DEPARTMENT
OF ECONOMICS

Essays on the Theory of Collective Action

Konuray Mutluer

Thesis submitted for assessment with a view to
obtaining the degree of Doctor of Economics
of the European University Institute

Florence, 07 July 2023

European University Institute
Department of Economics

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Abstract

This thesis is composed of three essays on the theory of collective action in various economic settings. These essays conduct positive analyses of the factors behind successful provision of a public good, with particular focus on uncertainty and social learning.

The first chapter considers the formation of a grassroots social movement and investigates the factors that determine whether the movement reaches the necessary size to achieve its common goal. The key aspect of the model is that while the number of active participants it takes for the movement to succeed is initially unknown, it becomes clearer as the movement keeps growing. I find that small changes in the environment, even seemingly detrimental ones, can trigger sudden and drastic jumps in the size of the movement. In particular, an increase in the personal cost of participation can lead to a greater number of participants.

The second chapter considers a duopoly competition model where demand is affected by past prices. Consumers are willing to buy more of a good at a given price if that price is the result of a discount, and less if it is higher than past prices. This behaviour by the consumers leads to overpricing. Furthermore, if the prices become too high, sudden and very large discounts are observed.

In the final chapter, I consider a dynamic game of volunteer's dilemma: A public good which is enjoyed by everyone in a group is produced only if one of its members bears the cost of producing it. The value of the public good is uncertain and all members receive private signals about it. Members infer others' signals by observing their actions. The focus of the analysis is the effect of group size on the provision probability of the public good. I find that this effect is not monotonic: The provision probability makes a discrete jump when population size reaches a threshold. Above this threshold, provision probability decreases with population. This suggests an optimal population size for maximizing provision.

Acknowledgements

I would like to express my gratitude to the people who made the work in this thesis possible. First of all, I am forever indebted to my advisors. Through his work, David K. Levine has been a role model since long before I met him personally. During my PhD, however, he helped me grow as a researcher, a teacher, and a person not only through his expertise, but also the immense effort and care he puts into the work and well-being of his mentee. In addition to the technical support he provided, I am thankful to him for refuting the phrase “Never meet your heroes". Andrea Mattozzi has been there to provide valuable help at every turn. His guidance has been crucial for the completion of my degree, and I feel privileged to have been advised by him. His encouragement in the more difficult times of the process particularly meant a lot to me.

I would like to thank Giacomo Calzolari for keeping an open door for me at all times. He has provided me with very insightful feedback, as well as support in various facets of the PhD. Discussions with Laurent Mathevet have greatly contributed to my research, and for this I am grateful. I am also grateful to Michalis Drouvelis and Ran Spiegler for generously agreeing to be members of the thesis committee. Your input will greatly improve my work and I look forward to our further interactions in the future.

I feel my PhD has taken place in an environment truly unique in its hospitality and sense of comradery. I am very thankful to every member of the Villa La Fonte community for providing this environment. This is particularly true for my friends Alaitz, Dalila, Gabriele, Josep, Lukas, Miguel, Mustafa, Nihan, Ozgen, Ozlem, Pietro, Thomas and Zheng, with whom I started and went through this process together. Their company made my stay in Florence some of the most enjoyable times of my life. I also want to thank Matteo Escude, Alessandro Ferrari and Junze Sun not only for their friendship, but also their help with my research over many interesting discussions.

Most importantly, I am grateful to my family; in particular my dear mother, father and sister. It is due to their many sacrifices that I had the chance to come so far. Perhaps the most difficult part of this PhD was not being able to see them for such long periods of time, but their unconditional support and love, as well as the hope to make them proud, is what kept me going. This will remain the case in any future endeavour. I would additionally like to thank my sister Aybike for the extended conversations about my work. Her insights were a great help, and provided a valuable cross-disciplinary perspective. Finally, I am extremely fortunate to have Alaitz as my companion through this journey. Your presence has always made the bad times good and the good times great. An important reason why I am grateful to have done this degree is because it gave me a chance to meet you. I am thrilled about being together through what is to come.

Contents

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Leading by Example Among Equals

Abstract I examine the factors that determine whether a grassroots social movement reaches the necessary size to achieve its goal. I study a collective action problem where identical individuals who value the common goal sequentially decide whether to join the movement. The model has two key ingredients: (i) The movement is facing a free-riding problem (i.e., while individuals want the movement to succeed, they would rather have others bear the cost of joining) and (ii) The necessary number of members to achieve success is ex-ante unknown but it can be revealed as the movement grows in size. The central insight is that a small change in the environment surrounding the collective action problem can trigger a surge in membership, drastically increasing the chances of success. Surprisingly, a higher cost of membership, such as harsher and more likely punishment for members of the movement or more effort intensive tasks, can trigger this surge.

1.1 Introduction

It is difficult to predict whether a grassroots social movement will succeed. While the supporters of a cause fail to mobilize in one case, they take active responsibility in mass under similar circumstances in another. When we look at movements that rapidly grow and reach the critical level of participation to achieve their common goal, we see that they often follow lengthy periods where supporters of the cause were unable to coordinate in collective action. A significant example of such an abrupt spark in participation is the fall of the Berlin Wall in 1989. In some cases, rapid mass mobilization is paradoxically triggered by changes that are meant to deter it. An example of this is repressive action against a movement's members reaching a point where it backfires. Such instances range from the Amritsar massacre of 1919 prompting the mass mobilization of those who oppose the British rule in India, to more recent cases such as those who oppose the ongoing deforestation of Istanbul taking to the streets after the use of excessive police force in Gezi Park in 2013.

In this paper, I study when and how the supporters of a cause overcome the problem of collective action. I argue that while the variation in a movement's ability to mobilize may appear arbitrary at first glance, it can be explained in a unified manner through a threshold phenomenon. A small change in the environment, even a seemingly detrimental one, can convince a large number of "ordinary" individuals who value the cause to pay the personal cost of taking action instead of free-riding off others.

To demonstrate this phenomenon, I model the formation of a movement as an ongoing process. As the movement forms, different individuals randomly encounter the decision of whether to participate at different points in time (e.g., as they come in contact with the local chapters of an organization). The movement succeeds if the necessary number of participants is reached. Participation comes with a personal cost independently of the outcome, such as the time dedicated to the movement or risk of being arrested in the process. These individuals are identical except for the time at which they encounter the decision: They value the cause equally, they have to pay the same cost to participate, and all information is public. The key ingredient of the setting is that initially, the necessary number of participants for success is unknown. As the movement continues to grow, however, their progress (or lack thereof) informs those who value the goal about how many participants are needed to achieve it. For instance, if politicians start holding meetings with the group or the policy demand is brought up for debate in legislative bodies, it can be inferred that the group is coming closer to the critical mass. In particular, when the goal itself is reached, it will be known. The lack of such developments despite a growing movement, on the other hand, make these individuals believe that more people are needed than they initially thought.

In this setting, the equilibrium chance of success is determined by when the initial participation takes place, which depends on the cost of participation and the (common) prior belief about how many participants are necessary. Once an individual decides to participate, those who follow her do so until either the goal is reached or the base of supporters is exhausted (i.e., all individuals who support the cause have made their decisions). Thus, the earlier participation starts, the higher the chance of success is for the movement. Individuals who decide before the initial participant choose to stay out, knowing that those who will be in their position later will bear the cost. The structure of the equilibrium suggests that due to free-riding, it can take time for a movement to get off the ground.

I capture the shifts in the environment surrounding the movement through changes in cost of participation. The central finding is that for a large set of prior beliefs about the necessary number of participants, there are critical intervals of cost where the equilibrium chance of success makes an upward jump. When there is an increase in cost such that it crosses the lower bound and enters one of these intervals, it becomes optimal for a large number of early decision makers to participate instead of free-riding off later movers. This is because in these intervals, they are not only responsible for their own participation in the movement. The participation of future decision makers depend on theirs as well. By participating, they can lead the future movers by example, even though they have no private information to signal and their preferences are the same as everyone else.

The dependence on the actions of earlier movers occurs because the participation decision of an individual is affected by the group size at time of her move. In particular, observing a larger group leading up to her move has two potential effects on her incentive to participate that counteract each other. First, joining a larger group can mean that the difference that her participation makes (i.e., her marginal contribution for success) is lower. We can consider this a deterrent “crowding out" effect. Second, if success has not yet been achieved despite a large group, she updates her beliefs about the number of participants it takes to achieve the common goal. This makes her think that her participation is more likely to be necessary. We can consider this an encouraging “information effect". If the latter effect dominates the former, then a bigger group incentivizes this individual to participate. Then, for some levels of cost, she participates if and only if sufficiently many have done so before her without achieving success. That is, her decision is “contingent" on a large enough group size for an interval of cost. Earlier movers anticipate this. If the cost is outside this interval, they know that the decision of later movers are independent of their own, so they free-ride. If, however, the cost lies in an interval where one or more future decision maker is contingent, these earlier movers start a chain of participation such that each participant puts the next one in a position where future individuals depend on her. Throughout the paper, I refer to this phenomenon as a “participation cascade".

I provide a sufficient condition on the prior beliefs about the necessary group size for success such that these cascade intervals exist. I show that when there there is a large population playing the game, this condition holds for a large set of commonly used distributions, such as the log-normal. Then I provide two examples to demonstrate the extent of the change in success probability caused by participation cascades, including an extreme case where a small increase in cost leads to a jump from a single participant to all individuals who value the goal participating. These examples also provide simple settings that can be easily replicated in a laboratory environment.

Relation to the Literature

The surge in group size I describe occurs through leading by example: The early movers join although their individual contribution is not high enough, in order to manipulate the decision of others who observe their action. Numerous experimental studies find evidence for this phenomenon in coordination games, as settings with sequential decisions yield higher contributions to the public good than simultaneous settings (Moxnes & van der Heijden (2003), Güth (2007), Levati et.al. (2007), Potters et.al. (2007) Gächter & Renner (2018)).

Theoretical literature involving leading by example in various coordination settings focuses predominantly on heterogeneity among the players. First, this phenomenon is attributed to asymmetric information. Hermalin (1998), Ginkel & Smith (1999) and Loeper et.al. (2014) describe leaders who possess better information regarding the value of the common goal than the followers. A second proposed source of leading by example is that observable participation can serve as an aggregator of private information dispersed across the population (Lohmann (1994a,b), Chwe (2000), Bueno de Mesquita (2010), Kricheli et.al. (2011), Battaglini (2016), Barbera & Jackson (2020)). Finally, heterogeneity in how much individuals value the public good relative to the cost can lead to this phenomenon (Kuran (1991), Winter (2009)). I describe an environment where leading by example facilitates coordination among individuals that are identical except for their order of move and all information is public. In other words, I show that "ordinary people" who possess no information to signal can still spark mass participation.

In the specific context of mass mobilization in social movements, three main contributions of my framework can be underlined. First, existing work that studies sudden sparks in mobilization focuses on "extremists" (i.e., those who value the common goal the most) mobilizing the moderates (Kuran (1991), Lohman (1994a,b), Kricheli et. al (2011)). As described above, I focus on identical individuals and show that the actions which mobilize the masses need not come from extremists or fringe groups. They can take place purely by the virtue of their observability.

Second, I provide a novel explanation as to how a successful movement can be triggered by adverse developments, such as intensifying persecution of its members.[1]

[1] Kricheli et. al. (2011) find that a more repressive government makes protests less likely to occur, but more likely to lead to a revolution when they do. In their model, however, a more repressive government unambiguously decreases the ex-ante probability of a revolution. Winter (2009) describes a related result for a different setting; namely, optimal reward schemes in a team project. He finds that if there is a degree of complementarity of efforts in the production technology, we can design two (heterogenous across members) reward schedules such that one induces more effort although it rewards all team members a lower amount for a successful project.

Third, I provide conditions under which the participation of others encourages or deters a supporter of the movement from participating herself. Much of existing work on mass mobilization abstracts away from a potential deterrence by imposing strategic complementarity. That is, they start with the premise that a supporter is more willing to participate when many others do so (e.g., imposed directly on preferences in Kuran (1991), and via costless participation in case of success in Kricheli et. al . (2011)). I make no a priori restriction on the complementarity of individual participations and allow free-riding incentives to arise through strategic substitutability. I show that the willingness of a supporter to participate can increase with the number of participants they observe, while decreasing with how many others they expect to participate in the future. This discrepancy between the effects of knowledge and expectation regarding the participation of others seems to be in line with recent empirical evidence on protest behavior. While improved information transmission is found to increase protest attendance (Enikopolov et. al. (2020), Manacorda & Tesei (2020)), when only the beliefs about the number of participants are varied, evidence of strategic substitutes is found (Cantoni et. al. (2019)).

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 provides a sufficient condition for participation cascades and discusses when it applies for large populations. Section 4 analyzes the model for two example threshold distributions. Section 5 concludes.

1.2 Model

Consider the following game where a finite number N of identical individuals are to form a group in order to produce a public good. The game consists of N periods of time. In each period, one individual is randomly selected to move without replacement (i.e., each individual is drawn in exactly one period). Let Player i denote the individual who is drawn to move in period $i \in \{1,...,N\}$. Player i chooses an action $a_i \in \{0,1\}$, where actions 0 and 1 stand for *pass* and *participate* respectively. Prior to her move, Player i observes the action of all players $\{1,...,i-1\}$ who have moved before her. After the move of Player N, the game ends.

If the number of players who choose to participate reaches threshold $t \in \mathbb{N}_+$, a public good is produced and all players obtain utility 1 regardless of their action. If a player participates, she pays cost $c \in (0,1)$ with no refunds. We can summarize the payoff of Player i as follows.

$$u_i = \mathbb{1}\left\{\sum_{j=1}^{N} a_j \geq t\right\} - \mathbb{1}\{a_i = 1\}c$$

Throughout the analysis, it is assumed that players choose to participate in case of indifference.[2]

Ex-ante, threshold t is unknown. It is common knowledge that this threshold will be revealed to all as soon as the number of players i who play $a_i = 1$ reaches t (before the remaining players make their move). The game effectively ends as soon as t players choose to participate, because it is revealed that the public good has already been produced and there is no more gains from participating for those who move later.

It is common knowledge that ex-ante, t follows prior probability distribution F with $supp(F) \subseteq \mathbb{N}_+$. Denote by p and F the probability mass function and cumulative distribution function of t respectively. That is, $p(n)$ dentoes the prior probability that $t = n$ for $n \in \mathbb{N}_+$. Given that t has not yet been reached, the belief of Player i is the Bayesian posterior as a function of the number of players who chose to participate up to her turn. In particular, if she observes $k \in \{1, ..., i-1\}$ participants prior to her turn without success (i.e., without the threshold having been reached), she conditions her belief on $t > k$. The solution concept is subgame perfect equilibrium.

1.3 General Case

In this section, I first discuss the structure of the unique equilibrium path in the above model. Then I argue that raising the cost c above certain levels can lead to an upward jump in the equilibrium group size (that is, the number of players who choose to participate on the equilibrium path), and thus the probability of producing the public good. I provide a sufficient condition on distribution F such that these upward jumps occur and discuss the underlying mechanism. Finally, I describe some cases where this condition holds when the game is played by a large number of players.

1.3.1 Equilibrium Path

As described in the previous section, once threshold t has been reached the public good has already been produced and there is no gain from participating. Thus, any strategy that chooses to participate after the threshold is reached is strictly dominated. Here I discuss the action of a player given that the threshold has not yet been reached by the time of her move. Let “potential participant" describe a player who participates if the threshold is not reached by the time of her move (i.e., Player i is a potential participant if she participates given that the threshold is not reached by period i). If a player is not a potential participant, then she passes regardless of whether the threshold has been reached.

[2]This assumption is required only for finitely many values of cost c in the interval $(0, 1)$. For generic cost, all results discussed in the analysis carry through without this assumption.

The ex-ante probability of success (i.e., producing the public good) is equal to the probability that the number of potential participants is greater than or equal to threshold t. That is, if the number of potential participants is $s \in \{1, ..., N\}$, the success probability is given by $F(s)$. The marginal return (in terms of added ex-ante success probability) to one further participant when there are s potential participants is given by $F(s+1) - F(s) = p(s+1)$: This additional participant will be pivotal to success if and only if the threshold is $s+1$.

While the equilibrium number of potential participants and the resulting success probability depends on the exact specification of distribution F, we can make the following observations on the structure of the unique equilibrium path. Proofs of all results can be found in the Appendix.

Proposition 1. *The following hold on the unique equilibrium path.*

(a) *If there are any potential participants, then there exists an initial participant* $n^* \in \{1, ..., N\}$ *such that Player* i *is a potential participant if and only if* $i \in \{n^*, ..., N\}$

(b) *If* $c \leq p(N - i + 1)$*, then Player* i *is a potential participant.*

The first observation states that on the equilibrium path, all players who move before some period n^* pass. Player n^* is the first one to participate and after her, the remaining players participate until either the threshold is reached or all N players have made their moves without reaching the threshold. In other words, once participation starts, it continues until success or the end of the game. The success probability of the group is determined by the order of this initial participant n^*. The earlier the initial participant (i.e., lower n^*), the more potential participants there are and the higher the ex-ante probability of success is. Note that with sufficiently high cost c, there is no initial participant and all N players pass on the equilibrium path.

The second observation states a sufficient condition for a given Player i to be a potential participant on the equilibrium path. If Player i knows that all of the later $N - i$ movers are potential participants and all of the earlier movers have passed, then the return to her participation is given by $p(N - i + 1)$. If this return is greater than the cost, then Player i is a potential participant in equilibrium. In other words, if Player i is willing to be the initial participant when those who move later are potential participants regardless of her action, then she herself is a potential participant in equilibrium. Note that this is not a necessary condition. The cascades described in the remainder of the paper occur because when the decision of a player influences later movers, she may participate even though the return to her individual participation does not cover the cost.

Finally, note that the two observations together mean there is at least $\max\{n \in \{1, ..., N\} : p(n) \geq c\}$ potential participants in equilibrium. That is, if the probability that a given participation level n is the threshold for success is greater than the cost, then that participation level will be met in equilibrium (unless success is already achieved before it is met). To see this, note that if $p(n) \geq c$ for some $n \in \{1, ..., N\}$, then observation (b) implies that Player $N - n + 1$ is a potential participant, which by observation (a) means that all n players in $\{N - n + 1, ..., N\}$ are potential participants. It is worth noting that lower bound $\max\{n \in \{1, ..., N\} : p(n) \geq c\}$ is the number of potential participants that would arise without the information externality caused by observing earlier players participate without success. In particular, it is the highest number of participants across all Nash Equilibria if the game is played with simultaneous decisions. Furthermore, it is the number of participants in the unique equilibrium of the game if the decisions are sequential but the threshold is revealed after all players have made their moves instead of being revealed as soon as it is reached. This level will be used as a benchmark for the analysis of the participation cascades in our setting.

1.3.2 Participation Cascades: A Sufficient Condition

Denote by s^* the equilibrium group size (i.e., number of potential participants on the equilibrium path). By Proposition 1, we have $s^* := N - n^* + 1$ and the ex-ante success probability in equilibrium is given by $F(s^*) = F(N - n^* + 1)$. The following result states a sufficient condition on threshold distribution F such that s^* makes an upward jump as a result of an increase in cost c.

Theorem 1. *If $p(N-2) > p(N-1) > p(N)$ and $\frac{p(N)}{p(N-1)} > 1 - p(1)$, then s^* is non-monotonic with respect to cost c.*

The main observation of this result is that if the ex-ante marginal return $p(n)$ to one additional participant is decreasing above $N - 2$ participants, but not too fast, then a higher cost c can yield a larger group size in equilibrium. In particular, when cost c is just below $p(N-1)$, Player 1 passes and Player 2 is the initial participant on the equilibrium path. This leads to equilibrium group size $s^* = N - 1$. If the cost is raised to just above $p(N-1)$, however, Player 1 becomes the initial participant, and the equilibrium group size increases to N.

For the argument behind this increase, consider the equilibrium action of the first two movers under two cases $c = p(N-1) - \epsilon$ and $c = p(N-1) + \epsilon$ with ϵ small. Theorem 1 requires that we have $c < p(N-2)$ in both of these cases. Thus, if Player 1 and Player 2 pass, which means the move of Player 3 is reached with zero past participants, all of the remaining $N-2$ players are potential participants in the continuation game by Proposition 1. Now suppose $c = p(N-1) - \epsilon$. If Player 1 has passed, then Player 2 decides based on her prior belief and concludes that the return to her participation is $p(N-1)$ (since the remaining $N-2$ players are potential participants), which is greater than the cost. Anticipating this, Player 1 then knows that her own participation would yield return $p(N)$. This return is smaller than the cost for small ϵ, so she passes and Player 2 is the initial participant in equilibrium. This yields group size $N-1$.

If the cost is raised to $c = p(N-1) + \epsilon$, the return $p(N-1)$ is not sufficient to convince Player 2 to participate. Thus, Player 1 knows that if she passes, Player 2 will pass as well. If Player 1 participates and the threshold is not reached, however, then Player 2 rules out the possibility that the threshold is equal to 1 and she updates her belief accordingly. Since Player 1 has already participated and there are $N-2$ more potential participants after her, Player 2 knows that her own participation is pivotal to success if and only if the threshold is N. The increase in success probability resulting from her participation is then given by $\frac{p(N)}{1-p(1)}$, which is greater than the cost (by the condition of Theorem 1). Hence, Player 2 is willing to participate if and only if she has observed Player 1 participate without success before her. Anticipating this, Player 1 knows that the participation of Player 2 depends on hers. She is effectively adding not 1 but 2 potential participants to the group: Herself and Player 2. Hence, Player 1 becomes the initial participant and the equilibrium group size increases to N.

To summarize, a player can be incentivized to participate herself by observing others participate without success. Through this observation, she rules out low group sizes as threshold candidates, which makes her believe that her own participation is more likely to be pivotal for reaching the threshold. When the cost is raised to a level where she needs this additional incentive, her participation becomes contingent on a certain number of past participants. Anticipating this, earlier movers are prompted to participate as well, knowing that this will induce the contingent player to do so. As a result, the group size is higher and success is more likely in equilibrium.

The sufficient condition in Theorem 1 only uses the contingency of one player (Player 2) on one past participant (Player 1). As such, it corresponds to a cascade of size 1. The examples in the next section show that more drastic upward jumps are possible. In particular, the first example demonstrates that multiple players can be contingent on a past participant for a given cost. The second example shows that a player can be contingent on a large number of past participants. This includes the the extreme case where a player who moves late in the game is contingent on all players up to her turn having participated.

1.3.3 Interpreting the Condition in Large Populations

Suppose F has full support over positive integers. In this case, if the convergence rate $\lim_{n\to\infty} \frac{p(n+1)}{p(n)}$ of p to 0 is sufficiently high (greater than $1-p(1)$), then there is a number $\underline{N}$ such that the condition presented in Theorem 1 is satisfied for all $N > \underline{N}$. In other words, if the threshold distribution decays sufficiently slowly and the game is played by sufficiently many players, then Theorem 1 applies.

The condition holds for large N regardless of initial value $p(1)$ if p converges to zero sublinearly. That is, if $\lim_{n\to\infty}\frac{p(n+1)}{p(n)}=1$. For example, if $p(n)=\int_{n-1}^{n}f(x)dx$ where f is the density of the log-normal distribution, then the property $\lim_{n\to\infty}\frac{p(n+1)}{p(n)}=1$ is satisfied and the upward jumps in group size will occur for sufficiently large N. Furthermore, any power law distribution (i.e., $p(n)$ follows a power law $n^{-\alpha}$ atfter some value of n), converges to zero at a sublinear rate. An example is the case where the threshold is Pareto distributed. Hence one implication of Theorem 1 is that in large games, participation cascades occur at certain cost intervals when the threshold distribution is sufficiently heavy tailed. Note that sublinear convergence of p is not a necessary condition. Example 2 in the following section analyses a case where $\lim_{n\to\infty}\frac{p(n+1)}{p(n)}<1$ but participation cascades are still observed.

1.4 Examples

This section analyzes the equilibrium of the model under two threshold distributions to illustrate the extent of the participation cascades introduced for the general case above. First, I discuss the case where there the threshold can only take one of two possible values. Second, I look at the case where the threshold follows a geometric distribution with an uncertain parameter.

1.4.1 Two Possible Thresholds

Suppose $supp(F)=\{\underline{t},\overline{t}\}$ with $\underline{t},\overline{t}\in\mathbb{N}_+$ and $\underline{t}<\overline{t}<N$. Assume $p(\underline{t})=1-p(\overline{t})>0.5$. The following result states the equilibrium group size as a function of cost c.

