ does not bind. The buyer's gains from trading at price $p > \overline{p}$ are $S(a, z, p) = W_o\left(a - (1 + \tau)\, p\, \hbar, z\right) - W_r(a, z)$. By Theorem 1, $S(a, z, p)$ is strictly decreasing in p. If $S(a, z, \overline{p}) \leq 0$ then $S(a, z, p) < 0$ for all $p > \overline{p}$, and non-participation is optimal in this case. Suppose that $S(a, z, \overline{p}) > 0$, so the gains from participation are positive. It is direct to check from the first-order condition of problem (D.4) that the Lagrange multiplier of the borrowing constraint is given by $\lambda(a, z) = m_b'(p)[S(a, z, p) - \tilde{S}(a, z, p)]$, where

$$\tilde{S}(a,z,p) = \frac{m_b(p)}{m_b'(p)} u_c\left(g_o^c\left(a - (1+\tau_b)\, p\, \hbar, z\right), \hbar\right)(1+\tau_b).$$

Hence, at an optimal solution, $S(a,z,p) \geq \tilde{S}(a,z,p)$, with equality if the constraint does not bind. By the Envelope Theorem,

$$W_o'\left(a-(1+\tau)\,p\,\hbar,z\right)=u_c\left(g_o^c\left(a-(1+\tau_b)\,p\,\hbar,z\right),\hbar\right), \text{ so}$$
$$\tilde{S}(a,z,p)=\frac{m_b(p)}{m_b'(p)}(1+\tau_b)W_o'\left(a-(1+\tau)\,p\,\hbar,z\right).$$

If $g_o^c(a,z)$ is non-decreasing then W_o is concave, since u is strictly concave. Since m_b is strictly increasing and strictly concave, this implies that $\tilde{S}(a,z,p)$ is strictly increasing in p and non-increasing in a. Also, $\tilde{S}\left(a,z,\overline{p}\right)=0$ regardless of the value of a, since $m_b\left(\overline{p}\right)/m_b'\left(\overline{p}\right)=0$. There is then a unique value p_T which solves $S(a,z,p_T)=\tilde{S}(a,z,p_T)$ (in line with Proposition 1), and $S(a,z,p_T)>0$. There are then two cases: (i) if $p_T\,\hbar \leq \left(a+(1-\zeta)\,\overline{p}\,\hbar\right)/(1+\tau_b)$ then $g_p(a,z)=p_T$, and (ii) otherwise, $g^p(a,z)\,\hbar=\left(a+(1-\zeta)\,\overline{p}\right)/(1+\tau_b)$.

We use the following algorithm to find $g^p(a,z)$. Given the value functions W_o, W_r and the policy function g_o^c:

1. Check that $S\left(a,z,\overline{p}\right)>0$, so the agent's gains from participation are positive. (Otherwise, $g^\theta(a,z)=\theta_0$).
2. Find the maximum price the agent is willing to pay. This is equal to $p_r=\widetilde{p}$ where $S\left(a,z,\widetilde{p}\right)=0$ if $\widetilde{p}\,\hbar \leq \left(a+(1-\zeta)\,\overline{p}\,\hbar\right)/(1+\tau_b)$. Otherwise, this maximum price satisfies $p_r\,\hbar=\left(a+(1-\zeta)\,\overline{p}\,\hbar\right)/(1+\tau_b)$.
3. If $\tilde{S}\left(a,z,p_r\right)>S\left(a,z,p_r\right)$ use any solver to find a price $p\in\left(\overline{p},p_r\right)$ for which $\tilde{S}\left(a,z,p\right)=S\left(a,z,p\right)$.
4. If $\tilde{S}\left(a,z,p_r\right)\leq S\left(a,z,p_r\right)$, set $p=p_r$.

If $S(a,z,p)$ is increasing in a, as in our quantitative model, the above arguments imply that both p_r and $g^p(a,z)$ increase with a (in line with Proposition 3). Agents with low assets are constrained and choose $p\,\hbar=\left(a+(1-\zeta)\,\overline{p}\,\hbar\right)/(1+\tau_b)$. Wealthier agents are unconstrained.