Theorem 2. *The equilibrium group size in the model with two possible thresholds is as follows.*

$$s^*=\begin{cases}\overline{t}; & c\in(0,p(\overline{t})]\\ \underline{t}; & c\in(p(\overline{t}),p(\underline{t})]\\ \overline{t}; & c\in(p(\underline{t}),1)\end{cases}$$

Early movers $\{1,...,N-\overline{t}\}$ free ride off the later $\overline{t}$ movers and never participate in equilibrium. If cost c is lower than the probability of both possible thresholds, then the group size is $\overline{t}$, which guarantees that the public good is produced. For interior levels where the cost c is lower than the probability of low threshold $\underline{t}$ but higher than that of high threshold $\overline{t}$, only the low threshold is met in equilibrium. This leads to ex-ante probability $p(\underline{t})$ of producing the public good. The main observation of this result is the third case. If the cost is above $p(\underline{t})$, then once again just enough players participate in equilibrium to guarantee the production of the public good. Hence, raising the cost above the level $p(\underline{t})$ increases the group size and the probability that the public good is produced. In short, the reason is that with high cost, participation by late movers becomes contingent on sufficiently many past participants. This leads to cascades of participation among earlier movers.

The equilibrium group size in the first two cases $c \leq p(\bar{t})$ and $c \in (p(\bar{t}), p(\underline{t})]$ are unsurprising: A candidate group size is met in equilibrium if and only if the likelihood that it is the necessary threshold for success is greater than the cost. That is, $s^* = \max\{n \in \{1, ..., N\} : p(n) \geq c\}$

If $c > p(\underline{t})$, then no individual participation yields high enough returns to cover the cost under the prior threshold distribution. If, however, a player follows a history where at least $\underline{t}$ earlier players have participated without success, she rules out the possibility that the threshold is $\underline{t}$ and is certain that it is $\bar{t}$. If she is in a position where she is pivotal to the group size reaching $\bar{t}$ after such a history, then her decision is between producing the public good for certain or with zero probability, which prompts her to participate at any cost. Thus, if the move of Player $N - (\bar{t} - \underline{t}) + 1$ is reached with $\underline{t}$ past participants and no success, the remaining $\bar{t} - \underline{t}$ players are all potential participants since they know for certain that the threshold is $\bar{t}$. Otherwise, they all pass since their decision is based on their prior belief. The previous $\underline{t}$ movers anticipate this. Starting from Player $N - \bar{t} + 1$, each of them participates and puts the next mover in a position where she must also do so in order the continue the chain of $\underline{t}$ participations that will change the belief of the last $\bar{t} - \underline{t}$ movers. Thus, every player from Player $N - \bar{t} + 1$ onwards is in a position where the potential participation of all later movers is contingent on hers. This makes it worth paying cost c and the number of potential participants is $\bar{t}$. Success is guaranteed.

1.4.2 Geometric Threshold

Suppose the threshold follows a geometric distribution with a parameter p that is unknown (i.e., F is a compound geometric distribution). In particular, p can take one of two possible values p_1 or p_2 with $p_1 > p_2$. The common prior belief of the players is that $p = p_1$ with probability q and $p = p_2$ with probability $1 - q$. Then the prior probability that threshold t is equal to $n \in \mathbb{N}_+$ is given by

$$p(n) = q((1 - p_1)^{n-1} p_1) + (1 - q)((1 - p_2)^{n-1} p_2)$$

It is possible to interpret this threshold distribution in two ways. One interpretation is that as the current group size becomes larger, the marginal contribution of a further member decreases at a constant but unknown rate. A second interpretation is that when there is a new participant, the group makes a new (independent) attempt at producing the public good.[3] However, the difficulty of the attempts is unknown to the players. Then, $p = p_1$ and $p = p_2$ correspond to the easy and the difficult states of the world respectively.

In the remainder of this section, I state the necessary and sufficient condition under which participation cascades occur in this setting. Then I discuss the equilibrium group size in cases where this condition is violated and satisfied, respectively.

Proposition 2. *Equilibrium group size s^* is monotonic with respect to cost c if and only if $p_1(1-p_1)^{N-2} \geq p_2(1-p_2)^{N-2}$. If s^* is monotonic, then $s^* = max\{n \in \{1,...,N\} : p(n) \geq c\}$*

This first implication of this result is that for any given $(p_1,p_2) \in (0,1)^2$, there exists a value $\underline{N}$ such that s^* is non-monotonic for all $N > \underline{N}$. Thus, if the game is played by sufficiently many players, upward jumps in group size with respect to cost c will be observed. The second implication is with respect to the possible parameters p_1 and p_2. Fixing N, we can conclude that if s^* is non-monotonic under some pair (p_1,p_2), then it is also non-monotonic under any (p_1',p_2') with $(p_1',p_2') \geq (p_1,p_2)$.[4] This can be interpreted as for a given N, cascades occurring if and only if the possible parameters of the distribution are sufficiently high. In other words, for a given population of players, cascades occur if the prior belief is that producing the public good is "sufficiently easy".

When s^* is monotonic with respect to c, a number $n \in \{1,...,N\}$ of potential participants is met in equilibrium if and only if the likelihood $p(n)$ that n is the threshold for success is greater than the cost. As a result, we have $s^* = \max\{n \in \{1,...,N\} : p(n) \geq c\}$ for the equilibirum group size. Clearly, s^* is decreasing with respect to cost c in this case.

The following result states the equilibrium group size for a range of parameters in the non-monotonic case.

Theorem 3. *Assume $p_1 + p_2 > 1$ and $p_2 \in (q, 1-q)$. For all $n \in \{2,...,N-1\}$ there is a $k_n \in \mathbb{N}$ such that an interval*

$$C_n^k := \Big(\frac{p(n+k-1)}{1-F(k-1)}, \min\Big\{\frac{p(n+k)}{1-F(k)}, \sum_{m=n}^{n+k} p(m)\Big\}\Big]$$

[3] With this interpretation, it can be argued that these attempts should not be independent but instead become more likely to succeed with more participants. Here I take parameter p as constant across number of participants for tractabiltiy of the equilibrium characterization. In general, it is confirmed by Theorem 1 that the cascades discussed here occur whenever parameter p is a non-decreasing function of the group size.

[4] The condition stated in Proposition 2 is under the assumption $p_1 > p_2$ and reversed for $p_2 > p_1$. The observation that the pairs (p_1,p_2) which violate the monotonicity condition constitutes an upper set of the allowed pairs holds in both cases.

exists for all $k \in \{1\} \cup \{k_n, k_n+1, ...\}$*. If* $c \in C_n^k$ *for some pair* (n,k) *with* $n+k \leq N$*, then* $s^* = n+k$*. Otherwise* $s^* = max\{n \in \{1,...,N\} : p(n) \geq c\}$*.*

Theorem 3 means that for $n \in \{2,...,N-1\}$ and for k either equal to 1 or sufficiently high, there is a cost interval C_n^k in which s^* makes an upward jump (of size $k+1$) with respect to the monotonic case group size $max\{n \in \{1,...,N\} : p(n) \geq c\}$. Outside these intervals, s^* is as in the monotonic case. Therefore, a small increase in cost c can cause a large surge in success probability, when it leads to c entering an interval C_n^k. This is in contrast to the monotonic case, where s^* decreases gradually with respect to c. There is a straightforward interpretation for these intervals: If cost c lies in interval C_n^k, then Player $N-n+1$ is willing to participate if and only if she follows a history where at least k earlier players have participated without success. Furthermore, for this cost interval the k players preceding her are willing to provide that history. Figure 1 is a visual representation of the equilibrium group size with respect to c in a 4 player game.

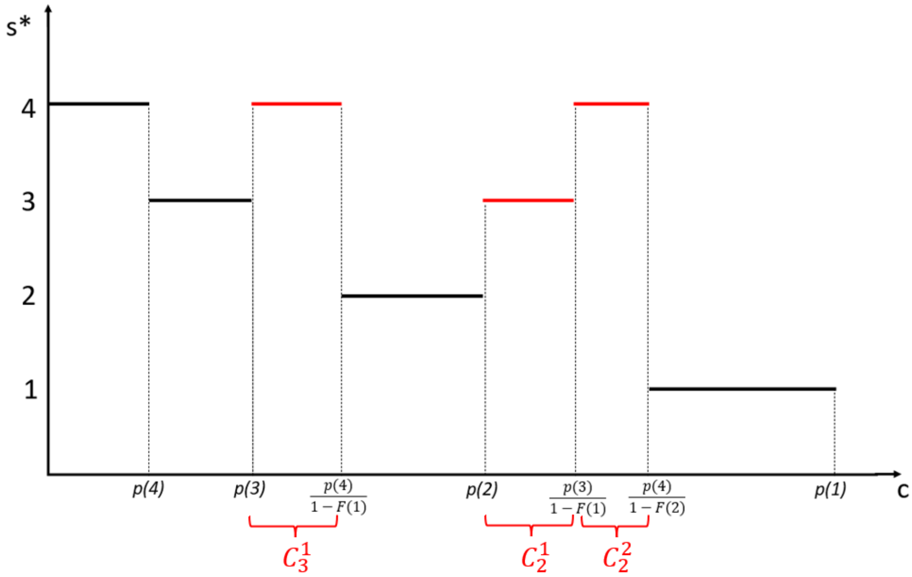

Figure 1.1 The equilibrium group size w.r.t. c under $(q,p_1,p_2) = (0.4, 0.8, 0.5)$ in a 4 player game. The red regions represent the "upward jump" intervals (from left to right: Player 2 contingent on 1, Player 3 contingent on 1 and Player 3 contingent on 2 past participants). The group size in the black regions are identical to the monotonic case. Note that depending on the parameters, regions C_2^1 and C_2^2 may or may not be adjacent.

To see the why a cascade occurs for $c \in C_n^k$, consider the decision of the contingent Player $N-n+1$. Given that all $n-1$ players after her are potential participants and there has been $k-1$ participants without success before her move, she knows that her participation is pivotal to success only if the threshold is $n+k-1$. Furthermore, she knows that the threshold is greater than $k-1$. In this case, the return to her participation is $\frac{p(n+k-1)}{1-F(k-1)}$. Similarly, if she follows k past participants, then this return is $\frac{p(n+k)}{1-F(k)}$. Since $c \in C_n^k$, the return is higher than the cost if and only if she is moving after at least k past participants. The player who moves k stages before the contingent player $N-n+1$ is responsible for starting the chain that makes her a potential participant. In effect, by participating, this player increases the number of potential participants from $n-1$ to $n+k$. Then the increase in success probability caused by the participation of this player is $\sum_{m=n}^{n+k} p(m)$, which is higher than the cost for $c \in C_n^k$. She starts the cascade of k participants aimed at encouraging the contingent player $N-n+1$ and the resulting group size is $n+k$.

Note that a cascade of size $k > 1$ occurs only if there is a pair (n,k) such that cost interval C_n^k exists and there are at least $n+k$ players in the game. The former condition is satisfied when k is sufficiently large (i.e., greater than lower bound k_n for given n). For the latter condition, we need N to be sufficiently large (i.e., greater than $n+k$). Combining these two observations, we see that drastic upward jumps in group size occur when the game is played by many players. For instance if N is sufficiently large, then for each $n \in \{2,...,N-1\}$ there is an interval C_n^{N-n}. When $c \in C_n^{N-n}$ this means Player $N-n+1$ participates if and only if every player before her did so, and Player 1 is indeed willing to start this cascade of size $N-n$. So when the cost is raised to this interval, the equilibrium group size can jump from $n-1$ (which can be as low as 1) to N (which means all players are potential participants). This demonstrates the extent to which the probability of success in large groups can be influenced by the common understanding that when the goal is reached, the players will know it.

Finally, turning to the parameter restrictions, it can be seen from Proposition 2 that $p_2 \in (q, 1-q)$ and $p_1 + p_2 > 1$ are not necessary conditions for cascades to occur. In particular, $p_1 + p_2 > 1$ implies that probability $\frac{p(n+k)}{1-F(k)}$ is increasing with k for all $n \in \{2,...,N-1\}$. This means that for any player in $\{2,...,N-1\}$, the return to participating is increasing with respect to the number of past participants without success. In general, observing one more past participant before her move has two opposing effects on a player's incentive to participate: First, a larger group size means higher chance of success without her participation and provides additional incentive to free ride.[5] Second, observing one more past participant without success allows her to rule out one more threshold value, increasing the probability she assigns to higher threshold values. This makes it more likely that the group will fail without her, making her participation more likely to be necessary. In the example here, if p_1, p_2 are sufficiently high, the second (positive) information effect outweighs the first (negative) free riding effect. As a result,

[5]Note that in our geometric example, the marginal return to a new participant is strictly decreasing with group size. In other words, there is decreasing returns to scale. This is the case whenever the threshold threshold probability $p(n)$ is a decreasing function. This means if distribution F has infinite support, decreasing returns to scale will always hold for a sufficiently large number of players.

there are intervals of cost where the participation of a player is contingent on sufficiently many past participants. Condition $p_1 + p_2 > 1$ guarantees that this is the case for all players in $\{2, ..., N-1\}$. [6]

1.5 Concluding Remarks

Successful grassroots movements are often characterized by sudden surges in mobilization, triggered by developments that make life more difficult for participants. I set out to examine this seemingly counter-intuitive phenomenon by focusing on the incentives of “ordinary people" who support the cause during the formation of the movement. To this end, I account for two natural aspects of movement formation. First, achieving the common goal benefits not only those who played an active role, but everyone who values it. Second, when individuals observe the movement’s current degree of success, they use this observation to make inferences about the difficulty of its task.

I find that an arbitrarily small change in the circumstances surrounding the movement can lead to a drastic rise in its membership. This rise can be to such an extent that a small increase in participation cost allows a virtually non-existent movement to mobilize all of its supporters. Furthermore, I describe a mechanism where by participating herself, an individual with no special characteristics or private information can induce others to do so. This mechanism suggests an informational foundation for conditional participation (described as “I will go if you go" by Chwe (2000)), a pattern commonly observed in grassroots movements and often imposed a priori through strategic complementarity in theoretical literature.

While the focus of this paper is grassroots movements aimed at political change, the mechanism I describe can be extended to other games of regime change where the benefits of leaving the status quo are not entirely restricted to the ones who contribute to it (i.e., where regime change has a public good component).[7] Examples of such cases include crowdfunding of an entrepreneurial venture whose success is enjoyed by a large group and investment in R&D for curing a widespread disease. A potentially useful direction of further analysis is to study the incentive effects of the information externality I describe in situations where regime change has a “public bad" component that needs to be prevented at a personal cost. Among possible instances are the consideration of adverse macroeconomic consequences of a large bank’s insolvency during a bank run and firms investing in measures against climate change.

[6] The other condition $p_2 \in (q, 1-q)$ guarantees that at most one player is contingent for a given cost, and is assumed for tractability.

[7] Games of regime change are described as “[...] coordination games in which a status quo is abandoned, causing a discrete change in payoffs, once a sufficiently large number of agents take an action against it" (Angeletos et. al. (2007)).

2

Tacit Collusion and Dynamic Reference Prices

Abstract This paper examines the extent of tacit collusion in an oligopoly market where consumers are affected by past prices. In particular, we study an infinite horizon Bertrand competition between two identical firms where today's demand for the good at a given price is higher if it is a discount relative to past prices and lower if the price has been raised. This history dependence has an ambiguous effect on collusion: On one hand, colluding on a higher price makes long run coordination more fruitful, as it yields higher demand at any given price tomorrow. On the other, higher prices make it more attractive to undercut the other firm today and obtain the entire demand in the short run. First, we find that history dependent demand leads to overpricing (relative to the myopic profit maximizing price), as the firms consider it an investment in future demand which they can take advantage of through discounts. Second, the firms are able to coordinate on monopoly behavior as long as an upper bound is not crossed. Prices that are too high are followed by very large discounts, after which the firms gradually raise it until a steady state is reached. Above this upper bound, a higher price today leads to a larger discount tomorrow and lower lifetime profits for the firms.

2.1 Introduction

A well established observation in the empirical marketing literature is the reference-pricing bias: When consumers make a purchasing decision, they are heavily influenced by some reference price which they use to contextualize the prices at which a good is offered to them. Furthermore, this reference price is formed to a large extent by the consumer's past observations. In their work, which surveys and generalizes the findings on the reference-price bias, Kalyanaram and Winer (1995) summarize as follows: "First, there is ample evidence that consumers use reference prices in making brand choices. Second, [...] consumers rely on past prices as part of the reference price formation process". In particular, buyers experience an additional sensation of gain (loss) when purchasing the good at below (above) the prices observed in the past. For instance, a consumer who makes a one-shot decision of how many units of a product to purchase at \$50 would buy more if she has observed it being sold for \$70 in the past than if she observed it being sold for \$30, even when she is certain of the intrinsic value of the good. We refer to this aspect of consumer behavior as "dynamic reference pricing".

This paper looks at the influence of such consumer behavior on the outcomes of oligopolistic price setting competition with repeated interactions. In this environment, dynamic reference pricing by consumers affects the competitiveness of the market, as forward looking firms have to take into account the future demand shifts caused by the current prices. This affects the sustainability and profitability of tacit collusion. For instance, if the firms coordinate on a very high price today, there is the additional benefit that the demand will be higher at any given price tomorrow, as tomorrow's consumer will compare the prices she faces to a high reference. On the other hand, the same dynamic reference pricing behavior can provide the firms additional incentive to undercut each other, which reduces the ability of the firms to collude. In this setting, our main focus is to study how the ability of firms to coordinate on collusive pricing patterns is determined by the current reference price. We check if and when a higher demand due to higher reference prices translates into higher profits for competing firms.

To this end, we present an infinite horizon Bertrand duopoly model with linear demand. In each round, two forward looking firms selling an identical good produced at zero cost set prices simultaneously and the firm that sets the lower price takes the entire demand. If they set the same price, they share the demand equally. The objective of the firms is to maximize their discounted lifetime profit. The addition in this paper is that the demand for the good in each period is not only determined by the price they set in the current round, but also by how it compares to a reference point that is elicited from the past prices. For any given current price, the demand is increasing in the reference point.

The consumer reference price in each period is given by the smaller of the two prices set by the firms in the previous period. First note that since they are selling identical goods, the same reference price applies for both firms. As a result, the price that one firm sets creates an additional externality on the other firm. Not only does the competitor's price determine the share of the market that a firm gets today, but it also determines the demand function that she will face tomorrow due to a change in consumer reference. The idea behind taking the smaller of the two previous prices as the reference is that in each period, the only firm who makes any sales is the one that sets the lower price. As a result, if a consumer is observing realized transactions, the only price she can base her reference price on is the lowest one.

For this model, we characterize the set of lifetime profits that a firm can obtain as a symmetric strategy subgame perfect equilibrium (sSPE) outcome. That is, we pin down the profits that the firms can obtain from colluding on a price path where they always set the same price and have no incentive to deviate following any pricing history. Allowing for public randomization, this set corresponds to all the values between the lowest and the highest sSPE payoffs. We show that as in the standard Bertrand model without reference effects, the lowest sSPE payoff is always zero, obtained when both firms set zero price after any history.

The highest payoff that the firms can obtain by colluding on a sSPE pricing strategy depends on the level of patience that firms have (i.e. how much they discount future profits), and the consumer reference price at the start of the game. We show that for low levels of patience, there is no sSPE that yields positive payoff for the firms, so the highest sSPE profit is zero. For intermediate and high levels of patience, the analysis takes the highest feasible payoff as a benchmark. The highest feasible payoff is sharing the lifetime profit of a monopolist who faces the same demand function and reference evolution rule as in the two firm competition. We call this the "long run monopolist" profit. Whenever the two firms can imitate the optimal pricing policy of the long run monopolist in a sSPE, the highest sSPE payoff is half of the long run monopolist profit. For sufficiently high levels of patience, the firms can collude on long run monopolist pricing with any initial reference price.

For intermediate values of firm patience, we show that collusion on long run monopolist pricing is possible (in a sSPE) if and only if the initial reference price is low enough. If the initial reference is above a certain threshold, the firms have incentive to deviate under the long run monopolist policy. In that case, the best they can coordinate on is setting a price that is low enough today, and then following the long run monopolist policy tomorrow onward.

The reason that high initial reference prices do not allow collusion on long run monopolist behavior can be described by two mechanisms. First, if the initial reference is very high, the long run monopolist policy follows a decreasing price path. That means the future demand will be decreasing over time as the reference price becomes lower. So the competing firms have an incentive to undercut today and take the entire demand before the market becomes small. This effect is similar to the incentive to undercut in the boom periods under exogenous business cycles. Another reason why high initial reference leads to deviation is that the long run monopolist always overprices relative to a myopic monopolist who would like to maximize the profit today. This can be seen as an "investment" in future reference points as it means making less profits in the current period. Then for the competing firms, coordinating on the long run monopolist policy requires forgoing immediate demand that they could extract by undercutting (and setting the myopic monopoly price). The overpricing, and thus the forgone profit is increasing with the initial reference price. After some level, the immediate profit the firms need to sacrifice by colluding becomes too large.

Finally, we look at the comparative statics of the highest sSPE payoff with respect to the initial reference. We show that for interior levels of patience, the highest sSPE payoff obtains a unique maximum at the highest initial reference that allows for coordination on long run monopolist pricing. Below this threshold, the highest sSPE profit is increasing in initial reference. If, on the other hand, the initial reference price is such that the firms are not able to coordinate on the long run monopolist policy, the highest sSPE payoff is strictly decreasing.

To the best of our knowledge, there is no study that looks at the effect of dynamic reference prices on the sustainability and profitability of tacit collusion in an infinite horizon setting. The effects of reference dependence on monopoly pricing (Heidhues & Koszegi (2004, 2014), Spiegler (2012)) and competition (Heidhues & Koszegi (2008, 2014), Karle & Peitz (2014), Hahn et. al. (2018)) in static environments have been explored by several studies. These papers focus on the effect of loss aversion on market outcomes. Piccolo & Pignataro (2018) model reference dependent consumers in an infinite horizon price competition setting as we do, but they impose a time-invariant reference point, and study the effects of loss aversion when the consumer is uncertain about the intrinsic value of the good. Note that loss aversion (i.e., asymmetry between the utility effect of a higher and lower price than the reference) is an empirically well documented aspect of consumer behavior and commonly included in theoretical modelling. Here, however, our focus is the effects of the change in the reference point itself over time. Therefore, we abstract away from loss aversion and consider instead a symmetric effect for tractability. A discussion on the potential implications of loss aversion for our results and possible extensions can be found in Section 6.

Dynamic pricing with history dependent reference prices has been studied predominantly in the operations research literature. Notable papers in the area include Kopalle et. al. (1996), Fibich et.al.

(2003), and Popescu & Wu (2007), who study optimal monopoly pricing when reference points are determined by past prices. Our result that the long run monopolist price policy converges to a steady state and overprices relative to a myopic monopolist is in line with the findings of Kopalle et.al. (1996), who postulate this behavior numerically, and Fibich et.al. (2003), who formally derive it for continuous time. Popescu and Wu (2007) study the problem for a more general class of demand functions (as opposed to linear) and derive a wide range of conditions such that this behavior holds. Furthermore, Kopalle et. al. (1996) look at an infinite horizon duopoly problem with product differentiation where the firms have two possible pricing strategies: constant or cyclical pricing. They show that with loss neutral consumers, the equilibrium given these two alternatives is constant pricing. Also note that they use a setting where the reference price is completely firm specific. Thus, their analytical results do not take into account the "prisoner's dilemma" caused by the reference externality firms can impose on each other. Other papers that look at competition with dynamic reference prices include Anderson et. al. (2005), Yang et. al. (2012) and Coulter et. al. (2014). However, they work in a finite horizon framework with some product differentiation. As such, they are unable to make observations regarding tacit collusion.

The rest of the paper is structured as follows. Section 2 introduces the model. Section 3 states the set of sSPE payoffs as a function of firm patience and the initial reference price. Section 4 outlines the argument underlying the highest sSPE payoffs. Section 5 presents comparative statics with respect to initial reference. Section 6 underlines some potentially useful further steps in the analysis and summarizes the findings.