D.2.2 The choice of financial assets

Let us focus on the problem solved by a renter at night. Her choice of housing is intratemporal and always satisfies

$$g_r^h(a,z)=\min\left\{\left(\frac{\phi\,\omega^{1-\sigma}}{r_h}\right)^{\frac{1}{\sigma}}\,g_r^c(a,z),\hbar\right\}. \tag{D.5}$$

To simplify the exposition let us assume that h denotes the services of rented housing. The expression for the Euler equation of the problem depends on whether the agent can participate in the frictional market in the next afternoon. Thus. there are two cases. If

$g_r^a(a,z) + (1-\zeta)\,\overline{p}\,\hbar < (1+\tau_b)\,\overline{p}\,\hbar$, the Euler equation is:

$$-u_c\left(g_r^c(a,z),h\right) + R\,\beta\,E_z\,u_c\left(g_r^c(a',z'),h\right) \leq 0, \tag{D.6}$$

with equality if $a' = g_r^a(a,z) > 0$. If $g_r^a(a,z) \geq (\zeta+\tau_b)\,\overline{p}\,\hbar$, the Euler equation becomes

$$\begin{aligned} &-u_c\left(g_r^c(a,z),h\right) + R\,\beta\,E_z\,m_b\left(g^p(a',z')\right)\,u_c\left(g_o^c\left(a' - (1+\tau_b)\,g^p(a',z')\,\hbar, z\right), \hbar\right) \\ &+ R\,\beta\,E_z\,\left(1 - m_b\left(g^p(a',z')\right)\right)\,u_c\left(g_r^c\left(a',z'\right), h\right) \\ &\qquad + R\,\beta\,E_z\,\frac{m_b'\left(g^p(a',z')\right)}{1+\tau_b}\,\left[S\left(a',z,g^p(a',z')\right) - \tilde{S}\left(a',z',g^p(a')\right)\right] \leq 0, \end{aligned} \tag{D.7}$$

with equality if $a' = g_r^a(a,z) > 0$. The problem solved by owners is similar, except for the fact that they can borrow up to $(1-\zeta)\,\overline{p}\,\hbar$. We build on Fella (2014) and solve for the optimal consumption rule using a modified version of his generalized endogenous grid method.

D.2.3 Solving the household's problem

The algorithm is as follows:

1. Choose an initial guess for $(W_o^j,\, W_r^j,\, g_o^{c,j},\, g_r^{c,j})$. For the owner's value function, we use the value function of an owner that is never hit by any shock. For the renter, we use that of a renter who never participates in the afternoon market. The consumption policy function of the renter will have a discontinuity point at $a_{part}(z)$. We choose $a_{part}^j(z) = (\zeta+\tau_b)\,\overline{p}$ as the first guess for this point.

2. Solve the afternoon problem as outlined in Section D.2.1 to find $g^p(a,z)$ and $W_b(a,z)$.

3. For a given grid for next period's assets, a', we use the Euler equation to find consumption today. We know that, if $a' < a_{part}^j(z)$, the Euler equation is (D.6); otherwise it is (D.7). We need to interpolate to obtain the consumption policy function as a function of the grid of assets today. We also need to be aware of the discontinuity at $a_{part}^j(z)$. This is key to use interpolation to find the policy function of consumption (as a function of assets today). To find the maximum in the region of assets that correspond to participation and non-participation, respectively, we conduct a Value Function Iteration Step. There is a cutoff point below which the renter will not participate in the afternoon market in next period. Save the node as $a_{part}^{j+1}(z)$. Save W_o^{j+1}, W_r^{j+1}, $g_o^{c,j+1}$, $g_r^{c,j+1}$. Notice that $a_{part}^{j+1}(z)$ may depend on the earnings state, z.

4. Go to step 2. Iterate until convergence.

A grid of 400 points in financial assets gives very high accuracy and is very fast.