2.2 Model Setup

Two identical firms ($i = 1, 2$) play an infinite horizon price setting game where they sell an identical good. In each stage $t \in \{1, 2, ..., \infty\}$, they set prices simultaneously. The firm that sets the lower price takes the entire market. If the prices are equal, demand is split equally. Let the period t demand for the good at price p be given by

$$d(p|r_t) := v - p + \lambda(r_t - p)$$

where $v \in \mathbb{R}_{++}$ is a constant and r_t is the consumer's period t reference price for the good, formed upon observing past prices. Then $\lambda \in \mathbb{R}_+$ corresponds to the relative weight of the reference effect. This demand function can be interpreted as the optimal purchase amount of one individual who lives for a single period, has a known intrinsic value of the good, suffers disutility from the price paid and experiences an additional loss (gain) for how much the transaction price exceeds (is lower than) the reference price. The fact that we use the same parameter λ for $p > r_t$ and $p < r_t$ implies loss neutrality.[1]

We take the initial period reference price r_1 as given. Denote the price set by firm $i \in \{1,2\}$ in period t by p_t^i. For $t \geq 2$, we set $r_t = min\{p_{t-1}^1, p_{t-1}^2\}$.[2] That is, we take the reference price equal to the smallest price in the previous period. One motivation for taking the minimum as the reference is that in our model, the firm that sets the lower price makes all the sales. Thus, the smallest of the two prices is the only one at which a transaction occurs. A consumer who only observes realized transactions in earlier stages would therefore take only the lowest price into account when forming reference prices. In other settings, it is possible to think of other measures of past prices as the reference price, such as the average of the two prices set in the previous period.

For the initial reference price, assume $r_1 \in [0, v]$. Let $p^c(r_t) := \frac{v+\lambda r_t}{1+\lambda}$ denote the choke price as a function of the reference price (i.e. the price at which the demand equals zero: $d(p^c(r_t)|r_t) = 0$). For both firms $i \in \{1,2\}$, we restrict the set of possible prices at time t given reference r_t to $p_t^i \in [0, p^c(r_t)]$. That is, given r_t, we only allow for prices that yield non-negative demand. Note that $r_1 \in [0, v]$ and $p_t^i \in [0, p^c(r_t)]$ together imply that $p_t^i, r_t \in [0, v]$ for all $t \in \mathbb{N}_+$ and $i \in \{1,2\}$.

Assume that both firms produce at zero marginal cost. The profit from selling $d(p|r_t)$ units at price p is thus given by $\pi(p|r_t) := pd(p|r_t)$. Denote the vector of prices set by the two firms at time t by $p^t := (p_t^1, p_t^2)$. The period t profit of firm i under price vector p^t and reference price r_t is as follows.

$$\pi^i(p^t|r_t) = \begin{cases} \pi(p_t^i|r_t); & p_t^i < p_t^j \\ \frac{1}{2}\pi(p_t^i|r_t); & p_t^i = p_t^j \\ 0; & p_t^i > p_t^j \end{cases}, \qquad i \neq j \in \{1,2\}$$

In the dynamic game, the payoff of firm i under the price vector path $\{p^t\}_{t=1}^{\infty}$ is then given by the discounted lifetime profit

$$\sum_{t=1}^{\infty} \delta^{t-1} \pi^i(p^t|r_t)$$

where discount factor $\delta \in (0,1)$ and initial reference $r_1 \in [0.v]$ are given and $r_t = \min\{p_{t-1}^1, p_{t-1}^2\}$ for all $t \geq 2$.

[1]This linear demand function is used for tractability. In general, one can derive the demand from a consumer utility function that satisfies these properties. For example, if we maximize utility $u(p,q|r) = vlnq - q(p + \lambda(p - r_t))$ where q denotes the quantity, we obtain a demand function that is decreasing and convex in p and increasing in r_t.

[2]This process for the reference price can be interpreted as the limit case of the exponential smoothing process $r_t = \alpha r_{t-1} + (1-\alpha)\min\{p_{t-1}^1, p_{t-1}^2\}$ as the memory parameter α converges to zero. Note that exponential smoothing is the most commonly used rule in the dynamic reference literature discussed in the previous section.

Let $s^i : H \to [0,v]$ describe a pure pricing strategy of firm i, where H is the set of all possible pricing histories with a time $t \geq 1$ history $h_t \in H$ of the form $h_t = \{p^\tau\}_{\tau=1}^t$ and h_0 the initial node. Then given strategy s^i, $s^i(h_t) \in [0, p^c(r_t)]$ denotes the price set at time $t+1$ under history h_t and $s^i(h_0) \in [0, p^c(r_1)]$ denotes the initial price at $t = 1$. Denote by S the set of all such strategies. Since the firms are identical, S denotes the set of all pure strategies for both firms.

2.3 Equilibrium Payoffs

Given the above environment, this section states the set of lifetime profits that can be obtained as the outcome of a subgame perfect equilibrium with symmetric strategies ($s^1(h_t) = s^2(h_t)$, for all $h_t \in H$). The set of such payoff vectors are determined by the initial reference price $r_1 \in [0,v]$ and the discount factor $\delta \in (0,1)$.

Since we are only looking at symmetric strategy subgame perfect equilibria (henceforth sSPE), any payoff vector will yield the same value for both firms. That is, a payoff vector resulting from symmetric strategies is of the form $(w,w) \in \mathbb{R}_+^2$.

Allowing for public randomization over such equilibria, the set of sSPE payoffs corresponds to all convex combinations of the lowest and the highest sSPE payoff. Let $\underline{w} : [0,v] \times (0,1) \to \mathbb{R}_+$ denote the lowest sSPE payoff as a function of the initial reference r_1 and discount factor δ. Similarly, define $\bar{w} : [0,v] \times (0,1) \to \mathbb{R}_+$ as the highest sSPE payoff. The set of all sSPE payoffs as a correspondence is then given by:

$$W(r_1,\delta) = \{w \in \mathbb{R}_+ : w = \alpha \underline{w}(r_1,\delta) + (1-\alpha)\bar{w}(r_1,\delta), \text{ for some } \alpha \in [0,1]\}$$

Propositions 1 and 2 characterize the functions $\underline{w}(r_1,\delta)$ and $\bar{w}(r_1,\delta)$ respectively. Complete proofs of all results can be found in the Appendix.

Proposition 1. $\underline{w}(r_1,\delta) = 0$ *for all* $r_1 \in [0,v]$ *and* $\delta \in (0,1)$.

That is, the lowest sSPE payoff is zero for any level of firm patience δ and any initial reference r_1, obtained by both firms setting zero price following any history. This result follows directly from the observation that $p_t^1 = p_t^2 = 0$ is the unique Nash Equilibrium (NE) of the one shot game under any reference r_t: Denote by $p^m(r_t) := argmax_p \pi(p|r_t) = \frac{v+\lambda r_t}{2(1+\lambda)}$ the optimal price of a myopic short-run (henceforth SR) monopolist that maximizes the profit in the current round under reference r_t. Then in the stage game, if the opponent is setting a price higher than $p^m(r_t)$, the best response of a firm is to set price $p^m(r_t)$. If the opponent is setting a price lower than $p^m(r_t)$, a firm has incentive to undercut the opponent by an arbitrarily small amount. Since zero is the lowest feasible lifetime payoff in the dynamic game and it is obtained by repeating the unique stage NE, it is the lowest sSPE payoff.

In identifying the highest sSPE payoff $\bar{w}(r_1,\delta)$, we take the highest feasible symmetric payoff as a benchmark. For given δ, the highest feasible payoff is a function of the initial reference price r_1 and corresponds to the firms equally splitting the lifetime profit of a monopolist with cost zero who faces the same demand function $d(p|r_t)$ and the reference evolution rule $r_t = p_{t-1}$ in an infinite horizon price setting problem. From now on we call this the long-run (henceforth LR) monopolist problem. [3]

As in the two firm case,the LR monopolist constrained to non-negative prices below the choke price at each stage. Formally, the lifetime profit of the LR monopolist under initial reference r_1 is given by:

$$\begin{aligned} V(r_1) = \quad & \max_{\{p_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1}\pi(p_t|r_t) \\ \text{s.t.} \quad & p_t \in [0, p^c(r_t)], \ \ \forall t \in \mathbb{N}_+ \end{aligned}$$

where $r_t = p_{t-1}$ for all $t \geq 2$. The highest feasible symmetric payoff in the two firm game is then $\frac{V(r_1)}{2}$, obtained when both firms follow the (unique) optimal pricing policy of the LR monopolist. Using this benchmark, the next result states the highest symmetric sSPE payoff $\bar{w}(r_1,\delta)$ for different values of $r_1 \in [0,v]$ and $\delta \in (0,1)$.

Proposition 2. *There exist $\underline{\delta}$, $\bar{\delta}$ with $1 > \bar{\delta} > \underline{\delta} > \frac{1}{2}$ such that:*

(i) *If $\delta \in (0, \frac{1}{2})$, then $\bar{w}(r_1,\delta) = 0$ for all $r_1 \in [0,v]$.*

(ii) *For each $\delta \in [\underline{\delta}, \bar{\delta})$, there exists a unique $\bar{r}(\delta) \in (0,v)$ that yields:*

$$\bar{w}(r_1,\delta) = \begin{cases} \frac{V(r_1)}{2}; & r_1 \in [0, \bar{r}(\delta)] \\ \delta V(p^*(r_1)); & r_1 \in (\bar{r}(\delta), v] \end{cases}$$

Where $p^*(r_1)$ is the unique value in the interval $(0, \min\{p^m(r_1), \bar{r}(\delta)\})$ that solves $\pi(p^*(r_1)|r_1) = \delta V(p^*(r_1))$.

(iii) *If $\delta \in [\bar{\delta}, 1]$ then $\bar{w}(r_1,\delta) = \frac{V(r_1)}{2}$ for all $r_1 \in [0,v]$*

[3] The long-run monopolist problem presented here is similar to the one analyzed in Kopalle et al. (1996) and Popescu and Wu (2007) for a more general class of demand functions. The findings on the optimal LR monopolist policy are in line with the authors' analysis of the loss-neutral demand case.

Proposition $2(i)$ states that if the firms have little patience (i.e. $\delta \in (0, \frac{1}{2})$), the firms cannot collude on any positive price sequence regardless of the initial reference r_1. As a result, the highest and the only sSPE outcome of the game is zero lifetime profit.

Proposition $2(ii)$ and (iii) state that there is an interval of firm patience levels $[\underline{\delta}, \bar{\delta})$ within $(\frac{1}{2}, 1)$ that we can use to describe the highest sSPE payoff as a function of the initial reference r_1. Given a δ in this interval, the firms can coordinate on LR monopoly pricing if and only if the initial reference is low enough. That is, there is an interior threshold $\bar{r}(\delta)$ such that if the initial reference r_1 is (weakly) lower than $\bar{r}(\delta)$, the LR monopoly profit $\frac{V(r_1)}{2}$ is indeed a sSPE outcome.

If the initial reference r_1 is above this threshold, then the firms cannot coordinate on LR monopoly pricing, and thus the payoff $\frac{V(r_1)}{2}$ is not sustainable in a sSPE. In that case, the best that the firms can coordinate on is initially setting a price that is sufficiently low (lower than the SR monopoly price $p^m(r_1)$), and following the LR monopolist path tomorrow onward. Starting with the highest initial price $p^*(r_1)$ that yields no incentive to deviate, the lifetime payoff from following such a price sequence is given by $\delta V(p^*(r_1))$. Finally, if the firms are patient enough ($\delta \geq \bar{\delta}$), sharing the LR monopoly profit can be sustained as a sSPE payoff with any initial reference $r_1 \in [0, v]$.

2.4 Discussion of Highest sSPE Payoffs

This section outlines the arguments underlying Proposition 2. That is, we discuss how the highest sSPE payoff is determined as a function of δ and r_1. First, we specify the class of strategy profiles that are relevant when determining whether a lifetime profit $w \in \mathbb{R}_+$ is a sSPE payoff. Using these strategy profiles, we derive the highest sSPE payoff $\bar{w}(r_1, \delta)$ for $\delta < 1/2$ and $\delta \geq 1/2$ respectively.

2.4.1 Strategies

As shown in Proposition 1, $p_t^1 = p_t^2 = 0$ is the unique NE of the stage game under any reference r_t and 0 is the lowest feasible lifetime payoff under any initial r_1 and δ. Now suppose we want to sustain a price sequence $\{p_t\}_{t=1}^{\infty}$ on the equilibrium path of a sSPE. Since 0 is the lowest feasible lifetime payoff, setting price 0 forever (grim trigger) upon deviation from $\{p_t\}_{t=1}^{\infty}$ is the harshest feasible punishment. Since both firms setting $p_t^1 = p_t^2 = 0$ is a NE of the stage game under any r_t, both firms setting price zero forever is a sSPE of any subgame following a deviation.

From these two observations we can conclude that given r_1 and δ, a sequence of symmetric price vectors $\{p^t\}_{t=1}^{\infty}$ is a sSPE equilibrium path if and only if the firms have no incentive to deviate at any stage under grim trigger punishment with price zero. A lifetime profit resulting from such an equilibrium path is a sSPE payoff.

Formally, $w \in \mathbb{R}_+$ is a sSPE payoff if and only if there exists a strategy $s^* \in S$ of the form $s^*(h_0) = p_1^*$ and for $t \geq 1$

$$s^*(h_t) = \begin{cases} p_{t+1}^*; & p^\tau = (p_\tau^*, p_\tau^*), \forall p^\tau \in h_t \\ 0; & otherwise \end{cases},$$

that satisfies the following. Given r_1, the price sequence $\{p_t^*\}_{t=1}^\infty$ yields

$$w = \sum_{t=1}^{\infty} \delta^{t-1} \frac{\pi(p_t^*|r_t)}{2}$$

where $r_t = p_{t-1}^*$ for all $t \geq 2$. Furthermore, sequence $\{p_t^*\}_{t=1}^\infty$ satisfies the incentive constraint

$$\sum_{t=\tau}^{\infty} \delta^{t-\tau} \frac{\pi(p_t^*|r_t)}{2} \geq \begin{cases} \pi(p_\tau^*|r_\tau); & p_\tau^* < p^m(r_\tau) \\ \pi(p^m(r_\tau)|r_\tau); & p_\tau^* \geq p^m(r_\tau) \end{cases} \tag{IC}$$

for all $\tau \in \{1, 2, ..., \infty\}$. The right hand side of the inequality corresponds to the lifetime payoff obtained by the optimal deviation from s^* at stage τ. If at τ the price p_τ^* is greater than the SR monopoly price $p^m(r_\tau)$, the optimal deviation is to $p^m(r_\tau)$. If $p_\tau^* < p^m(r_\tau)$, the optimal deviation is to undercut p_τ^* by an arbitrarily small amount. In both cases the deviator initially obtains the entire demand at the deviation price and makes zero profits forever after.

Thus, $\bar{w}(r_1, \delta)$ is the highest value in $\mathbb{R}_+$ that is the lifetime profit from a symmetric price vector sequence that satisfies the above incentive constraint at every stage. Note that r_1 plays a role in determining $\bar{w}$ in two ways. First, it affects the lifetime payoff obtained from a price sequence. Second, it determines whether the incentive constraint is violated in the initial round.

2.4.2 Highest Payoff under Low Patience

If $\delta \in (0, \frac{1}{2})$, then the highest sSPE payoff is zero and the firms cannot coordinate on any nonzero price sequence (Proposition $2(i)$). The argument behind this result is as follows. First recall that since we restricted the set of prices under reference r_t to $p_t^i \in [0, p^c(r_t)]$ and initial reference r_1 to $[0, v]$, any possible price sequence (as well as the sequence of reference points implied by the previous price) only takes values in $[0, v]$. Since a price sequence $\{p_t\}_{t=1}^{\infty}$ can only take values in the compact set $[0, v]$, the resulting sequence of stage profits $\{\frac{\pi(p_t|r_t)}{2}\}_{t=1}^{\infty}$ has a finite upper bound (supremum).

Second, note that if $\delta < \frac{1}{2}$, a price sequence that yields a constant stream of stage profits does not satisfy (IC) (i.e does not deter deviation).

Now take any candidate price sequence $\{\tilde{p}_t\}_{t=1}^{\infty}$ and consider the period τ where it yields the highest stage profit across all periods. Clearly, the continuation payoff from pursuing $\tilde{p}_t$ at time τ onward yields a (weakly) lower lifetime payoff than a constant stream of the stage profit obtained at time τ. However, even a constant stream of the stage profit obtained at time τ is not a high enough deter the deviation available. Thus, with the continuation payoff from $\{\tilde{p}_t\}_{t=\tau}^{\infty}$, the time τ incentive constraint will certainly be violated.

When the sequence of stage profits converges to the upper bound instead of obtaining a maximum, we can say that for any $\delta < \frac{1}{2}$ there is a period that yields a stage profit close enough to the upper bound that (IC) is violated, using the same argument as above.

To summarize, we can make the following observation for firms with low ($\delta < \frac{1}{2}$) patience. While at some periods the firms might have no immediate incentive to deviate from colluding on a given sequence of prices, they anticipate a period where the stage profit will be “as high as it ever will". At that period, $\delta < \frac{1}{2}$ guarantees that they have incentive to undercut. The anticipation of the future deviation leads to the unraveling of the collusion. This “highest stage profit" period exists because firms cannot set prices above the choke price, which confines them below a finite reference point. As a result, the only sSPE is setting price zero after any history.

2.4.3 Highest Payoff under High Patience

To determine the highest sSPE payoff when firms have patience $\delta \in [\frac{1}{2}, 1)$ (Proposition $2(ii)$ and (iii)), we first derive some important properties of the highest feasible payoff. Recall that the highest feasible payoff in the two firm game with initial reference price r_1 is given by half of the LR monopolist profit $V(r_1)$. Since for any given price path the future profit of the LR monopolist depends only on the current reference price, we can reformulate the problem in recursive form:

$$V(r) = \max_{r' \in [0, p^c(r)]} \{\pi(r'|r) + \delta V(r')\}$$

Here, state variable r corresponds to the initial reference r_1. The price set at a period with reference r is denoted by r' and is equal to the state in the next period. Let the optimal policy of the LR monopolist be given by the function $f : [0, v] \to [0, v]$. That is, the optimal price in any period with reference r is given by $f(r) \in [0, p^c(r)]$. There is a unique optimal policy function in the recursive problem and it is derived in the proof of Lemma 1 below.

Clearly, payoff $\frac{V(r_1)}{2}$ can be obtained by firms with symmetric strategies only if they both set prices according to the unique monopolist policy (starting from the given r_1). Denote by $f_n(r)$ the n$^{\text{th}}$ iterate of function f on state r (e.g. $f_2(r) = f(f(r))$). Then, the only symmetric price sequence that yields payoff $\frac{V(r_1)}{2}$ is $p_t^1 = p_t^2 = f_t(r_1)$ for all $t \in \mathbb{N}_+$. This means $\frac{V(r_1)}{2}$ is a sSPE payoff of the game starting from r_1 if and only if the price sequence $\{f_t(r_1)\}_{t=1}^{\infty}$ is a sSPE equilibrium path. Since it is the highest feasible payoff, we have that $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ whenever $\frac{V(r_1)}{2}$ is a sSPE payoff.

The next result presents some properties of the recursive LR monopolist problem that are useful in the analysis of the sSPE payoffs in the two firm game.

Lemma 1. *For the solution of the recursive LR monopolist problem, the following properties hold.*

(a) The unique optimal policy function $f(r)$ satisfies $f(r) \in (p^m(r), p^c(r)]$ for all $r \in [0, v]$

(b) For any initial $r \in [0, v]$, $f_n(r)$ monotonically converges to unique steady state $r^{ss} := \frac{v}{2+\lambda(1-\delta)}$ as n tends to infinity.

(c) If the choke price constraint $r' \leq p^c(r)$ is binding at $r = r_1 \in [0, v]$, then it is binding at $r = r_2$ for any $r_2 \in [0, r_1]$.

(d) $V(r)$ is continuously differentiable and strictly increasing.

Here it is helpful to discuss the intuition behind some of the above properties. Recall that we defined $p^m(r) = argmax_p \pi(p|r)$ as the price that a short-run (SR) monopolist would set in a one-period problem with reference r. Then, (a) states that under any reference point, the LR monopolist sets a strictly higher price than the SR monopolist who maximizes the stage profit. The intuition behind this result is simple. The stage profit $\pi(r'|r)$ is a strictly concave function that obtains its maximum at $r' = p^m(r)$. On the other hand, the continuation payoff $V(r')$ of the LR monopolist is strictly increasing in the price r' that she currently sets. Clearly, the LR monopolist has no incentive to set a price below the SR monopolist. When comparing prices $r' \in [p^m(r), p^c(r)]$, the LR monopolist faces a trade-off. In this region, setting a higher current price decreases the current profit but increases the future value through a higher reference point tomorrow. As a result, for any positive δ, it is optimal to set a strictly higher price than the SR monopolist price.

In other words, it is always optimal for the LR monopolist to forgo some current profit as an "investment" in future reference prices. This creates a wedge $f(r) - p^m(r)$ between the optimal LR and SR monopolist prices, respectively. This wedge plays an important role in determining the competing firms' ability to coordinate on the LR monopolist policy. The wedge implies that in addition to taking the current demand in its entirety, the optimal deviation (to $p^m(r)$) also leads to an increase in the current demand.

Result (b) states that that under the optimal LR monopolist policy, the reference point converges to the unique state $r^{ss} = \frac{v}{2+\lambda(1-\delta)}$, regardless of the initial reference r. Recall that since the price of today is the reference point of tomorrow, the n^{th} iterate $f_n(r)$ of the policy function corresponds to the optimal price in the n^{th} period and the reference price in the $(n+1)^{\text{th}}$ period. The convergence is monotonic in the sense that if the LR monopolist is facing a reference $r < r^{ss}$, she sets a price r' strictly higher than the current reference price and if she faces a reference with $r > r^{ss}$, she sets a price strictly lower than the current reference price. Thus, the LR monopolist chooses a decreasing price path if the initial reference is large (greater than r^{ss}) and increasing if it is small (smaller than r^{ss}). In the two firm game, this property affects the deviation decision of the firms through the comparison of current versus future profits when coordinating on the LR monopoly policy. Point (c) and the differentiability of $V(r)$ are more technical results that we use when deriving the sSPE payoffs of the two firm game.

Using the above properties of the LR monopoly problem and bearing in mind that the highest feasible symmetric lifetime payoff under initial reference r_1 is $\frac{V(r_1)}{2}$, we now determine $\bar{w}(r_1, \delta)$ under $\delta \in [\frac{1}{2}, 1)$ in two steps. First, we describe the cases when coordination on LR monopoly pricing is possible. Then, we pin down the highest sSPE payoff when it is not. Recall that a payoff is a sSPE outcome given initial reference r_1 if and only if it is the lifetime profit from a symmetric price sequence that that yields no incentive to deviate at any stage under grim punishment.

When is LR monopoly pricing sSPE?

Suppose that given r_1, we would like to support the highest feasible payoff $\frac{V(r_1)}{2}$ as a sSPE outcome. As stated above, the only way that the firms can obtain this payoff is if they coordinate on the optimal policy $\{f_t(r_1)\}_{t=1}^{\infty}$ of the LR monopolist. Lemma $1(a)$ tells us that the LR monopolist always prices above the SR monopolist price $p^m(r)$. As a result, the optimal deviation at a stage with reference r is to set $p^m(r)$. Thus, the incentive constraint at each stage states that the payoff from sharing the LR monopolist profit equally is higher than taking the entire SR monopolist profit under the current reference point (and zero tomorrow onward). This inequality can be expressed in terms of "positive net gain from cooperating" as $g(r) := V(r) - 2\pi(p^m(r)|r) \geq 0$.[4] For subgame perfection, the constraint $g(r) \geq 0$ has to hold for all reference points $r \in \{r_1\} \cup \{f_t(r_1)\}_{t=1}^{\infty}$ on the LR monopolist path starting from r_1.

[4]Cooperation is assumed in case of indifference.

In the Appendix, it is shown that the continuous function $g(r)$ cuts zero at no more than one value of $r \in [0, v]$. Furthermore, if it does cut zero, then this intersection must be from above. Finally, for any $\delta \geq \frac{1}{2}$, we have that $g(0) > 0$. Then depending on δ, we are in one of two possible cases: Either $g(r) > 0$ for all $r \in [0, v]$, or there is a unique "incentive threshold" $\bar{r}$ with $g(\bar{r}) = 0$ such that we have $g(r) > 0$ for all $r < \bar{r}$ and $g(r) < 0$ for all $r > \bar{r}$. In the first case, the firms never have incentive to deviate from LR monopolist policy. In the second, they have incentive to deviate if only if the reference point is above a certain interior threshold $\bar{r}$.