D.3 The stationary distribution

As we have already explained in Section 2.6, we cannot use Monte Carlo simulations because the curse of dimensionality. We thus solve for the stationary distribution as in Huggett (1993) and as explained in Ríos-Rull (1997). We use a much finer grid than the one used to solve the household's problem (750 points in our case) and guess the distribution of owners and renters at night. Then we use the policy functions for financial assets to integrate numerically and find the distribution of non-traders and potential buyers in the afternoon as shown in equations (2.15)–(2.18).

D.4 The outer fixed point problem and the algorithm to find the stationary equilibrium

1. Choose an initial guess for the Walrasian price $\overline{p}$ and obtain the price function in (D.2). This guess pins down the rental price, $r_h = \kappa/\hbar + (1 - 1/R + \delta)\,\overline{p}$.
2. Find the households's value and policy functions and the participation threshold $a_{part}(z)$ using the process described in Section D.2.
3. Use the policy functions to find the stationary distributions using (2.15)–(2.18).
4. For each (a, z) in the support of ψ_b, use $g^p(a, z)$ to calculate the probabilities of buying and selling in the submarkets where the buyers who participate search, $m_b(g^\theta(a, z))$ and $m_s(g^\theta(a, z))$.
5. Use (2.24) and (2.25) to find the amount of vacancies overnight, V.
6. Obtain H_o, H_r and use the market clearing condition in the frictionless morning market. Given the price find the amount built today, I_h. If I_h is greater than $\delta\ (H_o + H_r + V)$, update $\overline{p}$ downwards. Likewise, if $I_h < \delta\ (H_o + H_r + V)$, rise $\overline{p}$. Go back to step 1.

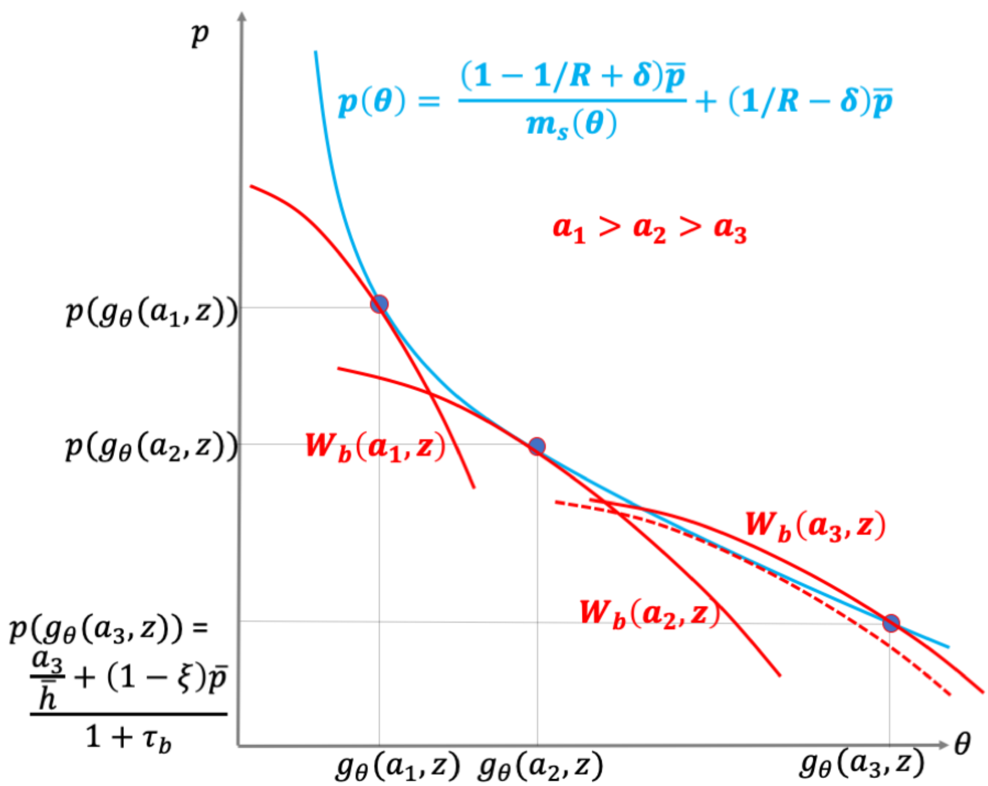


Figure 1: The choice of submarket.