By Lemma $1(b)$, we know that the LR monopolist policy monotonically converges to the unique steady state r^{ss}. Then starting from initial reference r_1, any reference price that the firms encounter on the LR monopolist path will be between r_1 and r^{ss}. Further note that the above observation on $g(r)$ implies convexity of the set of references that satisfy $g(r) \geq 0$. Thus, we can conclude that LR monopolist policy is a sSPE if and only if $g(r_1) \geq 0$ and $g(r^{ss}) \geq 0$. That is, if the incentive constraint holds at both the initial stage and the steady state, then we know that the firms have no incentive to deviate at any stage.

Clearly the incentive constraint becomes more slack as firms become more patient. Thus, the incentive threshold $\bar{r}$ is increasing with δ. It is shown that there is a value $\underline{\delta} \in (\frac{1}{2}, 1)$ that yields $\bar{r} = r^{ss}$. If the firms' patience is strictly below $\underline{\delta}$, we have $\bar{r} < r^{ss}$ and the incentive constraint does not hold at the steady state. In that case, the firms anticipate the eventual incentive to deviate (as the reference gets close enough to steady state), which leads to the unravelling of the collusion. Hence we know that for $\delta < \underline{\delta}$ it is impossible for the firms to coordinate on LR monopolist pricing, regardless of the initial reference.

Note that if the firms are infinitely patient ($\delta \to 1$), they will never want to deviate. As a result, we can say that there is an interior patience level $\bar{\delta} \in (\underline{\delta}, 1)$ that yields $\bar{r} = v$. Above this level of patience, the incentive constraint holds for all reference points in $[0, v]$, and thus, LR monopolist policy is a sSPE outcome from any initial reference r_1. That is, $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for any $r_1 \in [0, v]$ whenever $\delta \geq \bar{\delta}$.

When $\delta \in [\underline{\delta}, \bar{\delta})$, the threshold $\bar{r}$ is in $[r^{ss}, v)$. At the steady state, there is no incentive to deviate. This means that whenever the incentive constraint holds in the initial round, then LR monopoly pricing is a sSPE outcome. So if the initial reference price r_1 is smaller than $\bar{r}$, $\frac{V(r_1)}{2}$ is a sSPE payoff. If $r_1 > \bar{r}$, the firms would like to deviate in the initial round and the LR monopoly behavior cannot be sustained as a sSPE. Hence the result "LR monopoly pricing is a sSPE outcome if and only if the initial reference price is low enough".

Before describing the highest payoff when LR monopoly pricing is not a sSPE, it is helpful to briefly discuss the driving force behind the result that coordination on LR monopoly behavior is only possible if the firms start with a low reference price. There are two main channels through which the incentive constraint gets stricter as the initial reference price increases (i.e. $g(r)$ cuts zero from above).

First, the LR monopolist policy converges to the same steady state regardless of where it starts. Then, the higher the initial demand, the more sharply decreasing a path it follows. So for the two firms who want to coordinate on this path, a higher initial reference point makes today's market more valuable relative to the markets in future stages. This decreases the net gain from coordinating in the initial stage. Above a certain level, it becomes optimal to undercut in the initial period and take the whole market before the reference decreases. We can view this channel as the "dynamic effect" of the initial reference, since it determines how the firms compare today's market with the continuation value.

The second channel follows from the observation that the LR monopolist always prices strictly above the SR monopoly level. So if the firms want to coordinate on the LR monopolist policy, this also includes coordination on overpricing as an investment in tomorrow's reference point. If they deviate, however, the future markets do not matter under grim punishment. As a result, the optimal deviation is to the SR monopolist price, which maximizes the stage profit in the current round. Then, the deviator not only takes the entire demand instead of just half, but she also faces a higher total demand at the SR monopoly price. The wedge $f(r) - p^m(r)$ between LR monopoly price and the optimal deviation price is increasing in the reference point r. So if the initial reference r_1 is too high, cooperation means forgoing too much demand in the initial stage. Thus, collusion on the LR monopolist policy becomes harder when the initial reference is higher. We can think of this as the "static effect" of a high initial reference price: The increasing wedge would make deviation more attractive even if we kept the size of today's demand relative to future demand constant on the equilibrium path.

What is $\bar{w}(r_1, \delta)$ when LR monopoly pricing is not sSPE?

Now consider the case when $\delta \in [\underline{\delta}, \delta)$ and $r_1 > \bar{r}(\delta)$. As discussed above, in this case the initial reference is too high to coordinate on the LR monopolist policy. That is, $\frac{V(r_1)}{2}$ is not high enough to deter a deviation to $p^m(r_1)$ in the initial round. If even the highest feasible lifetime payoff cannot deter a downward deviation to $p^m(r_1)$, then no feasible payoff can. So we know that a price sequence that allows downward deviation to $p^m(r_1)$ in the initial round cannot be a sSPE outcome. As a result, any sSPE path should initially set a price $p < p^m(r)$.

Consider an initial reference r_1 with $r_1 > \bar{r}(\delta) \geq p^m(r_1)$. That is, LR monopolist pricing is not a sSPE starting from r_1, but it is a sSPE starting from any initial reference below the SR monopolist price $p^m(r_1)$. Then if the firms are setting some $p < p^m(r_1)$ in the initial round, the highest sSPE payoff of the game starting tomorrow is the LR monopoly payoff $\frac{V(p)}{2}$. Thus, we know that the highest sSPE payoff starting from r_1 is obtained by setting some $p < p^m(r_1)$ today and following the LR monopolist policy tomorrow onward. What remains to be found is the initial price p that yields the highest payoff among those that create no incentive to deviate in the first period.

Since $p < p^m(r_1)$, the optimal deviation in the first round is to undercut by an arbitrarily small amount, in which case the deviator takes the entire demand under reference r_1 and makes zero profit tomorrow onward. The payoff from cooperating is sharing the demand today at price p, and sharing the LR monopoly profit starting from initial reference p tomorrow onward. The initial period incentive constraint is that cooperation is weakly better than the optimal deviation. Again, this incentive constraint can be expressed in terms of "net gain from coordination" as $h(p|r_1) := \delta V(p) - \pi(p|r_1) \geq 0$.

The lifetime payoff from a price sequence that initially sets p and then follows the LR monopolist policy is given by $\frac{1}{2}(\pi(p|r_1) + \delta V(p))$ and is strictly increasing in p as long as $p < p^m(r_1)$. So the highest sSPE payoff from such a strategy is obtained by setting the highest initial price p that satisfies the incentive constraint. It is shown in the Appendix that $h(p|r_1)$ cuts zero exactly once for $p \in [0, p^m(r_1)]$ and it is from above. To see the intuition as to why the incentive constraint holds only for low initial prices, consider the two extremes. If the firms are setting initial price p very close to zero, then payoff from undercutting is essentially zero. On the other hand, the continuation value from collusion, which is the lifetime profit of a LR monopolist who starts at initial reference zero, is clearly positive. So the incentive constraint definitely holds. On the high extreme ($p = p^m(r_1)$), the incentive constraint is definitely violated by the above argument that even the highest feasible payoff is not good enough to prevent a deviation that yields $\pi(p^m(r_1)|r_1)$.

The highest sSPE profit is thus obtained by setting the unique $p^*(r_1) \in [0, p^m(r_1))$ that satisfies $h(p^*(r_1)|r_1) = 0$. If the firms set a higher initial price than $p^*(r_1)$, they have incentive to deviate in the first round. If they set a lower initial price, they can do better while still maintaining the incentive constraint by setting $p^*(r_1)$. When we plug in $h(p^*(r_1)|r_1) = 0$ into the lifetime payoff, we obtain $\bar{w}(r_1) = \delta V(p^*(r_1))$.

The result can be extended to initial references r_1 such that there is a range of values below $p^m(r_1)$ where LR monopoly pricing is not sSPE (i.e. $r_1 > p^m(r_1) > \bar{r}(\delta)$). If the initial price is in this range, it is shown that the first period incentive constraint can never hold. Therefore, the initial price has to be in the range $(0, \bar{r}(\delta))$, where the LR monopoly pricing can be sustained from the second period onward. Hence, we can generalize the result as $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ with $p^*(r_1) \in (0, \min\{p^m(r_1), \bar{r}(\delta)\})$ whenever $r > \bar{r}(\delta)$. To summarize, if the initial reference r_1 is too high to collude on the LR monopolist behavior today, then the best that the firms can collude on is setting a price that is low enough to sustain LR monopoly behavior starting tomorrow.

2.5 Comparative Statics

We now look at the behavior of $\bar{w}(r_1, \delta)$ with respect to r_1. From Proposition $2(i)$ we know that if $\delta < \frac{1}{2}$, then $\bar{w}(r_1, \delta)$ is constant at zero. If $\delta \geq \bar{\delta}$, we have $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ which is strictly increasing in r_1. For the intermediate region of patience $\delta \in [\underline{\delta}, \bar{\delta})$, the behavior of $\bar{w}(r_1, \delta)$ is ambiguous and depends on the level of initial reference r_1. In this case, we can make the following observation from Proposition $2(ii)$.

Corollary to Proposition 2. *If $\delta \in [\underline{\delta}, \bar{\delta})$, then $\bar{w}(r_1, \delta)$ obtains a unique maximum at $r_1 = \bar{r}(\delta)$.*

That is, if the firms' patience level is in the region $[\underline{\delta}, \bar{\delta})$, the highest sSPE payoff is obtained when starting from the largest initial reference that allows coordination on LR monopoly pricing. Note that $\bar{r}(\delta)$ is interior. So for $r_1 \in (\bar{r}(\delta), v)$, the highest sSPE payoff is strictly decreasing in the size of the first period demand.

For any $r_1 \leq \bar{r}(\delta)$, we have $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$. This means that as long as the firms are able to coordinate on LR monopolist pricing, the best they can do improves with a higher initial reference. On the other side of $\bar{r}(\delta)$, recall that if we have $r_1 > \bar{r}(\delta)$, then $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$. The reason why $\bar{w}(r_1, \delta)$ is decreasing in this region is that $p^*(r_1)$ is itself a strictly decreasing function. In other words, suppose the firms want to sustain a price sequence that initially sets a low price and follows the LR monopolist path tomorrow onward. Then the highest price that does not create incentive to deviate in the first round is decreasing in the size of the demand. The reason is that the gain from cooperation $h(p|r_1)$ is strictly decreasing in r_1 for any given $p < (0, \min\{p^m(r_1), \bar{r}(\delta)\})$.

For illustration, suppose the firms have incentive to deviate from setting a price $p < p^m(r_1)$ today and following LR monopolist pricing tomorrow onward if their initial reference is r_1. With a higher reference price $r_1' > r_1$, gain from deviation increases because the demand in the first period is higher. However, the gain from cooperation (which is the LR monopolist profit starting from p tomorrow) is the same. So if setting p is not incentive compatible in the first round under r_1, then it is also not incentive compatible under r_1'. Therefore, the highest price that is incentive compatible is decreasing in the initial reference.

Finally, note that $\bar{w}(\bar{r}(\delta), \delta) = \frac{V(\bar{r}(\delta))}{2}$ is the highest feasible payoff when starting from $\bar{r}(\delta)$. Then clearly, it is greater than $\lim_{r_1 \searrow \bar{r}(\delta)} \bar{w}(r_1, \delta)$. This observation, together with $\bar{w}(r_1, \delta)$ increasing in r_1 for $r_1 < \bar{r}(\delta)$ and decreasing for $r_1 > \bar{r}(\delta)$ allows us to conclude that the highest sSPE payoff $\bar{w}(r_1, \delta)$ is maximized at $r_1 = \bar{r}(\delta)$. Hence, for interior values of patience, the highest lifetime profit that the firms can collude on does not necessarily increase with the initial size of the market. It is increasing if the initial market is such that they are able to collude on the optimal pricing policy of the LR monopolist and decreasing otherwise.

2.6 Comments

In the above setting, we can think of other features to include in the model that can be helpful in understanding the effect of dynamic reference pricing on collusion. The most natural of such considerations is consumer loss aversion. It is a common modelling assumption based on substantial empirical evidence that the utility effect of a price increase (relative to the reference point) is greater than that of a price decrease for the consumers. In the our setting, this can be represented by a demand function which is kinked at the reference price. A rigorous discussion of how the collusion behavior is affected by this consideration can improve the robustness of our predictions, and therefore seems warranted. However, using findings from existing literature, we can make some conjectures as to why the main results will continue to hold qualitatively under loss aversion.

Fibich et. al. (2003) show that in the case of a kinked (linear) demand, the behaviour described Lemma $1(a)$ and $1(b)$ holds. [5] That is, from any initial reference, the LR monopolist policy monotonically converges to a steady state and overprices with respect to the SR monopolist level. One difference is that it is possible for the set of steady states to be a closed interval instead of a singleton. Even in that case, however, the LR monopolist policy converges to the upper (lower) bound of this set when the initial refence is above (below) this interval. This means the price path that provides the highest feasible payoff for the competing firms has the same form under loss aversion.[6]

Given these observations about loss aversion, consider for example our result that collusion on LR monopoly pricing is not possible when the initial reference is too high. [7] Since for higher initial references the reference still decreases more steeply over time, the dynamic channel we describe (i.e., deviate before the market becomes small) continues to apply. Since the optimal deviation under a high initial reference price is still a discrete downward jump to the SR monopolist price, the static channel (i.e., the "wedge" is too large to ignore) continues to apply as well. These effects are qualitatively unaffected when we include asymmetry around the reference price. Overall, as the strategic considerations by

[5]They do so in a continuous time framework. However, it can be seen from the monotonicity conditions of Popescu and Wu (2007) that this also holds in discrete time.

[6]Linear demand is sufficient but not necessary for monotonic convergence of the LR monopolist policy to a steady state under loss aversion. In their analysis of monopoly pricing for a more general class of demand functions, Popescu and Wu (2007) provide a wide range of relatively weak conditions such that this holds. From their findings, for instance, it follows that $\alpha = 0$ or the convexity of $V(r)$ both imply this behaviour.

[7]Recall that such an initial reference is higher than the steady state

the competing firms that drive our main results do not rely on symmetry around the reference point, we can reasonably predict their robustness to loss aversion. Nevertheless, the formal analysis of these incentives when allowing for asymmetry is a useful further step.

A second consideration is that the current model works with a very short consumer memory. While forming reference prices, the consumers only use the prices from the period immediately preceding the current one. An interesting observation would be the change in the possible profits from collusion with respect to the length of consumer memory. As stated above, we can interpret the results here as the limit case of the commonly used exponential smoothing process $r_t = \alpha r_{t-1} + (1 - \alpha) p_{t-1}$ as α tends to zero.

Some intuitive predictions can be made about the consequences of increasing the memory parameter α. Recall the intuition behind the result "the highest sSPE payoff is decreasing if LR monopoly pricing is not sustainable" is that if the best collusion is to initially set a low price and follow the LR monopolist path tomorrow onward, then for firms who are setting a given initial price, a higher initial reference increases only the current demand and does not affect the continuation payoff. As a result, the net gain from deviation is increasing with the initial reference. If the consumers have longer memory, however, a higher initial reference will directly mean a higher reference tomorrow. So with a longer consumer memory, we can expect the highest sSPE payoff to be "less decreasing" in initial reference. In general, we would expect the effects of dynamic reference pricing to become weaker as the consumer memory becomes longer. Note that we obtain the standard Bertrand model without reference effects if consumers have infinite memory ($\alpha = 1$).

Other promising extensions include the behavior of the sustainable collusion profits with respect to the number of firms as well as varying levels of product differentiation and cross-reference effect between the firms. Under product differentiation, we would expect some degree of independence between the reference price of the two firms to reduce the deterrence of punishment for deviating from a collusive price pattern. These, among other further steps in the analysis, may help deepen our understanding of the dynamics described in the model above.

In sum, the analysis of the price competition model presented in this paper gives us the following insights about the extent of tacit collusion in a market with dynamically reference pricing consumers. If the firms have very little patience, collusion is not possible. If they are sufficiently patient, they can always imitate a long run monopolist, which gives them the highest feasible profit in the market. If the firms are moderately patient, then the current reference of the consumers plays a critical role in determining the extent to which they can collude. If the they are starting with consumers with low enough reference, the firms can coordinate on setting long run monopolist prices. Otherwise, the best they can do is to lower the reference price by setting a low price initially, and imitate the long run monopolist thereafter. From this case, we also observe that a high reference price does not always translate to a higher collusion profit. If the current reference is high, increasing it further leads to a decrease in the highest profit that the firms can obtain through collusion.

3

Volunteer’s Dilemma with Social Learning

Abstract We study a volunteer’s dilemma game with sequential moves. If one individual exerts costly effort, a public good that benefits everyone equally is produced. The value of the public good is uncertain, and each individual receives a private signal about it. The central mechanism of the model is that individuals make inferences about each other’s private signal by observing their action. We focus on the behavior of the equilibrium probability of public good provision with respect to population size and show that it is not monotonic. If the population is small, increasing it beyond a certain threshold leads to an upward jump in provision probability. Above this threshold, provision probability gradually decreases as population grows. This allows two observations in the presence of social learning: (i) There is a unique and finite population level that maximizes provision probability and (ii) The “bystander effect" hinders provision only if the population is already large. A second result shows that the effect of effort cost on provision is ambiguous: A rise in cost can increase or decrease provision probability.

3.1 Introduction

Situations where a few or even only one member of a group has to take costly action for a goal that benefits the entire group are common in social interactions. When a motorist is stranded on a highway, it could be sufficient that one individual stops and allows her to use their phone, but doing so takes time and effort. When voting for a proposal that requires unanimity to pass, one veto can help an entire subset of voters that could potentially be hurt by the proposal. Vetoing, however, comes at the cost of backlash from another subgroup that supports the proposal. Such settings are known as the “Volunteer’s Dilemma" (Diekmann, (1985,1986)).

This paper studies the Volunteer’s Dilemma when there is uncertainty and dispersed private information about the value of the common good, and members of the group can learn about it by observing each other’s decision of whether to volunteer. For instance, passers-by may not be certain that the motorist on the side of the road is actually in need of help. The group that is considering to oppose a proposal may not be certain that it is indeed hurtful to them. In these situations, observing that others have refused to volunteer can cause group members to update their beliefs and push them towards the opinion that a volunteer is not needed after all. Our main focus is how the probability that a volunteer arises is affected by the number of potential volunteers (i.e., group size).

Specifically, we study a model where a finite number of individuals sequentially decide whether to volunteer at a given personal cost and produce a public good or refuse. Each individual observes the action of all those who move before her. As soon as one individual decides to volunteer, the public good is produced and the game ends. All individuals value the public good equally, which can either be high or low. The public good is worth the cost only if it is of high value. Importantly, individuals do not know the value of the good, but they each receive (conditionally independent) private signals about it. Thus, the choice of an individual informs others who observe it about her private signal.

Our main finding is that the highest public good provision probability (i.e., probability that a volunteer arises) across equilibria is not monotonic with respect to group size. When the group size is small, the provision probability is not affected by an increase. Once the group size reaches a certain threshold, however, the provision probability makes an upward jump. Above this threshold, increasing the group size further gradually decreases the provision probability. This behavior indicates that in the presence of social learning, there is a finite group size that maximizes the probability that the public good is produced.

The upward jump occurs because the members can coordinate on an equilibrium where they all volunteer with positive probability only if the group is sufficiently large. When the group size is below the threshold, the dominant strategy of the final mover is to volunteer for certain when she receives the high value signal. Anticipating this, all earlier movers free-ride off her in the unique equilibrium of the game. Since their strategy is refusing to volunteer regardless of their signal, their actions reveal no information. Therefore, there is no information aggregation and public good provision hinges on just one signal realization; that of the final mover.

If the group size is sufficiently large, however, another (mixed) equilibrium exists, and it yields higher provision probability: With a large group, it is no longer dominant for the last mover to volunteer. If she believes that the past movers would have volunteered had they received the high signal, then with many past refusals she can infer enough low signals to rationalize refusing despite having received a high signal herself. If such is the belief of the last mover, earlier movers cannot free-ride by always refusing, because their deviation will be mistaken for a low signal and deter the others from volunteering as well. Hence, when the group can generate a large enough number of refusals, an equilibrium can be sustained where players do not refuse for certain in order to free ride off the last mover. This equilibrium is an improvement in terms of provision probability.

The gradual decrease with respect to group size beyond this threshold occurs because in the mixed equilibrium, individuals' weakening incentive to volunteer (which arises from the expectation that someone else will do so) outweighs the increased number of potential volunteers.

A second result states that for a given group size, a higher cost of volunteering can lead to a higher provision probability. It follows from the same reasoning as the first result: The higher the cost of volunteering, the smaller the number of past refusals required to convince the last mover to refuse with positive probability. For every group size, there is a lower bound on volunteering cost, above which an equilibrium where all players may volunteer is possible.

The effect of group size on provision probability in Volunteer's Dilemma has been of interest for several studies (Weesie (1998), Harrington (2001), Osborne (2004), Bergstrom (2017)). An important result in the literature on Volunteer's Dilemma is that an increase in group size not only decreases the likelihood of volunteering for each individual, but also the overall probability that a volunteer arises (Diekmann (1985)). This prediction, however, is met with mixed evidence from experimental studies (Goeree et. al. (2017)). Our findings suggest that this result only starts to hold once the group is already large. For small groups, an increase in size can allow them to coordinate and drastically increase the provision probability. Our mechanism beyond the group size threshold is the similar to that of Diekmann (1985). Our description of an optimal group size and a lower bound for coordination, however, can contribute to the understanding the effects of population on prosocial behavior. It can also help bridge the gap between theoretical predictions and experimental patterns in the area.

Various sources of uncertainty (e.g., Bliss and Nalebeuff (1984), Weesie (1994), Hildebrand and Winter (2018)) as well as the consequences of sequential moves (e.g., Bergstrom (2017)) has been investigated in theoretical studies. For instance Hildebrand and Winter (2018) find that uncertainty about population size can foster cooperation. Bergstrom (2017) considers a dynamic setting and argues that under sufficient heterogeneity in volunteering costs, a higher frequency of arrivals by potential volunteers leads to a shorter time until someone chooses to volunteer. We combine these modelling ingredients and analyze the effects of social learning (in the sense of learning by observing others' actions). Our model shares the move sequence and information structure with the seminal study on herd behavior by Bikhchandani, Hirshleifer, and Welch (1992), and incorporates into their framework the decision of whether to invest in a public good instead of a private one.

The paper is structured as follows. Section 2 describes the model. Section 3 states and discusses the equilibria of interest. Section 4 observes the comparative statics of provision probability with respect to group size and cost. Section 5 concludes.

3.2 Setting

A finite number N of players $i \in \{1,...,N\}$ sequentially choose an action $a_i \in \{A,R\}$. Actions A and R denote "adopt" and "reject" respectively. The order of moves is exogenous and known to all. If Player i chooses R, the game continues and it is Player $i+1$'s turn to choose an action $a_{i+1} \in \{A,R\}$, having observed the actions of all players before her. If Player i chooses A, she pays cost c and a public good that provides the same utility for all N players is adopted. As soon as one player chooses A, the game ends. If all players including player N choose R, the game ends and there is no public good. The common utility from the public good depends on state $s \in \{H,L\}$ (high or low) of the world. If $s = H$, the value of the public good is 1 and if $s = L$ it is 0. We can summarize Player i's payoff function as follows.

$$u_i = \mathbb{1}\{a_j = A \text{ for some } j \in \{1,...,N\} \ \wedge \ s = H\} - \mathbb{1}\{a_i = A\}c$$

The state is unknown to the players and the common prior is $p(s = H) = p(s = L) = 0.5$. Before her move, each player receives a private signal $m_i \in \{h,l\}$ (high or low). Only player i observes m_i, but it is common knowledge that the signals are conditionally independent and $p(m_i = l|s = L) = p(m_i = h|s = H) = p > 1/2$ for all $i \in \{1,...,N\}$.

Throughout the analysis, assume $c \in (1-p, 0.5)$. That is, the expected value of the public good is greater than the cost of adopting under the prior beliefs, but it is smaller than the cost of adopting under the posterior following a low signal.[1] The solution concept is Perfect Bayesian Equilibrium.

[1]Note that this posterior is given by $p(s = H|m_i = l) = 1-p$.

3.3 Equilibria

This section describes the set of rationalizable strategy profiles, states the equilibria of interest, and discusses the underlying incentives.

First note that each player has only two information sets: high signal and low signal. This is because they do not observe the signals of the previous players and the only history of play that leads to their move is the one where all previous players reject. So for each player, a pure strategy simply assigns one action to each possible realization of her own signal.

When a player observes a rejection by all previous movers, she makes inferences about their private signals and updates her belief about the state of the world. She does so in line with the strategy profile. For instance, suppose the strategy profile is such that all players before i would adopt for certain if they received the high signal and reject with some probability if they received the low signal. Then Player i conditions her belief on $i-1$ low signals in addition to her own when it is her turn.

The following Lemma helps simplify the analysis by pinning down the low signal actions for strategy profiles that are candidates for an equilibrium.

Lemma 1. *For every player, each rationalizable strategy rejects with certainty following a low signal.*

This result implies that in equilibrium, we do not have any player adopt after receiving a low signal. Furthermore, it means that as long as every player plays a rationalizable strategy, all information sets are reached with positive probability ex-ante. Hence, there will be no off equilibrium path beliefs under any candidate strategy profile.

Lemma 1 is obtained simply by iterated elimination of strictly dominated strategies. Note that a player does not bear any cost if she rejects. Thus, rejecting always yields a non-negative payoff. If Player 1 receives a low signal, her Bayesian posterior of the high state is $1-p$. Since $c > 1-p$, adopting offers a negative expected payoff, and any strategy that adopts with positive probability under the low signal is strictly dominated by the strategy that prescribes the same action under the high signal but rejects with certainty under the low signal.

Player 1 rejecting for certain under the low signal implies that Player 2's high state posterior following a low signal of her own is (weakly) smaller than $1-p$: If Player 1 rejects for certain under the high signal as well, then her rejection reveals no information about the state and therefore the posterior belief of Player 2 is based solely on her own signal (i.e., equal to $1-p$ after a low signal). The higher the likelihood that Player 1 adopts after a high signal, the more likely it is that her rejection stems from a low signal, leading to a Player 2 posterior strictly lower than $1-p$. In either case, adopting under the low signal yields negative expected payoff for Player 2. Iterating this argument forward until player N yields Lemma 1.

With this observation in mind, we can discuss the equilibria of this game. From Lemma 1 we know that any candidate strategy profile will prescribe all players to reject after a low signal. Thus with some abuse of notation, we denote by σ_i the probability that Player i adopts after a high signal, under (possibly mixed) strategy profile $\sigma := (\sigma_1, ..., \sigma_N)$. Furthermore, in the discussion that follows, the words “adopt" and “reject" will refer to the action following a high signal unless indicated otherwise.

The following result states the unique pure strategy equilibrium of the game. The proofs for all results can be found in the Appendix.

Proposition 1. *The game yields a unique pure strategy equilibrium σ^P. The strategy profile is given by $\sigma_N^P = 1$ and $\sigma_i^P = 0$ for all $i \in \{1, ..., N-1\}$.*

The only pure strategy equibilibrium of this game is one where all players before the final mover reject regardless of their signal, and the final mover (i.e., Player N) adopts if and only if she receives the high signal. In short, all players free-ride off Player N even if they receive the high signal themselves. They do so projecting that the signals are positively correlated (via the state) and Player N is more likely to receive the high signal as well. Furthermore, they anticipate that Player N has no one to defer the responsibility, so it will be indeed optimal for her to adopt if she receives the high signal. Note that since all players up to N reject regardless of their signal, players cannot extract information from the actions of previous players. Therefore, there is no information accumulation in equilibrium and whether the public good is produced depends on the realization of only one signal: that of Player N.

More specifically, in this equilibrium each player $i < N$ compares the payoff $p(s = H|m_i = h) - c = p - c$ from adopting to payoff $p(m_N = h \wedge s = H|m_i = h)$ from rejecting (i.e., the probability that the state is high and Player N indeed adopts) when she receives signal $m_i = h$. Because $c > 1 - p$, we have that the payoff from adopting is smaller than $2p - 1$. Since the signals are conditionally independent, we have $p(m_N = h \wedge s = H|m_i = h) = p(m_N = h|s = H)p(s = H|m_i = h) = p^2$. The rejection payoff p^2 is greater than $2p - 1$ for all $p < 1$ and thus, rejecting and free riding off Player N is indeed optimal.

Next, we turn to the mixed strategy equilibria of the game.[2] In the main statement and the following discussion, we restrict our attention to mixed equilibria where each player adopts with strictly positive probability. This follows the same reasoning as the common focus on symmetric equilibria in the literature studying the effects of population size in Volunteer's Dilemma (starting with Diekmann

[2]Here, the term mixed strategy equilibrium is used to describe those that are supported by strategy profiles where least one player randomizes according to a non-degenerate distribution between her two actions at some information set.

(1985)): Exempting a subset of players from volunteering would require communication or additional coordination tools that are not present in our setting. Note that our game can yield other mixed equilibria where a subset of players reject for certain, and Lemma 2 in the Appendix provides a complete statement of all equilibria of the game. However, as can be observed in the following result, these only exists under conditions such that there is also an equilibrium where all players adopt with positive probability.

Proposition 2 gives conditions for existence of mixed equilibria and describes the form of equilibria where all players adopt with positive probability.

Proposition 2. *There exists a cost level $\underline{c} \in (1-p, 0.5)$ such that the following hold.*

1. *If $c \leq \underline{c}$, then pure equilibrium σ^P is the unique equilibrium of the game.*
2. *For every $c > \underline{c}$, there is a threshold number of players $\underline{N} > 2$ such that;*

   - For $N < \underline{N}$, pure equilibrium σ^P is the unique equilibrium of the game.

   - For $N \geq \underline{N}$, there exists a mixed equilibrium σ^N where $\sigma_1^N = 1$ and for all $i \in \{2, ..., N\}$, we have $\sigma_i^N \in (0,1)$ and $\sigma_i^N < \sigma_{i-1}^N$.

To rephrase this finding, a mixed equilibrium exists if and only if cost c and population size N are sufficiently large. If c is sufficiently low (below some $\underline{c}$), then the "free-riding" pure equilibrium σ^P is the only equilibrium of the game regardless of population size. For $c > \underline{c}$, there is a minimum population size $\underline{N}$ that depends on c. If the population is below $\underline{N}$, then σ^P is once again the only equilibrium. Above $\underline{N}$, there is an additional equilibrium σ^N. The existence of σ^N is important because it yields a higher probability of public good provision than the pure equilibrium σ^P. This will be discussed in the next section.

In equilibrium σ^N, Player 1 adopts for certain (if she receives the high signal). This is followed by a sequence $\{\sigma_i^N\}_{i=2}^N$ such that each Player $i \geq 2$ adopts with probability $\sigma_i^N \in (0,1)$ if she receives a high signal. This is the unique sequence that makes players $\{2, ..., N\}$ all indifferent between adopting and rejecting and is decreasing in i. In other words, the later a player moves, the lower the probability that she chooses to adopt.

Next we look at the intuition behind why N needs to be large for the existence of a sequence $\{\sigma_i^N\}_{i=2}^N$ that makes players $\{2, ..., N\}$ indifferent. In doing so, it is best to start with a 2 player game ($N = 2$). The potential adoption by Player 1 imposes an externality on Player 2: The more likely it is that Player 1 adopts after a high signal, the more likely that her rejection stems from a low signal and the less incentive Player 2 has to adopt herself. However, recall that $c < 0.5$ means Player 2 is willing to adopt under her prior belief. So even if Player 1's strategy is to adopt for certain after a high signal and perfectly reveal a low signal when she rejects, this information cancels out with Player 2's own high signal, which sets Player 2 back to her prior belief. Hence, with only 2 players, it is strictly dominant for Player 2 to adopt after a high signal. However, as discussed for the pure equilibrium, whenever Player 2's strategy is adopt for certain, Player 1 wants to reject and free-ride. The only equilibrium is therefore σ^P.

If we increase the population size, then under a strategy profile where all players adopt with positive probability, the final mover (Player N) will accumulate more low signals from observing rejections. With enough rejections, her posterior belief that the state is high can be lowered to a level where she is indifferent between adopting or rejecting despite receiving a high signal herself. Earlier players cannot deviate to rejecting for certain in order to ensure that the final mover adopts, because their deviation will be mistaken for a low signal by the later movers, who can rationally mix as prescribed by the strategy profile. So with sufficiently many players, we can find a sequence of adoption probabilities where each player is indifferent given the accumulation of low signals from observing past rejections and the probability that someone will adopt later if they reject themselves.

To summarize, in a small population the dominant strategy of the last mover is to adopt for certain after receiving a high signal. Earlier movers anticipate this, and the only equilibrium is a pure one where they free-ride. A larger population allows for a sufficient accumulation of "bad news" through observing others' actions, which rationalizes rejecting with positive probability for the last mover. If the earlier players are believed by others to adopt with positive probability, they cannot free-ride by deviating to "reject for certain", because their deviation will be interpreted as a low signal. In a sense, they are "trapped" into the strategy profile, because their deviation is not observable. Hence, a large population allows for a mixed "non free-riding" equilibrium where everyone adopts with positive probability.

The necessity of a sufficiently large c follows from the same reasoning: The higher the cost of adopting, the smaller the number of inferred low signals necessary for being indifferent despite a high signal of your own.

3.4 Probability of a Volunteer

This section makes observations regarding the ex-ante probability that one of the players adopts, given that the state is high.[3] The primary focus is on how this probability is affected by population size N in equilibrium. Denote this probability in an N player game as a mapping from the set of (rationalizable) strategy profiles by $\pi^N : [0,1]^N \mapsto [0,1]$. When the state is high, players receive the high signal with

[3] Since the game ends as soon as one player adopts, it is not possible for multiple players to do so. Therefore, "no one adopts" and "one of the players adopts" are complementary events and we are looking at the probability of the latter. In other words, we look at the probability that the public good is provided, given that it is of high value.

probability p. Since each player rejects for certain under the low signal and adopts with probability σ_i under the high signal, we have

$$\pi^N(\sigma) := 1 - \prod_{i=1}^{N}(1 - p\sigma_i)$$

The following observations relate to the behavior of the equilibrium probability of public good provision with respect to N. As before, when mixed equilibria exist, we only consider those where all N players adopt with positive probability. Recall that unless such an equilibrium exists, the unique equilibrium of the game is σ^P.

Remark 1. *For all $c \geq \underline{c}$ and $N \geq \underline{N}$ the following hold.*

1. *Pure versus mixed:* $\pi^N(\sigma^N) > \pi^N(\sigma^P) = p$.
2. *Mixed with respect to population size:* $\pi^N(\sigma^N) > \pi^{N+1}(\sigma^{N+1})$.
3. *Limit Behavior:* $\lim_{N\to\infty} \pi^N(\sigma^N) > p$

Remark 1.1. compares the public good provision probabilities from the pure equilibrium σ^P and the mixed equilibrium σ^N. For any population size N such that σ^N exists, σ^N yields a higher provision probability than σ^P. Recall that in σ^P, the public good is provided if and only if Player N receives the high signal, which happens with probability p under the high state. In equilibrium σ^N, this probability is reached with the possible adoption of Player 1 alone: since Player 1 adopts for certain after a high signal, the public good is produced immediately with probability p. In case Player 1 rejects, all remaining players also adopt with positive probability in equilibrium σ^N. Hence, the difference in provision probability between the two equilibira is the probability that it is one of the players in $\{2, ..., N\}$ that adopts in σ^N.

Recall that for $N < \underline{N}$, profile σ^P is the unique equilibrium of the game and once N crosses $\underline{N}$, an equilibrium of the form σ^N exists. From this, together with Remark 1.1, we can conclude that the maximum provision probability across all equilibria is constant in N and equal to p as long as $N < \underline{N}$. When the population size reaches $\underline{N}$, however, the provision probability makes an upward jump thanks to the ability to coordinate on an equilibrium where all players may adopt instead of simply free riding off the final mover.

Remark 1.2 states that above the threshold $\underline{N}$ of existence, the provision probability from equilibrium σ^N decreases with population size N. The argument is best understood by looking at the incentives of Player 2, who is indifferent between adopting and rejecting in equilibrium σ^N: Player 2 inherits one low signal from Player 1's rejection (as Player 1 adopts for certain if she gets a high signal) which cancels out her own high signal, and she is deciding based on her prior belief. The gain from adopting for Player 2 is then determined by the probability that if she rejects, everyone else will also reject in the continuation game (i.e., no public good). Regardless of the population size, the level of this probability that makes her indifferent is the same. Thus, the probability that no one adopts after Player 2 has to be constant in N. However, the remaining Players $\{3, ..., N\}$ also have to be indifferent. With higher N, future players in the sequence accumulate a larger number of low signals by observing more rejections. To maintain indifference for them, Player 2 must decrease the probability that she adopts after a high signal and make her rejection less informative. The resulting public good provision probability is lower.

Finally, Remark 1.3 states that as the population size goes to infinity, the provision probability in σ^N is bounded away from the provision probability p in σ^P. So while the provision probability in σ^N is decreasing in N, it still yields a strictly positive ”improvement" over provision by a small group ($N < \underline{N}$) as N becomes arbitrarily large.

We can summarize the behavior of provision probability with respect to population size as follows. If N is below a threshold $\underline{N}$, then the unique equilibrium provision probability is constant at p. At $N = \underline{N}$, the highest provision probability across all equilibria makes an upward jump due to the emergence of equilibrium σ^N. After this level (i.e., for $N > \underline{N}$) the provision probability from equilibrium σ^N is decreasing in N, but bounded away from the “small group" provision probability p. This behavior means the population size that yields the highest provision probability is $N = \underline{N}$.

The following observation relates adoption cost c to provision probability.

Remark 2. *Threshold $\underline{N}$ is decreasing with respect to cost c over interval $c \in (\underline{c}, 0.5)$. In particular, $\lim_{c \to \underline{c}^+} \underline{N} = \infty$ and there exists a value $\epsilon > 0$ such that $\underline{N} = 3$ for interval $c \in [0.5 - \epsilon, 0.5)$.*

That is, the higher the cost of adoption, the smaller the minimum population size required for sustaining mixed equilibrium σ^N. Furthermore, for any finite $N \geq 3$ there is a cost level in $(1-p, 0.5)$ above which σ^N is an equilibrium. Hence, we can conclude that for a given population size, increasing the adoption cost to a sufficiently high level will increase the maximum probability of public good provision across equilibria by making it possible to coordinate on a mixed strategy equilibrium where all players adopt with positive probability.

While a higher cost of adoption leading to a higher provision probability may seem counter-intuitive at first, it is explained via the argument discussed in the previous section. To reiterate: To avoid free-riding off the final mover, the own high signal of the final mover must be offset by the observation of sufficiently many past rejections. The higher the cost of adopting, the less past rejections are needed to make her indifferent between adopting and rejecting. For any given population size (of at least 3 players), a high enough cost will allow for a mixed equilibrium where she is not the only individual adopting with positive probability.

3.5 Concluding Remarks

The analysis looks at the problem of coordinating on a volunteer in the presence of social learning. We find conditions for an important result in the study of public goods; a result that is not universally supported by experimental evidence: The ”bystander effect", in the sense of aggravated free-rider incentives, determines the effect of population growth only if the group is already large. For small groups, a population increase can greatly improve the capability of producing a public good by shifting the structure of the volunteering process from one where each individual free-rides as long as there is someone left to decide, to one where anyone can volunteer. Furthermore, making it more costly to volunteer can paradoxically increase the chances of provision by allowing the latter structure in a small group. The best chance of provision is obtained when the population is at the smallest level that allows coordination on this structure.

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A

Appendix to Chapter 1

The following notation will be used in all proofs. Denote by $x_i \in \{0, ..., i-1\}$ the number of participants prior to the turn of player $i \in \{1, .., N\}$. That is, $x_i := \sum_{j=1}^{i-1} a_j$. Using this notation, each subgame where the threshold has not yet been reached can be described by a starting player i and starting history x_i.[1]

Define function $a_i^* : \{i, ..., N\} \times \{0, ..., i-1\} \mapsto \{1, 0\}$ where $a_i^*(j, x_i)$ denotes the equilibrium path action of Player $j \in \{i, ..., N\}$ in the subgame that starts from Player i's move under history x_i, conditional on the threshold not having been reached by the time of the move of Player j. Finally, let $s_i(x_i)$ denote the number of potential participants on the equilibrium path of the subgame that starts with Player i and history x_i. That is,

$$s_i(x_i) := \sum_{j=i}^{N} a_i^*(j, x_i)$$

Note that $a_1^*(i, 0)$ corresponds to the action of Player i (conditional on the threshold not having been reached before her move) on the equilibrium path of the supergame and $s_1(0) = s^*$ corresponds to group size on the equilibrium path of the supergame. That is, Player i is a potential participant in

[1] Note that history enters players' beliefs only through the number of past participants, so the identity of those who joined does not matter for optimal action. Thus, all subgames starting from player i under x_i past participants are payoff equivalent and have the same unique subgame perfect equilibrium.

equilibrium if and only if $a_1^*(i,0) = 1$ and the number of equilibrium potential participants is given by $s_1(0)$.

Now we can state the condition such that it is optimal for Player i to play $a_i = 1$ under history x_i, given that $x_i < t$ and any possible continuation game will follow its equilibrium path. If Player i plays $a_i = 1$, then we have $x_{i+1} = x_i + 1$ for the history of the following player, and since Player i plays $a_i = 1$ herself, there will be $s_{i+1}(x_{i+1}) + 1 = s_{i+1}(x_i + 1) + 1$ potential participants in the subgame. This yields success probability $\frac{F(x_i+s_{i+1}(x_i+1)+1)}{1-F(x_i)}$. If Player i plays $a_i = 0$, we have $x_{i+1} = x_i$ and thus, the total number of potential participants in the subgame will be $s_{i+1}(x_{i+1}) = s_{i+1}(x_i)$. This yields success probability $\frac{F(x_i+s_{i+1}(x_i))}{1-F(x_i)}$. Since the gain from playing $a_i = 1$ is the increase in success probability and the loss is the participation cost c, we can write the optimality condition for Player i to partcipate under history $x_i < t$ (given equilibrium continuation) as

$$a_i^*(i,x_i) = 1 \iff \frac{F(x_i + s_{i+1}(x_i+1)+1) - F(x_i + s_{i+1}(x_i))}{1 - F(x_i)} \geq c \qquad (IC_{i,x_i})$$

Where (IC_{i,x_i}) stands for the incentive constraint of Player i under history x_i. The following Lemma will be used in all proofs.

Lemma 1. *For any $i \in \{1,...,N-1\}$ and $x_i \in \{0,...,i-1\}$, $\frac{p(x_i+N-i+1)}{1-F(x_i)} \geq c$ implies $s_i(x_i) = N - i + 1$.*

Proof. Consider Player N under history $x_N = x_i + N - i$. Then IC_{N,x_N} yields

$$\frac{p(x_i+N-i+1)}{1-F(x_i+N-i)} \geq \frac{p(x_i+N-i+1)}{1-F(x_i)}) \geq c$$

where the first inequality holds by $N \geq i$ and the second inequality is the condition of the Lemma. This means $a_N^*(N, x_i+N-i) = 1$ and $s_N(x_i+N-i) = 1$.

Now suppose by induction for some $j \in \{i,...,N-1\}$ that $s_{j+1}(x_i+j-i+1) = N-j$. Then under history $x_j = x_i + j - i$, IC_{j,x_j} yields

$$\begin{aligned}
&\frac{F(x_i+j-i+s_{j+1}(x_i+j-i+1)+1) - F(x_i+j-i+s_{j+1}(x_i+j-i))}{1-F(x_i+j-i)} \\
&= \frac{F(x_i+N-i+1) - F(x_i+j-i+s_{j+1}(x_i+j-i))}{1-F(x_i+j-i)} \\
&\geq \frac{F(x_i+N-i+1) - F(x_i+N-i)}{1-F(x_i+j-i)}
\end{aligned}$$

$$= \frac{p(x_i+N-i+1)}{1-F(x_i+j-i)}$$
$$\geq \frac{p(x_i+N-i+1)}{1-F(x_i)} \geq c$$

where the first equality is by the induction hypothesis, the first inequality holds since $s_{j+1}(x_i+j-i) \leq N-j$ by definition of s (i.e. there are only $N-j$ remaining movers after Player j), the second inequality holds from $j \geq i$ and the third inequality is the condition of the Lemma. This means $a_j^*(j, x_i+j-i) = 1$ which together with the induction hypothesis yields $s_j(x_i+j-i) = N-j+1$. The induction hypothesis is shown above to hold for $j = N-1$. Plugging in i for j in $s_j(x_i+j-i) = N-j+1$ yields $s_i(x_i) = N-i+1$, completing the statement of the Lemma. □

General Case

Proposition 1. *The following hold on the unique equilibrium path.*

(a) *If there are any potential participants, then there exists an initial participant $n^* \in \{1,...,N\}$ such that Player i is a potential participant if and only if $i \in \{n^*,...,N\}$.*

(b) *If $c \leq p(N-i+1)$, then Player i is a potential participant.*

Proof. **Observation (a):** I prove this by showing that for all $i \in \{1,...,N-1\}$ and $j \in \{i,..,N\}$, $a_i^*(i,x_i) = 1$ implies $a_i^*(j,x_i) = 1$. That is, if the optimal action of Player i is to participate after history x_i, then in the continuation game the optimal action of all later movers must be to participate as well.

If Player i participates under history x_i, then we have $x_{i+1} = x_i+1$. Thus the observation is equivalent to $a_i^*(i,x_i) = 1 \Rightarrow a_{i+1}^*(i+1,x_i+1) = 1$. I show the counterpositive of this. First note that $a_{i+1}^*(i+1,x_i+1) = 0$ implies $s_{i+1}(x_i+1) = s_{i+2}(x_i+1)$. Then we have two cases:

Case (i) $a_{i+1}^*(i+1,x_i) = 1$: This implies $s_{i+1}(x_i) = s_{i+2}(x_i+1)+1$. Then we have

$$\frac{F(x_i+s_{i+1}(x_i+1)+1)-F(x_i+s_{i+1}(x_i))}{1-F(x_i)}$$
$$= \frac{F(x_i+s_{i+2}(x_i+1)+1)-F(x_i+s_{i+2}(x_i+1)+1)}{1-F(x_i)}$$
$$= 0 < c$$

which by (IC_{i,x_i}) means $a_i^*(i,x_i) = 0$.

Case (ii) $a_{i+1}^*(i+1,x_i) = 0$: this implies $s_{i+1}(x_i) = x_{i+2}(x_i)$. Then we have

$$\frac{F(x_i+s_{i+1}(x_i+1)+1)-F(x_i+s_{i+1}(x_i))}{1-F(x_i)}$$

$$= \frac{F(x_i + s_{i+2}(x_i+1)+1) - F(x_i + s_{i+2}(x_i))}{1-F(x_i)} < c$$

which once again by (IC_{i,x_i}) means $a_i^*(i,x_i) = 0$. The inequality holds because if it is violated, by (IC_{i+1,x_i}) we must have $a_{i+1}^*(i+1,x_i) = 1$, which contradicts Case (ii).

From the two cases we conclude that $a_{i+1}^*(i+1,x_i+1) = 0 \Rightarrow a_i^*(i,x_i) = 0$, and thus $a_i^*(i,x_i) = 1 \Rightarrow a_i^*(i+1,x_i) = a_{i+1}^*(i+1,x_i+1) = 1$. Iterating this statement forward yields $a_i^*(i,x_i) = 1 \Rightarrow a_i^*(j,x_i) = 1$ for all (i,j) such that $i \in \{1,..,N-1\}$ and $j \in \{i,...,N\}$.

Observation (b): Setting $x_i = 0$ in the statement of Lemma 1, we obtain that $p(N-i+1) \geq c$ implies $s_i(0) = N-i+1$. Since there are only $N-i$ movers after Player i, this means $a_i^*(i,0) = 1$. If in equilibrium we have $\sum_{j=1}^{i-1} a_1^*(j,0) = 0$ then Player i is indeed reached with history $x_i = 0$ and $a_i^*(i,0) = a_1^*(i,0) = 1$. If $\sum_{j=1}^{i-1} a_1^*(j,0) > 0$, then it must be the case that $a_1^*(j,0) = 1$ for some $j \in \{1,...,i-1\}$ which by Observation (a) means $a_1^*(i,0) = 1$ □

Theorem 1. *If $p(N-2) > p(N-1) > p(N)$ and $\frac{p(N)}{p(N-1)} > 1 - p(1)$, then s^* is non-monotonic with respect to cost c.*

Proof. First suppose $c \leq p(N)$. By Lemma 1, this implies $s^* = s_1(0) = N$.