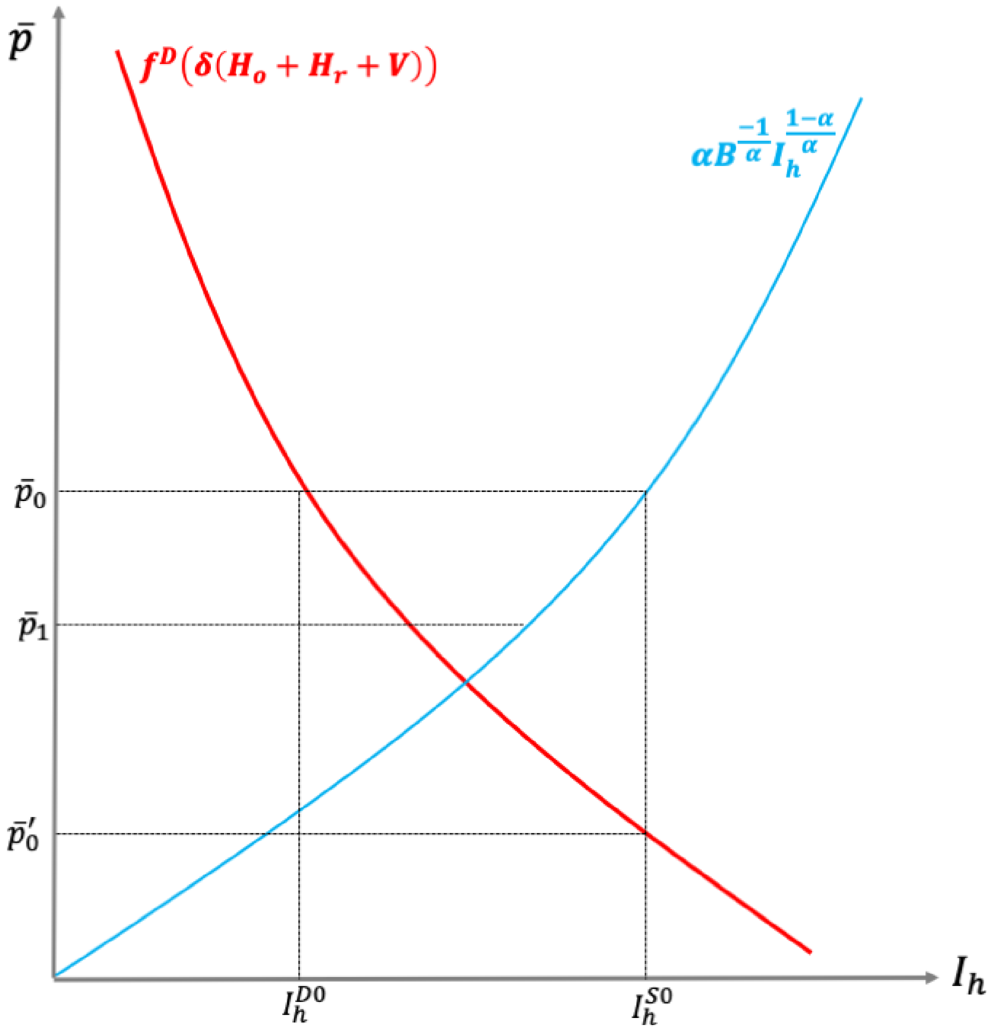
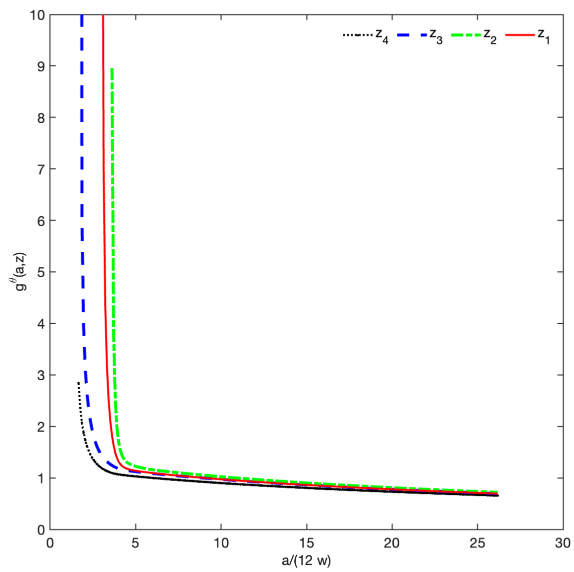
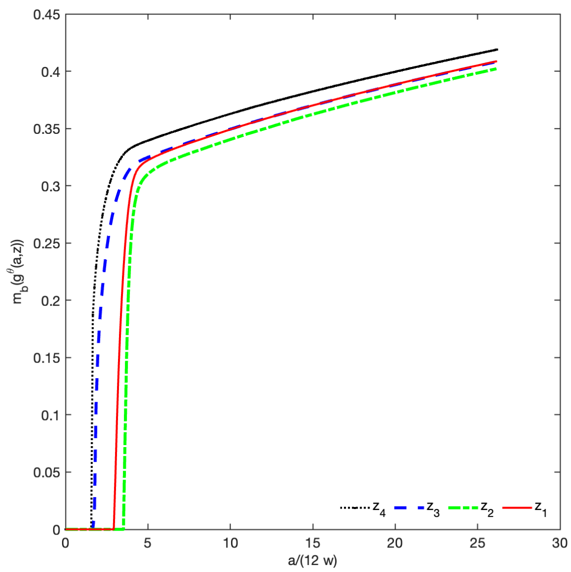