Second, let $c = p(N) + \epsilon$ for ϵ small. Since Theorem 1 requires $p(N) < p(N-1)$, this implies $p(N) < c < p(N-1)$. By Lemma 1, we then have $s_2(0) = N-1$ and $(IC_{1,0})$ reads:

$$\begin{aligned}
&F(s_2(1)+1) - F(s_2(0)) = F(s_2(1)+1) - F(N-1) \\
&\leq p(N) < p(N) + \epsilon = c
\end{aligned}$$

where the first inequality holds because $s_2(x_2) \leq N-1$ for all x_2 by definition. Thus, the optimal action of Player 1 is $a_1^*(1,0) = 0$ and we have $s^* = s_1(0) = s_2(0) = N-1$ when $c = p(N) + \epsilon$ with ϵ small.

Finally, let $c = p(N-1) + \epsilon$. From the condition $p(N)/p(N-1) > 1 - p(1)$ of Theorem 1, we have

$$\frac{p(N)}{1-p(1)} > p(N-1) + \epsilon = c$$

for ϵ sufficiently small. This means by Lemma 1 that $s_2(1) = N-1$.

Furthermore, since Theorem 1 requires $p(N-2) > p(N-1)$, we have $c = p(N-1)+\epsilon < p(N-2)$. Then, again by Lemma 1, we have $s_3(0) = N-2$. This yields the following for $(IC_{2,0})$.

$$F(s_3(1)+1) - F(s_3(0)) = F(s_3(1)+1) - F(N-2) \leq p(N-1) < c = p(N-1)+\epsilon$$

The first inequality holds because $s_3(1) \leq N-2$ by definition of s. Hence, $a_2^*(2,0) = 0$ and $s_2(0) = s_3(0) = N-2$.

Given the above continuation game, Player 1 faces the following $(IC_{1,0})$.

$$\begin{aligned} &F(s_2(1)+1) - F(s_2(0)) = F(N) - F(N-2) = p(N) + p(N-1) \\ &> p(N-1)+\epsilon = c \end{aligned}$$

This means $a_1^*(1,0) = 1$ and thus $s^* = s_1^*(0) = N$.

To summarize, we have shown that for sufficiently small ϵ, the equilibrium group size is given by

$$s^* = \begin{cases} N; & c \leq p(N) \\ N-1; & c = p(N)+\epsilon \\ N; & c = p(N-1)+\epsilon \end{cases}$$

Hence, s^* is non-monotonic with respect to c.

$\square$

Example 1

Theorem 2. *The equilibrium group size in the model with two possible thresholds is as follows.*

$$s^* = \begin{cases} \bar{t}; & c \in (0, p(\bar{t})] \\ \underline{t}; & c \in (p(\bar{t}), p(\underline{t})] \\ \bar{t}; & c \in (p(\underline{t}), 1) \end{cases}$$

Proof. First note that for all $n \in \{1,...,N\}$ and $k \in \{1,...,N-n\}$ we have

$$\frac{p(k+n)}{1-F(k)} = \begin{cases} q; & n+k = \underline{t} \\ 1-q; & n+k = \bar{t} \ \wedge \ k < \underline{t} \\ 1; & n+k = \bar{t} \ \wedge \ k \geq \underline{t} \\ 0; & otherwise \end{cases} \tag{A.1}$$

These probabilities will be used throughout the proof.

Observation (i): Equilibrium action $a_1^*(i,0) = 0$ for all $\{1,...,N-\bar{t}\}$. That is, all players $\{1,...,N-\bar{t}\}$ pass in equilibrium. Thus, $s^* = s_{N-\bar{t}+1}(0)$.

To see this, suppose $a_1^*(i,0) = 1$ for some $i \leq N-\bar{t}$. By Proposition 1(a), this implies $a_1^*(j,0) = 1$ for all $j > i$ and thus, on the equilibrium path we have $x_N > \bar{t}$ (i.e. the move of Player N is reached with more than $\bar{t}$ past participants). This yields the following decision (IC_{N,x_N}) for Player N.

$$\frac{p(x_N+1)}{1-F(x_N)} = 0 < c$$

where the equality holds by Equation (1). So Player N has incentive to deviate to $a_N = 0$.

Regarding the equilibrium action of the remaining players $\{N-\bar{t}+1,...,N\}$, there are 3 possible cases.

Case 1 ($c \leq 1-q$):. Since $p(\bar{t}) = 1-q \geq c$, Lemma 1 directly implies $s^* = s_{N-\bar{t}+1}(0) = \bar{t}$.

Case 2 ($c \in (1-q, q]$): First note that since $p(\underline{t}) = q \geq c$, Lemma 1 implies $s_{N-\underline{t}+1}(0) = \underline{t}$. Consider the decision of Player $N-\underline{t}$ under history $x_{N-\underline{t}} = 0$. Her decision $(IC_{N-\underline{t},0})$ is given by

$$\begin{aligned}&F(s_{N-\underline{t}+1}(1)+1) - F(s_{N-\underline{t}+1}(0)) = F(s_{N-\underline{t}+1}(1)+1) - F(\underline{t})\\&\leq p(\underline{t}+1) \leq 1-q < c\end{aligned}$$

where the first inequality holds since $s_{N-\underline{t}+1}(1) \leq \underline{t}$ by definition of s and the second inequality holds since $p(\underline{t}+1) = 1-q$ if $\bar{t} = \underline{t}+1$ and $p(\underline{t}+1) = 0$ otherwise. Then $(IC_{N-\underline{t},0})$ yields $a_{N-\underline{t}}^*(N-\underline{t},0) = 0$ and $s_{N-\underline{t}}(0) = s_{N-\underline{t}+1}(0) = \underline{t}$.

By induction, assume for some $i \in \{N-\bar{t}+1,...,N-\underline{t}\}$ that $s_{i+1}(0) = \underline{t}$. Under history $x_i = 0$, she faces the following $(IC_{i,0})$.

$$F(s_{i+1}(1)+1) - F(s_{i+1}(0)) \leq F(N-i+1) - F(\underline{t}) \leq 1-q < c$$

where the first inequality holds by the induction hypothesis and the definition of s (i.e. there are only $N-i$ movers after Player i). Thus we have $a_i^*(i,0) = 0$ and $s_i(0) = s_{i+1}(0) = \underline{t}$. The induction hypothesis is shown to hold for $i = N-\underline{t}-1$ above. Replacing i by $N-\bar{t}+1$ yields $s^* = s_{N-\bar{t}+1}(0) = \underline{t}$

Case 3 ($c > q$): Observation (i) implies $x_i \in \{0,...,i-(N-\bar{t}+1)\}$ for all $i \in \{N-\bar{t}+1,...,N\}$ on the equilibrium path: That is, since all players$\{1,...,N-\bar{t}\}$ pass, there can be at most $\{0,...,i-$

$(N-\bar{t}+1)\}$ participants before Player i. First consider some player $i \in \{N-\bar{t}+1,...,N\}$ under some history $x_i < i-(N-\bar{t}+1)$. Her decision is given by (IC_{i,x_i}), which is as follows.

$$\begin{aligned}
&\frac{F(x_i+s_{i+1}(x_i+1)+1)-F(x_i+s_{i+1}(x_i))}{1-F(x_i)} \\
&\leq \frac{F(x_i+N-i+1)-F(x_i)}{1-F(x_i)} \\
&= \frac{\sum_{k=1}^{N-i+1} p(x_i+k)}{1-F(x_i)} \\
&= \begin{cases} q; & \underline{t}-(N-i+1) < x_i < \underline{t} \\ 0; & otherwise \end{cases}
\end{aligned}$$

where the inequality holds since $s_{i+1}(x_i+1) \leq N-i$ by definition of s and the second equality holds by the $x_i < i-(N-\bar{t}+1)$ assumption (i.e. there are not enough players remaining to reach $\bar{t}$ after such a history x_i so the added success probability by the participation of Player i is no higher than q). Since $c > q$, this inequality tells us that $a_i^*(i,x_i) = 0$ for all $x_i < i-(N-\bar{t}+1)$.

As $a_i = 0$ means $x_{i+1} = x_i < i-(N-\bar{t}+1) < i+1-(N-\bar{t}+1)$, the same holds for the decision of Player $i+1$ in the continuation game, so she passes as well. Thus, $a_i^*(i,x_i) = 0$ implies $a_i^*(i+1,x_i) = 0$. Iterating this argument forward, we obtain $a_i^*(j,x_i) = 0$ for all $j \in \{i,...,N\}$. Hence, for any $x_i < i-(N-\bar{t}+1)$ we have $s_i(x_i) = 0$. That is, all remaining players pass if the move of Player i is reached with a history of less than $i-(N-\bar{t}+1)$ participants.

Next, note that since $\frac{p(\bar{t}-t)}{1-F(\underline{t})} = 1 > c$, Lemma 1 implies $s_{N-(\bar{t}-\underline{t})+1}(\underline{t}) = \bar{t}-\underline{t}$. That is, if the move of Player $N-(\bar{t}-\underline{t})+1$ is reached with history $\underline{t}$, all of the remaining $\bar{t}-\underline{t}$ players are potential participants. Summarizing these two observations, we have the following history dependent continuation after the move of Player $N-(\bar{t}-\underline{t})$.

$$s_{N-(\bar{t}-\underline{t})+1}\left(x_{N-(\bar{t}-\underline{t})+1}\right) = \begin{cases} \bar{t}-\underline{t}; & x_{N-(\bar{t}-\underline{t})+1} = \underline{t} \\ 0; & otherwise \end{cases} \tag{A.2}$$

Now assume by induction for some $i \in \{N-\bar{t}+1,...,N-(\bar{t}-\underline{t})\}$ that we have $s_{i+1}(i-N+\bar{t}) = N-i$ (this is shown to hold for $i = N-(\bar{t}-\underline{t})$ in Equation (2)). Then under history $x_i = i-N+\bar{t}-1$, decision of Player i is given by (IC_{i,x_i}), which is as follows.

$$\begin{aligned}
&\frac{F(x_i+s_{i+1}(x_i+1)+1)-F(x_i+s_{i+1}(x_i))}{1-F(x_i)}\\
&=\frac{F(\bar{t})-F\left(i-N+\bar{t}-1+s_{i+1}(i-N+\bar{t}-1)\right)}{1-F(i-N+\bar{t}-1)}\\
&=\frac{F(\bar{t})-F(i-N+\bar{t}-1)}{1-F(i-N+\bar{t}-1)}\\
&=F(\bar{t})=1>c
\end{aligned}$$

where the first equality is the induction hypothesis and the second equality is the application of the above observation that $s_i(x_i)=0$ for all $x_i<i-(N-\bar{t}+1)$. The third equality is because our restriction on i implies $i-N+\bar{t}-1<\underline{t}-1$, which means $F(i-N+\bar{t}-1)=0$.

Hence, it is optimal for Player i to participate and we have $a_i^*(i,i-N+\bar{t}-1)=1$ and thus $s_i(i-N+\bar{t}-1)=N-i+1$. This verifies that the induction hypotheses also holds for $i-1$. Setting $i=N-\bar{t}+1$ yields $s_{N-\bar{t}+1}(0)=\bar{t}$. By Observation (i), this means $s^*=\bar{t}$, the equilibrium group size. $\square$

Example 2

For the results of Example 2 (i.e. Proposition 2 and Theorem 3), the following notation will be used. Define function $g_k:\mathbb{N}^2\mapsto[-1,1]$ as the difference in public good provision probability between having n and m potential participants under a history of k (past) participants where the threshold has not been reached. That is,

$$g_k(n,m):=\frac{F(k+n)-F(k+m)}{1-F(k)}$$

With this notation, the optimality condition of Player i to participate under history x_i is given by

$$a_i^*(i,x_i)=T \iff g_{x_i}(s_{i+1}(x_i+1)+1,s_{i+1}(x_i))\geq c \qquad (IC'_{i,x_i})$$

Under the compound geometric distribution $p(n)=q((1-p_1)^{n-1}p_1)+(1-q)((1-p_2)^{n-1}p_2)$, it is useful to underline three properties of function g.

Property A1:

$$g_k(n+m,n)>g_k(n+1+m,n+1),\ \ \forall n,m,k\in\mathbb{N}$$

That is, the the probability gain from having m additional participants is decreasing in the number of potential participants they are added to.

Property A2:

$$g_{k+1}(n,m) > g_k(n+1,m+1), \;\; \forall n,m,k \in \mathbb{N} \;\text{ s.t. }\; n > m$$

That is, the probability gain from a given number of additional participants increases if we increase the number of past participants and decrease the number of potential participants by one.

Property A3: For any $k,n \in \mathbb{N}$, we have

$$g_k(n+1,n) \geq g_{k+1}(n+1,n) \iff (1-p_1)^n p_1 \geq (1-p_2)^n p_2$$

That is, the probability gain from an $n+1^{st}$ participant is decreasing with the number of past participants if and only if $(1-p_1)^n p_1 \geq (1-p_2)^n p_2$.

Proposition 2. *Equilibrium group size s^* is monotonic with respect to cost c if and only if $p_1(1-p_1)^{N-2} \geq p_2(1-p_2)^{N-2}$. If s^* is monotonic, then $s^* = max\{n \in \{1,...,N\} : p(n) \geq c\}$*

Proof. **First Direction of Proposition 2:** If $(1-p_2)^{N-2}p_1 \geq (1-p_2)^{N-2}p_2$, then the equilibrium group size is $s^* = max\{n \in \{1,...,N\} : p(n) \geq c\}$, which is monotonically decreasing w.r.t. c.

It will be shown by induction that under this condition, the equilibirium action of Player i is $a_1^*(i,0) = 1$ if and only if $p(N-i+1) = g_0(N-i+1,N-i) \geq c$. Since $p(n)$ is strictly decreasing over all $n \in \mathbb{N}_+$, this implies that $s^* = \sum_{i=1}^{N} a_1^*(i,0) = max\{n \in \{1,...,N\} : p(n) \geq c\}$.

Base Case: Consider Player N. Since she is the final mover, (IC'_{N,x_N}) yields

$$a_N^*(N,x_N) = 1 \iff g_{x_N}(1,0) = g_{x_N}(N-i+1,N-i) \geq c$$

for all $x_N \in \{0,...,N-1\}$.

Inductive Step: Consider some player $i \in \{1,...,N-1\}$. Suppose for any player $j \in \{i+1,...,N\}$ and under any history $x_{i+1} \in \{0,...,i\}$, we have $a_{i+1}^*(j,x_{i+1}) = 1$ if and only if $g_{x_{i+1}}(N-j+1,N-j) \geq c$. We want to show that this hypothesis implies for any $x_i \in \{0,...,i-1\}$ that we have $a_i^*(i,x_i) = 1$ if and only if $g_{x_i}(N-i+1,N-i) \geq c$. That is, we need to show the induction hypothesis implies

$$g_{x_i}(s_{i+1}(x_i+1)+1, s_{i+1}(x_i)) \geq c \iff g_{x_i}(N-i+1,N-i) \geq c \tag{A.3}$$

for all $x_i \in \{0, ..., i-1\}$.

Direction $\Rightarrow$ of (3): By the induction hypothesis, we have:

$$s_{i+1}(x_{i+1}) = \left|\{j \in \{i+1, ..., N\} : g_{x_{i+1}}(N-j+1, N-j) \geq c\}\right|$$

Note that $(1-p_1)^{N-2}p_1 \geq (1-p_2)^{N-2}p_2$ implies $(1-p_1)^n p_1 \geq (1-p_2)^n p_2$ for all $n \leq N-2$. By property A3, this means that $g_{x_i+1}(N-j+1, N-j) \leq g_{x_i}(N-j+1, N-j)$ for all $j \in \{2, ..., N\}$. Then under the induction hypothesis we have $s_{i+1}(x_i+1) \leq s_{i+1}(x_i)$. In particular, we have $s_{i+1}(x_i+1) < s_{i+1}(x_i)$ if $c \in (g_{x_i+1}(N-j+1, N-j), g_{x_i}(N-j+1, N-j)]$ for some $j \in \{i+1, ..., N\}$ and we have $s_{i+1}(x_i+1) = s_{i+1}(x_i)$ otherwise.

Now suppose c is such that $s_{i+1}(x_i+1) < s_{i+1}(x_i)$. Since the number of participants takes values in natural numbers this means $s_{i+1}(x_i+1)+1 \leq s_{i+1}(x_i)$ and we have

$$g_{x_i}(s_{i+1}(x_i+1)+1, s_{i+1}(x_i)) \leq g_{x_i}(s_{i+1}(x_i), s_{i+1}(x_i)) = 0 < c$$

which contradicts the left-hand side of (3). Thus, if $s_{i+1}(x_i+1) < s_{i+1}(x_i)$, participating cannot be optimal for Player i. If participating is optimal for Player i, we must have $s_{i+1}(x_i+1) = s_{i+1}(x_i)$. In that case, we can write the optimality condition as

$$g_{x_i}(s_{i+1}(x_i)+1, s_{i+1}(x_i)) \geq c \tag{A.4}$$

By definition, we know that $s_{i+1}(x_i) \in \{0, ..., N-i\}$. Now suppose $s_{i+1}(x_i) < N-i$. Then the induction hypothesis and property A1 together imply that $a^*_{i+1}(j, x_i) = 1$ for any $j \in \{N - s_{i+1}(x_i) + 1, ..., N\}$ and $a^*_{i+1}(j', x_i) = 0$ for any $j' \in \{i+1, ..., N - s_{i+1}(x_i)\}$. Under the induction hypothesis, the latter observation can be written as

$$\begin{aligned}
&c > g_{x_i}(N-(N-s_{i+1}(x_i))+1, N-(N-s_{i+1}(x_i))) \\
&= g_{x_i}(s_{i+1}(x_i)+1, s_{i+1}(x_i))
\end{aligned}$$

which contradicts (4). Thus, if participating is optimal for Player i, we cannot have $s_i(x_i+1) < N-i$. The only remaining case for $a^*_i(i, x_i) = 1$ is $s_i(x_i) = s_i(x_i+1) = N-i$. Thus, we can write

$$g_{x_i}(s_{i+1}(x_i+1)+1, s_{i+1}(x_i)) \geq c \Rightarrow g_{x_i}(N-i+1, N-i) \geq c$$

which is the first direction of (3)

Direction $\Leftarrow$ of (3): If $g_{x_i}(N-i+1,N-i) = \frac{p(N-i+1)}{1-F(x_i)} \geq c$, then Lemma 1 implies $s_i(x_i) = N-i+1$. Since there are only $N-i$ movers after Player i, it must be the case that $a_i^*(i,x_i) = 1$, which is equivalent by (IC'_{i,x_i}) to $g_{x_i}(s_{i+1}(x_i+1)+1, s_{i+1}(x_i)) \geq c$. Thus, we shown that (3) holds for Player i under the induction hypothesis.

Note that setting $i = N-1$ in the induction hypothesis yields the base case. Setting $i = 1$ (for whom the only possible history is $x_1 = 0$), shows that for all $i \in \{1,...,N\}$, we have $a_1^*(i,0) = 1$ if and only if $g_0(N-i+1,N-i) \geq c$. Since $g_0(n,n-1)$ is strictly decreasing in n by property A1, only the final $\max\{n \in \{1,...,N\} : g_0(n,n-1) \geq c\}$ movers are potential participants in equilibrium.

Second Direction of Proposition 2: If $(1-p_2)^{N-2}p_2 < (1-p_1)^{N-2}p_1$, then s^* is non-monotonic w.r.t. c.

This statement is obtained immediately by plugging in $p(n) = q((1-p_1)^{n-1}p_1) + (1-q)((1-p_2)^{n-1}p_2)$ into the non-monotonicity condition of Theorem 1. $\square$

For the proof, Theorem 3 will be stated equivalently to the text using function g.

Theorem 3. *Assume $p_1 + p_2 > 1$ and $p_2 \in (q, 1-q)$. For all $n \in \{2,...,N-1\}$ there is a $k_n \in \mathbb{N}$ such that an interval*

$$C_n^k := (g_{k-1}(n,n-1), \min\{g_k(n,n-1), g_0(n+k,n-1)\}]$$

exists for all $k \in \{1\} \cup \{k_n, k_n+1, ...\}$. If $c \in C_n^k$ for some pair (n,k) with $k+n \leq N$, then $s^* = n+k$. Otherwise $s^* = max\{n \in \{1,...,N\} : g_0(n,n-1) \geq c\}$.

Proof. First, I partition the possible values of c into two cases under the parameter restrictions of Theorem 3, using Lemma 2 below. Second, I separately derive the equilibrium group size s^* for these two cases. Finally, I show that interval C_n^k exists for given n when $k = 1$ or k sufficiently large.

Lemma 2. *If $p_2 > \max\{1-p_1, q\}$, then the values of $g_k(n,n-1)$ at $n \geq 2$ are lexigoraphically ordered; first strictly decreasing with n, then strictly increasing with k. That is for all $n \geq 2$ and $k, k' \in \mathbb{N}$, we have $g_k(n,n-1) < g_{k+1}(n,n-1)$ and $g_{k'}(n+1,n) < g_k(n,n-1)$.*

Proof. First note that from $p_1 > p_2$, we have that $p_2 > 1-p_1$ if and only if $p_1(1-p_1) < p_2(1-p_2)$. This implies $p_1(1-p_1)^n < p_2(1-p_2)^n$ for all $n \geq 2$. Therefore, by property A3, we know that $g_k(n,n-1)$ is strictly increasing in k for any $n \geq 2$.

Second, additionally assuming $p_2 > q$ implies $g_k(n,n-1) > g_{k'}(n+1,n)$ for all $n \geq 2$ and $k, k' \in \mathbb{N}$. The reason is as follows. Since $g_k(n,n-1)$ is strictly increasing in k as shown above, we have $g_{k'}(n+1,n) < \lim_{k\to\infty} g_k(n+1,n)$ and $g_k(n,n-1) \geq g_0(n,n-1)$ for all $k,k' \in \mathbb{N}$. Furthermore

$$\lim_{k\to\infty} g_k(n+1,n) < g_0(n,n-1)$$
$$\Leftrightarrow p_2(1-p_2)^n < qp_1(1-p_1)^{n-1} + (1-q)p_2(1-p_2)^{n-1}$$
$$\Leftrightarrow (q-p_2)p_2(1-p_2)^{n-1} < qp_1(1-p_1)^{n-1}$$

A sufficient condition for this inequality to hold is $p_2 > q$. So for all $n \geq 2$ and $k,k' \in \mathbb{N}$, we have $g_k(n,n-1) \geq g_0(n,n-1) > \lim_{k\to\infty} g_k(n+1,n) > g_{k'}(n+1,n)$. We thus obtain the second statement of the lemma. □

Lemma 2 leaves us with two possible cases regarding cost c.[2]

Case 1: There is a unique $\tilde{n} \in \{1,...N-1\}$ such that $c \in \Big(g_{N-(\tilde{n}+1)}(\tilde{n}+1,\tilde{n}), g_0(\tilde{n},\tilde{n}-1)\Big]$.

Case 2: There is a unique pair $(\tilde{n},k)$ with $\tilde{n} \in \{2,...N-1\}$ and $k \in \{1,...,N-\tilde{n}\}$ such that $c \in (g_{k-1}(\tilde{n},\tilde{n}-1), g_k(\tilde{n},\tilde{n}-1)]$

Next, I separately derive s^* under these two cases.

Equilibrium in Case 1: First note that $c \leq g_0(\tilde{n},\tilde{n}-1)$ implies $c \leq g_k(n,n-1) = \frac{p(k+n)}{1-F(k)}$ for all $n \leq \tilde{n}$ and all k by Lemma 2. By Lemma 1, this means $s_{N-\tilde{n}+1}(x_{N-\tilde{n}+1}) = \tilde{n}$ for all $x_{N-\tilde{n}+1} \in \{0,...,N-\tilde{n}\}$. That is, the final $\tilde{n}$ movers participate in equilibirium regardless of the history.