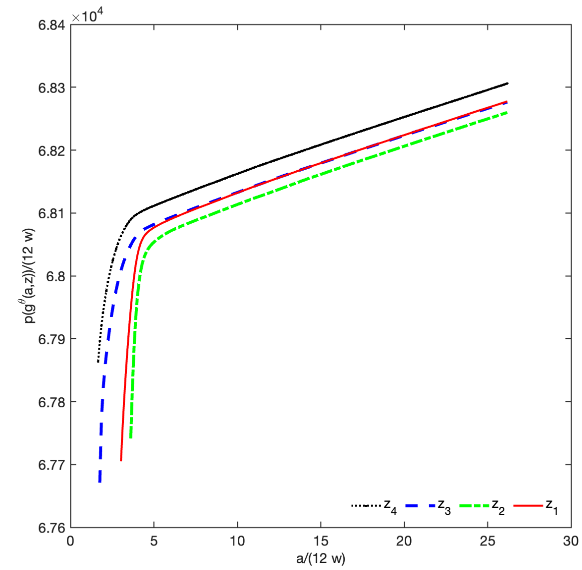
Figure 2: Demand and supply of housing.



(a) Afternoon policy function $g^{\theta}(a, z)$



(b) Probability of buying $m_b\left(g^{\theta}(a, z)\right)$



(c) House price $p\left(g^{\theta}(a, z), \overline{p}\right)$

Figure 3: Policy functions

Table 1: The earnings process

z			
0.8773	1.0000	2.1933	4.8107
Π_z			
0.0500	0.2375	0.4750	0.2375
0.0500	0.9461	0.0039	0.0000
0.0500	0.0019	0.9461	0.0019
0.0500	0.0000	0.0039	0.9461
Stationary distribution			
0.0500	0.2376	0.4748	0.2376