Next, I show that all the earlier movers $\{1,...,N-\tilde{n}\}$ pass regardless of history. That is, $s_i(x_i) = \tilde{n}$ for all $i \in \{1,...,N-\tilde{n}\}$ and all $x_i \in \{0,...,i-1\}$. As shown above, $s_{N-\tilde{n}+1}(x_{N-\tilde{n}}) = s_{N-\tilde{n}+1}(x_{N-\tilde{n}}+1) = \tilde{n}$. Since Case 1 dictates that $c > g_{x_{N-\tilde{n}}}(\tilde{n}+1,\tilde{n})$ for all $x_{N-\tilde{n}}$, we have by $(IC'_{N-\tilde{n},x_{N-\tilde{n}}})$ that Player $N-\tilde{n}$ passes under any history. That is, $a^*_{N-\tilde{n}}(N-\tilde{n},x_{N-\tilde{n}}) = 0$ for all $x_{N-\tilde{n}}$. Since this is true for any history, $a^*_1(N-\tilde{n},0) = 0$ must hold on the equilibrium path. By Proposition 1(a), this means all previous players $i \in \{1,...,N-\tilde{n}-1\}$ pass as well, which yields

$$s^* = s_1(0) = s_{N-\tilde{n}+1}(0) = \tilde{n} = max\{n \in \{1,...,N\} : g_0(n,n-1) \geq c\}$$

where the last equality holds by Case 1.

Equilibrium in Case 2: For players $i \in \{N-\tilde{n}+2,...,N\}$ we have $c \leq g_{x_i}(i,i-1) = \frac{p(x_i+i)}{1-F(x_i)}$ for all x_i by Lemma 2. Then once again by Lemma 1, we have $s_{N-\tilde{n}+2}(x_{N-\tilde{n}+2}) = \tilde{n}-1$ for all $x_{N-\tilde{n}+2}$.

For Player $N-\tilde{n}+1$, the case $c \in (g_{k-1}(\tilde{n},\tilde{n}-1), g_k(\tilde{n},\tilde{n}-1)]$ implies the following: In the continuation game, she knows $s_{N-\tilde{n}+2}(x_{N-\tilde{n}+1}) = s_{N-\tilde{n}+2}(x_{N-\tilde{n}+1}+1) = \tilde{n}-1$. By Lemma 2 and

[2] There are two additional cases that are trivial and not discussed here: It is straightforward to see that $c > g_0(1,0)$ and $c \leq g_0(N,N-1)$ yield equilibrium group size $s^* = 0$ and $s^* = N$ respectively.

Case 2, we have $g_{x_{N-\tilde{n}+1}}(\tilde{n},\tilde{n}-1)$ is greater than c if $x_{N-\tilde{n}+1} \geq k$ and smaller than c otherwise. Then by $(IC'_{N-\tilde{n}+1,x_{N-\tilde{n}+1}})$ we have $a^*_{N-\tilde{n}+1}(N-\tilde{n}+1, x_{N-\tilde{n}+1}) = 1$ if and only if $x_{N-\tilde{n}+1} \geq k$. That is, participating is optimal of Player $N-\tilde{n}+1$ if and only if she follows a history of at least k past participants. This means

$$s_{N-\tilde{n}+1}(x_{N-\tilde{n}+1}) = \begin{cases} \tilde{n}; & x_{N-\tilde{n}+1} \geq k \\ \tilde{n}-1; & otherwise \end{cases}$$

I derive the equilibrium actions of earlier players $\{1,...,N-\tilde{n}\}$ by induction.

Base Case: Consider player $N-\tilde{n}$ under different histories:

If $x_{N-\tilde{n}} < k-1$, then $s_{N-\tilde{n}+1}(x_{N-\tilde{n}}+1) = s_{N-\tilde{n}+1}(x_{N-\tilde{n}}) = \tilde{n}-1$. Case 2 yields $c > g_{x_{N-\tilde{n}}}(\tilde{n},\tilde{n}-1)$ for any $x_{\tilde{n}+1} < k$. Therefore by $(IC'_{N-\tilde{n},x_{N-\tilde{n}}})$, Player $N-\tilde{n}$ passes. That is, $a^*_{N-\tilde{n}}(N-\tilde{n}, x_{N-\tilde{n}}) = 0$ for all $x_{N-\tilde{n}} < k-1$.

If $x_{N-\tilde{n}} > k-1$, then $s_{N-\tilde{n}+1}(x_{N-\tilde{n}}+1) = s_{N-\tilde{n}+1}(x_{N-\tilde{n}}) = \tilde{n}$. Case 2 yields $c > g_{x_{N-\tilde{n}}}(\tilde{n}+1,\tilde{n})$ for all $x_{N-\tilde{n}}$. Therefore by $(IC'_{N-\tilde{n},x_{N-\tilde{n}}})$, Player $N-\tilde{n}$ passes. That is, $a^*_{N-\tilde{n}}(N-\tilde{n}, x_{N-\tilde{n}}) = 0$ for all $x_{N-\tilde{n}} > k-1$.

If $x_{N-\tilde{n}} = k-1$, then $s_{N-\tilde{n}+1}(x_{N-\tilde{n}}) = \tilde{n}-1$ and $s_{N-\tilde{n}+1}(x_{N-\tilde{n}}+1) = \tilde{n}$. Therefore, Player $N-\tilde{n}$ participates if and only if $c \leq g_{k-1}(\tilde{n}+1,\tilde{n}-1)$. Since $g_{k-1}(\tilde{n}+1,\tilde{n}-1) > g_{k-1}(\tilde{n},\tilde{n}-1)$, there is always a range of c within Case 2 where this holds.

We can summarize the optimal action of Player $N-\tilde{n}$ as follows.

$$a^*_{N-\tilde{n}}(N-\tilde{n}, x_{N-\tilde{n}}) = 1 \Leftrightarrow x_{N-\tilde{n}} = k-1 \;\; \wedge \;\; c \leq g_{k-1}(\tilde{n}+1,\tilde{n}-1) \tag{A.5}$$

Inductive Step: Fix a player $i \in \{N-(\tilde{n}+k)+1,...,N-\tilde{n}-1\}$. Suppose for all $j \in \{i+1,...,N-\tilde{n}\}$ we have

$$a^*_j(j,x_j) = 1 \Leftrightarrow x_j = k+\tilde{n}-(N-j+1) \;\; \wedge \;\; c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-j+1-\tilde{n}}$$

which is shown to hold for $i = N-\tilde{n}-1$ in Equation (5). Our goal is to show that this induction hypothesis implies

$$a^*_i(i,x_i) = 1 \Leftrightarrow x_i = k+\tilde{n}-(N-i+1) \;\; \wedge \;\; c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-i+1-\tilde{n}} \tag{A.6}$$

The hypothesis yields that unless $x_i = \tilde{n}+k-(N-i+1)$ and $c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-i-\tilde{n}}$ both hold, we have $a^*_{i+1}(i+1, x_i+1) = 0$. By Proposition 1(a), this implies $a^*_i(i,x_i) = 0$.

If $x_i = \tilde{n}+k-(N-i+1)$ and $c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-i-\tilde{n}}$ both hold, we can make the following observations.

(i) If Player i participates, we have $x_{i+1} = x_i+1$. Furthermore, we know by the induction hypothesis that $a_{i+1}^*(i+1,x_i+1) = 1$. By Proposition 1(a), this implies all later movers participate as well and we have $s_{i+1}(x_i+1) = N-i$.

(ii) If Player i passes, then we have $x_{i+1} = x_i = \tilde{n}+k-(N-i+1)$. This means $x_j < \tilde{n}+k-(N-j+1)$ for all $j \in \{i+1,...,N-\tilde{n}\}$. Therefore, by the induction hypothesis, these players all pass and we have $s_{i+1}(x_i) = \tilde{n}-1$.

From observations **(i)** and **(ii)**, we can see that when $x_i = \tilde{n}+k-(N-i+1)$ and $c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-i-\tilde{n}}$ both hold, the optimality condition (IC_{i,x_i}) of Player i is given by $c \leq g_{\tilde{n}+k-(N-i+1)}(N-i+1,\tilde{n}-1)$. This cost threshold can be obtained by substituting $(N-i+1)-\tilde{n}$ for l in $g_{k-l}(\tilde{n}+l,\tilde{n}-1)$.

Hence, we can summarize the necessary and sufficient conditions for $a_i^*(i,x_i) = 1$ as $x_i = \tilde{n}+k-(N-i+1)$ and $c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{N-i+1-\tilde{n}}$. This confirms that the induction hypothesis implies equation (6).

Setting $i = N-(\tilde{n}+k)+1$, we get

$$\begin{aligned}
&a_{N-(\tilde{n}+k)+1}^*(N-(\tilde{n}+k)+1, x_{N-(\tilde{n}+k)+1}) = 1 \\
&\Leftrightarrow x_{N-(\tilde{n}+k)+1} = 0 \;\;\wedge\;\; c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{k}
\end{aligned}$$

Note that Player $N-(\tilde{n}+k)+1$ passes under any positive history. So if any of the previous players participate, Player $N-(\tilde{n}+k)+1$ will pass. By Proposition 1(a), this implies that all previous players $\{1,...,N-(\tilde{n}+k)\}$ pass and we indeed have $x_{N-(\tilde{n}+k)+1} = 0$ on the equilibrium path. This, together with the optimality condition we derived for players $\{N-(\tilde{n}+k)+1,...,N-\tilde{n}\}$ in the above induction yields

$$s^* = s_1(0) = \begin{cases} \tilde{n}+k; & c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{k} \\ \tilde{n}-1; & otherwise \end{cases}$$

where $\tilde{n}-1 = \max\{n \in \{1,...,N\} : g_0(n,n-1) \geq c\}$ by Case 2. Finally, since $g_{k-l}(\tilde{n}+l,\tilde{n}-1)$ is quasi-concave in l and $c \leq g_k(\tilde{n},\tilde{n}-1)$ by Case 2, the condition $c \leq \min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{k}$ is equivalent to just $c \leq g_0(\tilde{n}+k,\tilde{n}-1)$. The restriction of c satisfying Case 2 and $c \leq$

$\min\{g_{k-l}(\tilde{n}+l,\tilde{n}-1)\}_{l=1}^{k}$can be summarized together as $c \in C_{\tilde{n}}^{k}$, where $C_{\tilde{n}}^{k}$ is given by

$$C_{\tilde{n}}^{k} := \left(g_{k-1}(\tilde{n},\tilde{n}-1), \min\{g_k(\tilde{n},\tilde{n}-1), g_0(\tilde{n}+k,\tilde{n}-1)\}\right]$$

When this restriction is violated, we have shown that $s^* = \max\{n \in \{1,...,N\} : g_0(n,n-1) \geq c\}$.

The final step of the proof is to show that for given $n \in \{2,...,N-1\}$, C_n^k exists when $k=1$ or k is sufficiently large. To show that C_n^k exists is to show that $g_{k-1}(n,n-1) < \min\{g_k(n,n-1), g_0(n+k,n-1)\}$. We know by Lemma 2 that $g_{k-1}(n,n-1) < g_k(n,n-1)$. Thus, given n, the conditions for $g_{k-1}(n,n-1) < g_0(n+k,n-1)$ will be derived.

First note that this clearly holds for $k=1$, since

$$g_0(n,n-1) = p(n) < p(n)+p(n+1) = g_0(n+1,n-1)$$

Thus, C_n^1 exists for all $n \in \{2,...,N-1\}$. For $k>1$, we are looking for k such that the inequality below holds

$$\begin{aligned}
&g_{k-1}(n,n-1)) = \frac{p(n+k-1)}{1-F(k-1)} \\
&= \frac{q(1-p_1)^{n+k-2}p_1}{q(1-p_1)^{k-1}+(1-q)(1-p_2)^{k-1}} + \frac{(1-q)(1-p_2)^{n+k-2}p_2}{q(1-p_1)^{k-1}+(1-q)(1-p_2)^{k-1}} \\
&< g_0(n+k,n-1) \\
&= F(n+k)-F(n-1) \\
&= q[(1-p_1)^{n-1}-(1-p_1)^{n+k}]+(1-q)[(1-p_2)^{n-1}-(1-p_2)^{n+k})] \\
&= q(1-p_1)^{n-1}[1-(1-p_1)^{k+1}]+(1-q)(1-p_2)^{n-1}[1-(1-p_2)^{k+1}]
\end{aligned}$$

Since $p_1+p_2>1$ implies $p_1(1-p_1)^{n-1} < p_2(1-p_2)^{n-1}$, we have $g_k(n,n-1) < p_2(1-p_2)^{n-1}$. That means $g_{k-1}(n,n-1) < g_0(n+k,n-1)$ is satisfied for any k that satisfies the following inequality

$$\begin{aligned}
&p_2(1-p_2)^{n-1} - g_0(n+k,n-1) \\
&= p_2(1-p_2)^{n-1} - \left[q(1-p_1)^{n-1}[1-(1-p_1)^{k+1}]+(1-q)(1-p_2)^{n-1}[1-(1-p_2)^{k+1}]\right] < 0
\end{aligned}$$

Note that the left-hand side of this inequality is continuous and strictly decreasing with respect to k. Furthermore,

$$\lim_{k\to\infty} p_2(1-p_2)^{n-1} - \left[q(1-p_1)^{n-1}[1-(1-p_1)^{k+1}]+(1-q)(1-p_2)^{n-1}[1-(1-p_2)^{k+1}]\right]$$

$$= (1-p_2)^{n-1}[p_2-(1-q)] - q(1-p_1)^{n-1} < 0$$

where the inequality holds from the restriction $p_2 < 1-q$ of Theorem 3. Thus, we know that for for sufficiently large k, we have $g_{k-1}(n, n-1) < g_0(n+k, n-1)$. Hence, for any given $n \in \{2, ..., N-1\}$, interval C_n^k exists for sufficiently large k.

$\square$

B

Appendix to Chapter 2

We first prove Proposition 1, Proposition $2(i)$ and Lemma 1 respectively. The results of Lemma 1 are then used to prove Proposition $2(ii)$ and (iii) jointly.

Proposition 1. $\underline{w}(r_1, \delta) = 0$ *for all* $r_1 \in [0, v]$ *and* $\delta \in (0, 1)$.

Proof. First, note that the restriction $p_t^i \in [0, p^c(r_t)]$ implies that the stage profit $\pi^i(p^t|r_t)$ of firm i is non-negative. Thus, for any r_1 and δ, the lowest feasible lifetime profit is zero.

As in the standard Bertrand model, the unique Nash Equilibrium (NE) of the stage game is $p_t^1 = p_t^2 = 0$ under any reference price r_t: Denote the short-run monopoly price induced by r_t by $p^m(r_t) := argmax_p \pi(p|r_t) = \frac{v+\lambda r_t}{2(1+\lambda)}$. If the opponent is setting price $p_t^j \in (0, p^m(r_t)]$, firm i has incentive to undercut j by an arbitrarily small amount and obtain $\pi(p_t^j|r_t) > \frac{1}{2}\pi(p_t^j|r_t)$. If j is setting $p_t^j > p^m(r_t)$, then it is optimal for firm i to set $p^m(r_t)$ and obtain $\pi(p^m(r_t)|r_t) > \frac{1}{2}\pi(p_t^j|r_t)$.

Since both firms setting price zero is the unique stage NE, the strategy profile $s^1(h_t) = s^2(h_t) = 0$ for all $h_t \in H$ is a sSPE for any r_1 and δ. Since this sSPE yields the lowest feasible payoff 0, we have $\underline{w}(r_1, \delta) = 0$ for all $r_1 \in [0, v]$ and $\delta \in (0, 1)$.

□

Proposition 2 (i). *If* $\delta \in (0, \frac{1}{2})$, *then* $\bar{w}(r_1, \delta) = 0$ *for all* $r_1 \in [0, v]$.

Proof. Given initial reference $r_1 \in [0, v]$, consider a symmetric price sequence $\{\tilde{p}_t\}_{t=1}^{\infty}$ with $\tilde{p}_t \in [0, p^c(r_t)]$ for all $t \in \mathbb{N}_+$ under the evolution rule $r_t = \tilde{p}_{t-1}$ for $t \geq 2$. Suppose $\tilde{p}_t > 0$ for some t. Recall that $r_1 \in [0,v]$ and $\tilde{p}_t \in [0,p^c(r_t)]$ together imply $\tilde{p}_t, r_t \in [0,v]$ for all t. Then the sequence of stage profits $\{\pi(\tilde{p}_t|r_t)\}_{t=1}^{\infty}$ resulting from r_1 and $\{\tilde{p}_t\}_{t=1}^{\infty}$ takes values in the compact set $[0,\pi(p^m(v)|v)]$ for all t. Let $\bar{\pi}(\tilde{p}|r_1) := \sup_{t\in\mathbb{N}_+}\{\pi(\tilde{p}_t|r_t)\}_{t=1}^{\infty}$.

Since 0 is the lowest feasible lifetime profit and $p_t^1 = p_t^2 = 0$ for all $t \in \mathbb{N}_+$ is a sSPE of any subgame following a deviation, $\{\tilde{p}_t\}_{t=1}^{\infty}$ is a sSPE path if and only if it yields no incentive to deviate under grim trigger strategies with price zero punishment. Under grim trigger strategies with price zero punishment, the period τ incentive constraint to sustain $\{\tilde{p}_t\}_{t=1}^{\infty}$ is

$$\sum_{t=\tau}^{\infty} \delta^{t-\tau} \frac{\pi(\tilde{p}_t|r_t)}{2} \geq \begin{cases} \pi(\tilde{p}_\tau|r_\tau); & \tilde{p}_\tau < p^m(r_\tau) \\ \pi(p^m(r_\tau)|r_\tau); & \tilde{p}_\tau \geq p^m(r_\tau) \end{cases} \tag{B.1}$$

which has to hold for all $\tau \in \mathbb{N}_+$ for subgame perfection. Since the sequence $\{\pi(\tilde{p}_t|r_t)\}_{t=1}^{\infty}$ takes values from a compact set in $\mathbb{R}_+$, we are in one of two cases. Either $\{\pi(\tilde{p}_t|r_t)\}_{t=1}^{\infty}$ obtains a maximum at some finite period $\tau \in \mathbb{N}_+$, or it converges to $\bar{\pi}(\tilde{p}|r_1)$ as t tends to infinity. If the sequence of stage profits obtains a maximum at $t = \tau$, we have

$$\pi(p^m(r_\tau)|r_\tau) \geq \pi(\tilde{p}_\tau|r_\tau) = \bar{\pi}(\tilde{p}|r_1) > \frac{\bar{\pi}(\tilde{p}|r_1)}{2(1-\delta)} \geq \sum_{t=\tau}^{\infty} \delta^{t-\tau} \frac{\pi(\tilde{p}_t|r_t)}{2}$$

which contradicts (1). The first inequality holds as $p^m(r_\tau)$ is by definition the price that maximizes stage profit under reference r_τ. The second inequality holds by $\delta < \frac{1}{2}$. The third inequality holds by the definition of the supremum.

Now suppose that $\pi(\tilde{p}_t|r_t)$ converges to $\bar{\pi}(\tilde{p}, r_1)$. Then for any $\delta < \frac{1}{2}$ there exists a finite $\tau \in \mathbb{N}_+$ with $\pi(\tilde{p}_\tau|r_\tau) = \bar{\pi}(\tilde{p}|r_1) - \epsilon$ where $\epsilon > 0$ is small enough such that

$$\pi(p^m(r_\tau)|r_\tau) \geq \pi(\tilde{p}_\tau|r_\tau) = \bar{\pi}(\tilde{p}|r_1) - \epsilon > \frac{\bar{\pi}(\tilde{p}|r_1)}{2(1-\delta)} \geq \sum_{t=\tau}^{\infty} \delta^{t-\tau} \frac{\pi(\tilde{p}_t|r_t)}{2}$$

which contradicts (1). Thus, if $\delta < \frac{1}{2}$, there is no symmetric price sequence that sets a positive price in some period and satisfies the incentive constraint in every period. As a result, the only sSPE payoff is zero, obtained when both firms play the unique stage NE in each period.

$\square$

Lemma 1. *For the solution of the recursive LR monopolist problem, the following properties hold.*

(a) The unique optimal policy function $f(r)$ satisfies $f(r) \in (p^m(r), p^c(r)]$ for all $r \in [0,v]$

(b) For any initial $r \in [0,v]$, $f_n(r)$ monotonically converges to unique steady state $r^{ss} := \frac{v}{2+\lambda(1-\delta)}$ as n tends to infinity.

(c) If the choke price constraint $r' \leq p^c(r)$ is binding at $r = r_1 \in [0,v]$, then it is binding at $r = r_2$ for any $r_2 \in [0,r_1]$.

(d) $V(r)$ is continuously differentiable and strictly increasing.

Proof. First, we derive the unique optimal pricing policy function of the problem. Denote this function by $f : [0,v] \to [0,v]$. That is, $f(r)$ is the optimal price by the LR monopolist under state r. We then use function f to prove Lemma 1. In deriving the optimal policy of the original problem, we use the optimal policy of the same problem without imposing the the constraints $0 \leq r' \leq p^c(r)$ (henceforth “unconstrained LR monopolist problem").

We can formulate the unconstrained recursive LR monopolist problem as

$$\begin{aligned} V^u(r) :&= \max_{r' \in \mathbb{R}} \pi(r'|r) + \delta V^u(r') \\ &= \max_{r' \in \mathbb{R}} r'(v - r' + \lambda(r - r')) + \delta V^u(r') \end{aligned}$$

Recall that $\pi(r'|r)$ is strictly concave and obtains a unique maximum at $r' = p^m(r)$. Denoting the optimal policy function of the unconstrained problem by $f^u : [0,v] \to \mathbb{R}$, the resulting Euler equation is:

$$f^u(r) = \frac{v + \lambda r + \delta\lambda f^u(f^u(r))}{2(1+\lambda)} \tag{B.2}$$

This functional equation yields 2 roots for $f^u(r)$, namely:

$$\begin{aligned} f^{u1}(r) &:= \frac{v}{(1+\lambda) - \lambda\delta + \sqrt{(1+\lambda)^2 - \lambda^2\delta}} + \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - \lambda^2\delta}}{\lambda\delta} r \\ f^{u2}(r) &:= \frac{v}{(1+\lambda) - \lambda\delta - \sqrt{(1+\lambda)^2 - \lambda^2\delta}} + \frac{(1+\lambda) + \sqrt{(1+\lambda)^2 - \lambda^2\delta}}{\lambda\delta} r \end{aligned}$$

However, we have $f^{u2}(0) < 0$ for any $\lambda \geq 0$ and $\delta \in (0,1)$. The optimal policy under any state (including $r = 0$) cannot be negative. To see this, consider the sequential formulation $V(r_1) =$

$\max_{\{r_t\}_{t=2}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} \pi(r_{t+1}|r_t)$ of the problem. The function $\pi(r_{t+1}|r_t)$ is strictly increasing in r_{t+1} at any $r_{t+1} < 0$ and strictly increasing in r_t at any $r_t \in \mathbb{R}$. Then for any given r_1 and continuation sequence $\{r_t\}_{t=3}^{\infty}$, setting $r_2 = 0$ yields a strictly higher lifetime value than setting any $r_2 < 0$. As a result the optimal policy cannot set a negative price, and $f^{u2}(r)$ cannot be an optimal policy in the unconstrained problem.

Note that we have $f^{u1}(0) > 0$ and $f^{u1}(r)$ strictly increasing. Thus, the unique optimal policy function of the unconstrained problem is $f^{u1}(r)$. That is, we know that $f^u(r) = a + br$ where

$$a := \frac{v}{(1+\lambda) - \lambda\delta + \sqrt{(1+\lambda)^2 - \lambda^2\delta}} \quad \text{and} \quad b := \frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - \lambda^2\delta}}{\lambda\delta}$$

Then in the constrained problem, the optimal policy is $f(r) = \min\{p^c(r), f^u(r)\} = \min\{\frac{v+\lambda r}{1+\lambda}, a + br\}$. The optimal policy is equal to the choke price if the upper bound is binding and to the unconstrained policy otherwise.

Lemma 1(a)**:** Note that $p^c(r) = \frac{v+\lambda r}{1+\lambda} > \frac{v+\lambda r}{2(1+\lambda)} = p^m(r)$ for all $r \in [0, v]$. Furthermore, we have $a > \frac{v}{2(1+\lambda)}$ and $b > \frac{\lambda}{2(1+\lambda)}$. This means $f^u(r) = a + br > \frac{v+\lambda r}{2(1+\lambda)} = p^m(r)$ for all $r \in [0, v]$. These two observations allow us to conclude that $f(r) = \min\{p^c(r), f^u(r)\} \in (p^m(r), p^c(r)]$.