Table 2: Calibration of the benchmark economy

Param.	Observation	Value
Financial parameters		
w	Monthly wage	1000.0000
$R^\star - 1$	Díaz and Luengo-Prado (2010)	0.0300
τ_b	Díaz and Luengo-Prado (2008)	0.0250
τ_s	Díaz and Luengo-Prado (2008)	0.0600
ζ	Median LTV ratio = 43%	0.2600
Technological parameters		
κ	Price-to-rent ratio = 12.5	199.3954
α	Kaplan et al. (2020)	0.6000
B	Median H/E for owners = 2.57	17.7963
γ	Median TTB (NAR) 11 weeks	0.6500
$\delta^\star$	Sommer and Sullivan (2018)	0.0150
Mobility and productivity parameters		
π_μ	NAR: Median tenure of 9 years	0.0068
ξ_o	Annual mobility of owners = 3.2%	0.0025
ξ_r	Annual mobility of renters = 12%	0.0100
$\rho^\star$	Storesletten et al. (2004)	0.9520
$\sigma_\epsilon^\star$	Storesletten et al. (2004)	0.1700
φ	Average US unemployment rate	0.0500
Preference parameters		
σ	Risk aversion parameter	2.0000
$\beta^\star$	Median A/E for renters = 0.2257	0.8900
$\hbar$	Owner occupied housing services	$w\, mean(z)$
ϕ	Homeownership rate = 69.15%	0.1700
ω	Relative house size 1.5	0.8300

Notes: The model period is a month. *Annualized values.

Table 3: Calibration of alternative economies

Param.	Observation	Value
	Low elasticity economy	
α	Supply elasticity = 0.6, Saiz (2010)	0.3750
B	Median H/E for owners = 2.57	10.6392
	Very low elasticity economy	
α	new housing supply elasticity = 0.1, Baum-Snow & Han (2019)	0.0909
B	Median H/E for owners = 2.57	4.4272
	Exogenous rental market and $\alpha = 0.6$	
B	Median H/E for owners = 2.57	16.0418
	Exogenous rental market and $\alpha = 0.3750$	
B	Median H/E for owners = 2.57	9.0465
	Exogenous rental market and $\alpha = 0.0909$	
B	Median H/E for owners = 2.57	3.4969

Notes: The model period is a month. *Annualized values.

Table 4: The benchmark steady state

Target	Data	Bench.	Walras[1]	Walras[2]	Walras[3]
$\overline{p}\,\hbar/(12\,w)$	-	5.64	5.54	5.41	4.74
Home. rate	69.15	69.17	66.47	67.55	69.26
Median H/E owners	2.57	2.57	2.53	2.47	2.16
Median LTV ratio (%)	43.00	46.74	40.07	40.08	41.05
*(%) of indebted owners	65.00	68.15	40.32	39.87	36.83
Mean $\hbar/g_r^h$	1.50	1.48	1.52	1.51	1.46
Median A/E renters	0.23	0.22	0.20	0.26	0.23
Price-to-Rent ratio (%)	12.50	12.50	12.41	12.28	11.54
Median TTB	[10-12]	11.30	-	-	-
*Mean TOM	[4-17]	9.89	-	-	-
For sale rate	-	2.27	-	-	-
*Vacancy rate	1.59	1.35	-	-	-
$^\star\sigma_p$ (%)	2.25	0.14	0.00	0.00	0.00
Participation rate	-	7.68	2.33	2.41	2.57
A/E buyers†	-	0.86	0.81	0.81	0.77
A/E med./mean††	-	0.63	0.42	0.42	0.41

Notes: σ_p is the standard deviation of the deviation of the log prices as a fraction of its mean. Walras[1]: Walrasian equilibrium with $\alpha = 0.6$. Walras[2]: Walrasian equilibrium with $\alpha = 0.375$. Walras[3]: Walrasian equilibrium with $\alpha = 0.0909$. † median A/E for potential buyers who participate in the frictional market. †† Median to mean ratio of wealth to earnings ratio for potential participating buyers. *: Non targeted moments.

Table 5: Credit expansion when markets are not segmented

Target	Bench.	$\alpha = 0.6$		$\alpha = 0.375$		$\alpha = 0.09$	
		Search	Walras[1]	Search	Walras[2]	Search	Walras[3]
$\Delta\,\overline{p}$ (%)	-	3.88	3.63	9.19	7.92	23.74	21.63
Homeownership rate	69.17	88.29	92.64	87.60	90.71	80.69	83.22
Med. H/E owners	2.57	2.67	2.66	2.81	2.77	3.18	3.13
Med. LTV ratio (%)	46.74	66.15	57.20	66.19	56.53	64.44	53.57
(%) of indebted owners	68.15	80.24	69.23	81.56	69.11	84.59	66.77
Mean $\hbar/g_r^h$	1.48	1.71	1.99	1.73	1.93	1.71	1.81
Med. A/E renters	0.22	0.10	0.04	0.11	0.06	0.15	0.13
Price-to-Rent ratio (%)	12.50	12.71	12.70	12.98	12.92	13.66	13.57
Med. TTB	11.30	11.50	-	11.85	-	15.48	-
*Mean TOM	9.89	9.52	-	9.12	-	8.65	-
For sale rate	2.27	2.22	-	2.15	-	2.03	-
*Vacancy rate	1.35	1.29	-	1.21	-	1.09	-
Δ for sale units (%)	-	24.38	-	19.53	-	4.01	-
$\Delta\,I_h$ (%)	-	5.87	5.49	5.42	4.68	2.15	1.98
σ_p (%)	0.14	0.15	-	0.16	-	0.17	-
$\Delta\,\sigma_p$ (%)	-	13.56	-	29.37	-	39.45	-
Participation rate	7.68	27.04	10.99	29.22	8.88	19.84	4.97
A/E buyers†	0.86	0.34	0.66	0.34	0.65	0.44	0.65
A/E med./mean††	0.63	0.35	0.46	0.38	0.44	0.47	0.40

Notes: In all cases $\zeta = 5\%$. $\Delta\,\overline{p}$ refers to the increase in the Walrasian price as percentage of its value in the benchmark economy. $\Delta\,\sigma_p$ is the increase in the standard deviation with respect to its value in the benchmark economy. †: median A/E for potential buyers who participate in the frictional market. ††: Median to mean ratio of wealth to earnings ratio for potential participating buyers.

Table 6: Credit expansion when markets are segmented

Target	Bench.	$\alpha = 0.6$		$\alpha = 0.375$	$\alpha = 0.09$
		Search	Walras	Search	Search
$\Delta\,\overline{p}$ (%)	-	12.68	15.95	18.28	22.77
Homeownership rate	69.17	82.97	82.89	76.36	70.95
Med. H/E owners	2.57	2.90	2.98	3.04	3.16
Med. LTV ratio (%)	46.74	64.80	53.81	61.74	59.89
(%) of indebted owners	68.15	82.10	65.42	82.51	82.98
Mean $\hbar/g_r^h$	1.48	1.61	1.68	1.55	1.51
Med. A/E renters	0.22	0.15	0.13	0.19	0.21
Price-to-Rent ratio (%)	12.50	14.09	14.49	14.78	15.35
Med. TTB	11.30	15.95	-	15.62	15.87
*Mean TOM	9.89	8.54	-	8.33	8.11
For sale rate	2.27	2.03	-	2.01	1.94
*Vacancy rate	1.35	1.08	-	1.04	0.98
Δ for sale units (%)	-	6.74	-	19.53	4.01
$\Delta\, I_h$	-	19.62	24.86	10.39	2.56
$\Delta\,\sigma_p$ (%)	-	35.25	-	22.94	22.76
σ_p (%)	0.14	0.17	-	0.14	0.13
Participation rate	7.68	28.04	4.89	16.76	13.86
A/E buyers†	0.86	0.36	0.60	0.47	0.57
A/E med./mean††	0.63	0.43	0.38	0.51	0.59

Notes: In all cases $\zeta = 5\%$. $\Delta\,\overline{p}$ refers to the increase in the Walrasian price as percentage of its value in the benchmark economy. $\Delta\,\sigma_p$ is the increase in the standard deviation with respect to its value in the benchmark economy. †: median A/E for potential buyers who participate in the frictional market. ††: Median to mean ratio of wealth to earnings ratio for potential participating buyers.

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