Lemma 1(b)**:** First note that as $f(r)$ is the minimum of two continuous functions, it is continuous. Furthermore, we have $f(0) > 0$ and $f(v) < v$. These observations imply that $f(r)$ obtains at least one fixed point r^{ss} with $f(r^{ss}) = r^{ss}$ in the interval $(0, v)$.

Whenever the choke price constraint is binding, we have $f(r) = \frac{v+\lambda r}{1+\lambda} > r$ for any $r < v$. Thus, at any fixed point, the choke price constraint is non-binding and we have $f(r) = a + br^{ss} = r^{ss}$. The unique solution the this equality is $r^{ss} = \frac{v}{2+\lambda(1-\delta)}$, which we can conclude is the unique fixed point of the policy function $f(r)$ for $r \in [0, v]$.

As $f(0) > 0$, $f(v) < v$, and r^{ss} is the unique fixed point, we know that $f(r)$ cuts the $45°$ line from above at r^{ss}. That is, $f(r) > r$ for all $r < r^{ss}$ and $f(r) < r$ for all $r > r^{ss}$. Since $f(r)$ is strictly increasing, we have $f(r) < f(r^{ss}) = r^{ss}$ for all $r < r^{ss}$ and $f(r) > f(r^{ss}) = r^{ss}$ for all $r > r^{ss}$. Combining these two observations, we have $f(r) \in (r, r^{ss})$ for all $r < r^{ss}$ and $f(r) \in (r^{ss}, r)$ for all $r > r^{ss}$. Denoting the n^{th} iterate of the policy function on state r by $f_n(r)$ (e.g. $f_2(r) = f(f(r))$), this (together with the continuity of f) implies that for all $r \in [0, v]$, $f_n(r)$ monotonically converges to r^{ss} as n tends to infinity.

Lemma 1(c)**:** Suppose the upper bound constraint $r' \leq p^c(r)$ is binding at some $r = r_1 \in [0, v]$. This means $f^u(r_1) = a + br_1 \geq \frac{v+\lambda r_1}{1+\lambda}$. Note that $b < \frac{\lambda}{1+\lambda}$. That is, the slope of the unconstrained policy function is lower than the slope of the choke price. This implies

$$a+br_1 \geq \frac{v+\lambda r_1}{1+\lambda} \Rightarrow a+br_2 > \frac{v+\lambda r_2}{1+\lambda}, \quad \forall r_2 \leq r_1$$

Thus for any $r_2 \in [0,r_1]$, we have that $f^u(r_2) > p^c(r_2)$ and the upper bound constraint is binding.

Lemma 1(d)**:** First we show that $V(r)$ is continuous at any $r \in (0,v)$. Then we show that the first difference $\frac{dV(r)}{dr}$ is also continuous.

Continuity of $V(r)$**:** Clearly, the unconstrained value function $V^u(r)$ is continuously differentiable. Furthermore, we know from Lemma $1(c)$ that the choke price constraint is non-binding only for a convex interval $(\tilde{r},v]$ with $\tilde{r} \in [0,r^{ss})$ given by $p^c(\tilde{r}) = f^u(\tilde{r})$. In this region, we have $V(r) = V^u(r)$. Thus, the value function $V(r)$ in the constrained problem is continuously differentiable in the region $(\tilde{r},v)$ where the optimal policy $f(r)$ is interior. We additionally need to show that $V(r)$ is continuous for $r \in (0,\tilde{r}]$, where the choke price constraint is binding and we have $f(r) = p^c(r)$.

Let $p^c_n(r)$ denote the n^{th} iterate of $p^c(r)$ on state r. Since $lim_{n\to\infty} p^c_n(r) = v > \tilde{r}$ for all $r \in [0,v]$, we know that for any $r \in [0,v]$ there is a finite $n \in \mathbb{N}_+$, such that $p^c_n(r) > \tilde{r}$. That is, starting from any state $r \in [0,v]$, a state high enough that the constraint is not binding can be reached through a finite number iterations of the choke price. Until such a state is reached, the optimal policy is to set the choke price which yields zero stage profit. Then for any r where the choke price is binding, we can say:

$$p^c_n(r) > \tilde{r} > p^c_{n-1}(r) \quad \text{for some } n \in \mathbb{N}_+ \Rightarrow V(r) = \delta^n V(p^c_n(r)) = \delta^n V^u(p^c_n(r)) \tag{B.3}$$

which is continuous since $V(r)$ is continuous at any $r > \tilde{r}$.

Next, we show the continuity of $V(r)$ at $r = \tilde{r}$ and at states $r \in [0,\tilde{r})$ with $p^c_n(r) = \tilde{r}$ for some $n \in \mathbb{N}_+$. Note that $p^c(r) > r$ for all $r \in [0,v]$ implies that there is an interval $(\tilde{r}-\epsilon,\tilde{r})$ that yields $p^c(r) > \tilde{r}$ for all $r \in (\tilde{r}-\epsilon,\tilde{r})$. From Equation (3) we know that $V(r)$ is continuous for this interval. Then we have

$$\begin{aligned}\lim_{r\nearrow\tilde{r}} V(r) = \lim_{r\nearrow\tilde{r}} \delta V(p^c(r)) = \delta V(p^c(\tilde{r})) = \pi(p^c(\tilde{r})|\tilde{r}) + \delta V(p^c(\tilde{r})) \\ = \pi(f^u(\tilde{r})|\tilde{r}) + \delta V(f^u(\tilde{r})) = \lim_{r\searrow\tilde{r}} V(r) = V(\tilde{r})\end{aligned} \tag{B.4}$$

which implies that $V(r)$ is continuous at $\tilde{r} = r$. The first equality follows from Equation (3). The second equality follows from continuity of $V(r)$ at $r = p^c(\tilde{r}) > \tilde{r}$. The third equality holds by the definition of choke price ($\pi(p^c(r)|r) = 0$). The fourth equality holds by the definition of $\tilde{r}$ ($f^u(\tilde{r}) = p^c(\tilde{r})$). The fifth equality holds by the continuity of $V^u(r)$ at $r = \tilde{r}$.

Finally, consider a $\tilde{\tilde{r}} \in [0, \tilde{r})$ with $p_n^c(\tilde{\tilde{r}}) = \tilde{r}$ for some $n \in \mathbb{N}_+$. There we have

$$\lim_{r \nearrow \tilde{\tilde{r}}} V(r) = \lim_{r \nearrow \tilde{\tilde{r}}} \delta^{n+1} V(p_{n+1}^c(r)) = \delta^{n+1} V(p^c(\tilde{r}))$$
$$\lim_{r \searrow \tilde{\tilde{r}}} V(r) = \lim_{r \searrow \tilde{\tilde{r}}} \delta^{n} V(p_{n}^c(r)) = \delta^{n} V(\tilde{r})$$

The two limits are equal if and only if $\delta V(p^c(\tilde{r})) = V(\tilde{r})$, which is shown in equation (4). Thus, $V(r)$ is continuous at any such point $\tilde{\tilde{r}}$. Function $V(r)$ is now shown to be continuous at $r = \tilde{r}$, for r such that $\tilde{r} \in (p_n^c(r), p_{n+1}^c(r))$ for some $n \in \mathbb{N}_+$ and r such that $p_n^c(r) = \tilde{r}$ for some $n \in \mathbb{N}_+$. These values span the interval $(0, \tilde{r}]$. Therefore we can conclude that $V(r)$ is continuous at any $r \in (0, v)$.

Continuity of $\frac{dV(r)}{dr}$: Omitting the non-negativity constraint (which is shown above to never bind), the Lagrangian of the LR monopolist problem is given by

$$L = \pi(r'|r) + \delta V(r') + \mu\left(\frac{v + \lambda r}{1+\lambda} - r'\right)$$

where μ is the multiplier for the choke price constraint. Then if we denote by $\mu(r)$ the marginal (shadow) value of relaxing the choke price constraint under state r and the optimal policy $f(r)$, the derivative of $V(r)$ with respect to r is given by

$$\frac{dV(r)}{dr} = \lambda\left(f(r) + \frac{\mu(r)}{1+\lambda}\right)$$

If $r > \tilde{r}$ the choke price is not binding ($\mu(r) = 0$) and we have $\frac{dV(r)}{dr} = \lambda f^u(r) = \lambda(a + br)$ which is continuous. If we have state $r = \tilde{\tilde{r}}$ such that $p_{n-1}^c(\tilde{\tilde{r}}) < \tilde{r} < p_n^c(\tilde{\tilde{r}})$ for some $n \in \mathbb{N}_+$, then $V(\tilde{\tilde{r}}) = \delta^n V(p_n^c(\tilde{\tilde{r}}))$ (Equation (3)). In that case:

$$\frac{dV(r)}{dr}\Big|_{r=\tilde{\tilde{r}}} = \delta^n \frac{dV(p_n^c(r))}{dr}\Big|_{r=\tilde{\tilde{r}}} = \delta^n\left(\frac{dV(r)}{dr}\Big|_{r=p_n^c(\tilde{\tilde{r}})} \cdot \frac{dp_n^c(r)}{dr}\right)$$
$$= \left(\frac{\delta\lambda}{1+\lambda}\right)^n \cdot \frac{dV(r)}{dr}\Big|_{r=p_n^c(\tilde{\tilde{r}})} = \frac{\delta^n \lambda^{n+1}}{(1+\lambda)^n}(a + b p_n^c(\tilde{\tilde{r}})) \tag{B.5}$$

which is continuous since $p_n^c(r)$ is continuous for all n. For state $r = \tilde{r}$ recall that there is an interval $(\tilde{r} - \epsilon, \tilde{r})$ with $p^c(r) > \tilde{r}$. Then for the one-sided limits of $\frac{dV(r)}{dr}$ at $r = \tilde{r}$, we have

$$\lim_{r \nearrow \tilde{r}} \frac{dV(r)}{dr} = \frac{\delta\lambda^2}{1+\lambda}(a + bp^c(\tilde{r})) = \frac{\delta\lambda^2}{1+\lambda}\left(a + b\frac{v + \lambda\tilde{r}}{1+\lambda}\right) \tag{B.6}$$

$$\lim_{r \searrow \tilde{r}} \frac{dV(r)}{dr} = \lambda(a + b\tilde{r}) \tag{B.7}$$

We would like to show that these two limits are equal. Setting Equation (6) equal to Equation (7), and using the definition of $\tilde{r}$ which implies $f^u(\tilde{r}) = a + b\tilde{r} = \frac{v+\lambda\tilde{r}}{1+\lambda} = p^c(\tilde{r})$ we get

$$\frac{\delta\lambda^2}{1+\lambda}\left(a + b\frac{v+\lambda\tilde{r}}{1+\lambda}\right) = \lambda(a + b\tilde{r})$$

$$\Leftrightarrow \quad \delta\lambda(a + b(a + b\tilde{r})) = v + \lambda\tilde{r}$$

$$= 2(1+\lambda)(a + b\tilde{r}) - (v + \lambda\tilde{r})$$

$$\Leftrightarrow a + b\tilde{r} = \frac{v + \lambda\tilde{r} + \delta\lambda(a + b(a + b\tilde{r}))}{2(1+\lambda)}$$

which is exactly the unconstrained Euler equation (Equation (2)), and holds by the definition of optimal unconstrained policy $f^u(r) = a + br$. Thus we know that $\lim_{r \nearrow \tilde{r}} \frac{dV(r)}{dr} = \lim_{r \searrow \tilde{r}} \frac{dV(r)}{dr} = \left.\frac{dV(r)}{dr}\right|_{r=\tilde{r}}$. So $\frac{dV(r)}{dr}$ is continuous at $r = \tilde{r}$. Finally, consider a state $\tilde{\tilde{r}}$ such that $p_n^c(\tilde{\tilde{r}}) = \tilde{r}$ for some $n \in \mathbb{N}_+$. In that case, by Equation (5), the one-sided limits are

$$\lim_{r \nearrow \tilde{\tilde{r}}} \frac{dV(r)}{dr} = \lim_{r \nearrow \tilde{\tilde{r}}} \frac{\delta^{n+1}\lambda^{n+2}}{(1+\lambda)^{n+1}}(a + bp_{n+1}^c(r)) = \frac{\delta^{n+1}\lambda^{n+2}}{(1+\lambda)^{n+1}}(a + bp^c(\tilde{r})) \tag{B.8}$$

$$\lim_{r \searrow \tilde{\tilde{r}}} \frac{dV(r)}{dr} = \lim_{r \searrow \tilde{\tilde{r}}} \frac{\delta^n\lambda^{n+1}}{(1+\lambda)^n}(a + bp_n^c(r)) = \frac{\delta^n\lambda^{n+1}}{(1+\lambda)^n}(a + b\tilde{r}) \tag{B.9}$$

The two limits are equal if and only if $\frac{\delta\lambda}{(1+\lambda)}(a + bp^c(\tilde{r})) = a + b\tilde{r}$. This condition is identical to the necessary and sufficient condition for the equality of the expressions (6) and (7), which holds as shown above. Thus, $\frac{dV(r)}{dr}$ is continuous at any r such that $p_n^c(r) = \tilde{r}$ for some $n \in \mathbb{N}$.

We have shown continuity of $\frac{dV(r)}{dr}$ for $r = \tilde{r}$, all r such that $p_n^c(r) > \tilde{r} > p_{n-1}^c(r)$ for some $n \in \mathbb{N}_+$ and all r such that $p_n^c(r) = \tilde{r}$ for some $n \in \mathbb{N}_+$. These values span the interval $(0, \tilde{r})$. As a result, we can conclude that $\frac{dV(r)}{dr}$ is continuous at any $r \in (0, v)$.

Since $\frac{dV(r)}{dr} = \lambda\left(f(r) + \frac{\mu(r)}{1+\lambda}\right) > 0$ for all $r \in (0, v)$, $V(r)$ is increasing.

$\square$

Proposition 2 (ii) and (iii). *There exist $\underline{\delta}$, $\bar{\delta}$ with $1 > \bar{\delta} > \underline{\delta} > \frac{1}{2}$ such that:*

(ii) *For each $\delta \in [\underline{\delta}, \bar{\delta})$, there exists a unique $\bar{r}(\delta) \in (0, v)$ that yields:*

$$\bar{w}(r_1, \delta) = \begin{cases} \frac{V(r_1)}{2}; & r_1 \in [0, \bar{r}(\delta)] \\ \delta V(p^*(r_1)); & r_1 \in (\bar{r}(\delta), v] \end{cases}$$

Where $p^*(r_1)$ is the unique value in the interval $(0, \min\{p^m(r_1), \bar{r}(\delta)\})$ that solves $\pi(p^*(r_1)|r_1) = \delta V(p^*(r_1))$.

(iii) *If $\delta \in [\bar{\delta}, 1]$ then $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for all $r_1 \in [0, v]$*

Proof. This proof follows two steps. First we show the existence of $\underline{\delta}, \bar{\delta}$ with $1 > \bar{\delta} > \underline{\delta} > \frac{1}{2}$ that yield $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ when either $\delta > \bar{\delta}$ or $\delta \in [\underline{\delta}, \bar{\delta})$ with r_1 smaller than a threshold $\bar{r}(\delta) \in [r^{ss}, v)$. Second, we show that if $\delta \in [\underline{\delta}, \bar{\delta})$ and $r > \bar{r}(\delta)$, then $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ with $p^*(r_1)$ as defined in the Proposition.

Conditions for $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$:

First note that by construction, the highest feasible lifetime payoff a firm can obtain under a symmetric strategy profile (given initial reference r_1) is $\frac{V(r_1)}{2}$. So whenever $\frac{V(r_1)}{2}$ is a sSPE payoff, we have $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$.

It is shown above that in the recursive LR monopolist problem with initial state r, profit $V(r)$ is obtained by following the unique optimal policy $f(r)$. This implies that in the two firm game given initial reference r_1, the only symmetric price sequence that yields payoff $\frac{V(r_1)}{2}$ for the firms is $\{f_t(r_1)\}_{t=1}^{\infty}$ (as before, $f_t(r_1)$ denotes the t^{th} iterate of function f on state r_1). Therefore, we have $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ if and only if symmetric price sequence $\{f_t(r_1)\}_{t=1}^{\infty}$ is a sSPE path.

Recall that a symmetric price sequence is a sSPE path if and only if it yields no incentive to deviate at any stage under grim trigger strategies with price zero punishment. From Lemma1(a), we know that $f(r) > p^m(r)$ for all $r \in [0, v]$. Then at a period with reference r, the optimal deviation from the LR monopolist policy is to set $p^m(r)$. Assuming no deviation in case of indifference, this yields the following incentive constraint under reference r:

$$\frac{V(r)}{2} \geq \pi(p^m(r)|r) \Leftrightarrow V(r) - 2\pi(p^m(r)|r) \geq 0$$

Let $g(r) := V(r) - 2\pi(p^m(r)|r)$. For subgame perfection, we need $g(r) \geq 0$ to hold for all reference points $r \in \{r_1\} \cup \{f_t(r_1)\}_{t=1}^{\infty}$ on the equilibrium path. Since both $V(r)$ and $\pi(p^m(r)|r)$ are continuous, $g(r)$ is also continuous. Next we show that for any $\delta \geq \frac{1}{2}$, $g(r)$ cuts zero for at most one $r \in [0, v]$ and if it does, the intersection is from above. Together with the continuity of g, this implies that the set of references r that satisfy $g(r) \geq 0$ is convex.

First note that $\delta \geq \frac{1}{2}$ implies $g(0) > 0$:

$$
\begin{aligned}
V(0) = \max_{r' \in [0, p^c(0)]} \{\pi(r'|0) + \delta V(r')\} &\geq \pi(p^m(0)|0) + \delta V(p^m(0)) \\
&> \pi(p^m(0)|0) + \delta V(0)
\end{aligned}
$$

The first inequality holds as the left hand side is the highest feasible payoff and the right hand side is feasible. The second inequality holds as $V(r)$ is increasing and $p^m(0) = \frac{v}{2(1+\lambda)} > 0$. Then if $\delta \geq \frac{1}{2}$, we have:

$$
V(0) > \frac{\pi(p^m(0)|0)}{1-\delta} \geq 2\pi(p^m(0)|0) \qquad \Rightarrow g(0) > 0
$$

Differentiating $g(r)$, we obtain

$$
g'(r) = \frac{dV(r)}{dr} - 2\frac{d\pi(p^m(r)|r)}{dr} = \lambda \left(f(r) + \frac{\mu(r)}{1+\lambda} \right) - \lambda \frac{v + \lambda r}{1+\lambda}
$$

where function $\frac{dV(r)}{dr}$ is as shown in the proof of Lemma $1(d)$. Again, $\mu(r)$ denotes the shadow value of relaxing the choke price constraint $r' \leq p^c(r) = \frac{v+\lambda r}{1+\lambda}$ under state r and optimal policy $f(r)$. If the choke price constraint is binding, we have $f(r) = \frac{v+\lambda r}{1+\lambda}$ and $\mu(r) \geq 0$, which yields $g'(r) \geq 0$. If the constraint is non-binding, we have $f(r) < \frac{v+\lambda r}{1+\lambda}$ and $\mu(r) = 0$, which yields $g'(r) < 0$. That is, $g(r)$ is increasing if the choke price constraint is binding and strictly decreasing if it is non-binding. Then by Lemma $1(c)$, if $g(r)$ is strictly decreasing at some reference $r = r_1$ then it is strictly decreasing at any greater value $r = r_2 \in (r_1, v)$. This, together with $g(0) > 0$ implies that for any $\delta \in [\frac{1}{2}, 1)$, if $g(r_1) \leq 0$, then $g(r_2) < 0$ for all $r_2 \in (r_1, v]$. Therefore, we are in one of two cases: either $g(r) > 0$ for all $r \in [0, v]$, or there exists a unique threshold $\bar{r} \in (0, v]$ such that $g(\bar{r}) = 0$, $g(r) > 0$ for all $r < \bar{r}$ and $g(r) < 0$ for all $r > \bar{r}$. In either case, the set of references r that satisfy $g(r) \geq 0$ is convex.

By Lemma $1(b)$, we know that $f_t(r_1) \in [r_1, r^{ss})$ for all t if $r_1 < r^{ss}$ and $f_t(r_1) \in (r^{ss}, r_1]$ for all t if $r_1 > r^{ss}$. Furthermore, $f_t(r_1)$ converges to r^{ss} as t tends to infinity. Since the set of references r that satisfy $g(r) \geq 0$ is convex, a necessary and sufficient condition for $g(r) \geq 0$ for all $r \in \{r_1\} \cup \{f_t(r_1)\}_{t=1}^{\infty}$ is $g(r_1) \geq 0$ and $g(r^{ss}) \geq 0$. That is, the LR monopolist profit is a sSPE outcome if and only if there is no incentive to deviate from the optimal LR monopolist policy at the initial reference price and the steady state. Then if a threshold $\bar{r} \in [r^{ss}, v)$ with $g(\bar{r}) = 0$ exists, we have $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for all $r_1 \in [0, \bar{r}]$.

To see when this interior threshold exists, first note $g(r)$ is continuous and strictly increasing with respect to δ for any $r \in [0, v]$. This implies that the value $\bar{r}$ that yields $g(\bar{r}) = 0$ is strictly increasing in δ. Next, observe that $\delta \leq \frac{1}{2}$ implies $g(r^{ss}) < 0$:

$$V(r^{ss}) = \pi(r^{ss}|r^{ss}) + \delta V(r^{ss}) < \pi(p^m(r^{ss})|r^{ss}) + \delta V(r^{ss})$$

The equality follows the definition of the steady state. The inequality holds by the definition of the SR monopoly price $p^m(r^{ss})$. So if $\delta \leq \frac{1}{2}$:

$$V(r^{ss}) < \frac{\pi(p^m(r^{ss})|r^{ss})}{1-\delta} \leq 2\pi(p^m(r^{ss})|r^{ss}) \Rightarrow g(r^{ss}) < 0$$

Furthermore, we have $\lim_{\delta \to 1} g(r) = \infty$ for all $r \in [0, v]$. This, together with $g(r^{ss}) < 0$ for all $\delta \leq \frac{1}{2}$ implies that there exists a unique $\underline{\delta} \in (\frac{1}{2}, 1)$ such that $g(r^{ss}) \geq 0$ if and only if $\delta \geq \underline{\delta}$. If $\delta < \underline{\delta}$, then $g(r^{ss}) < 0$ and the LR monopolist policy is not sSPE from any initial reference r_1.

Since $g(r)$ is strictly increasing in δ and tends to infinity for all r as δ tends to one, we know that there is a unique $\bar{\delta} \in (\underline{\delta}, 1)$ that yields $g(v) = 0$ when $\delta = \bar{\delta}$. If $\delta \geq \bar{\delta}$, we have $g(r) \geq 0$ for all $r \in [0, v]$ and thus $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for all $r_1 \in [0, v]$.

Finally if $\delta \in [\underline{\delta}, \bar{\delta})$, there is a unique $\bar{r} \in [r^{ss}, v)$ that satisfies $g(\bar{r}) = 0$. In that case, since $g(r_1) \geq 0$ if and only if $r_1 \leq \bar{r}$, we have $\bar{w}(r_1, \delta) = \frac{V(r_1)}{2}$ for all $r_1 \in [0, \bar{r}]$.

Highest sSPE Payoff when $\bar{w}(r_1, \delta) < \frac{V(r_1)}{2}$:

As shown above, the LR monopolist policy is not sSPE when $\delta \in [\underline{\delta}, \bar{\delta})$ and $r_1 > \bar{r}(\delta)$[1]. Here, we pin down $\bar{w}(r_1, \delta)$ under such parameter values.

First, recall that when $r_1 > \bar{r}$, we have $\frac{V(r_1)}{2} < \pi(p^m(r_1)|r_1)$. By construction, $\frac{V(r_1)}{2}$ is the highest per firm payoff that is feasible under symmetric strategies. Then for any feasible payoff $w \in [0, \frac{V(r_1)}{2}]$, we have $w < \pi(p^m(r_1)|r_1)$. So if a price sequence is such that it allows deviation to $p^m(r_1)$ in the

[1]In the remainder of this proof, we suppress the input value of the function $\bar{r}(\delta)$. That is, $\bar{r}$ corresponds to the value of the function $\bar{r}(\delta)$ evaluated at a fixed $\delta \in [\underline{\delta}, \bar{\delta})$

initial round, it cannot be a sSPE outcome. In Period 1, any price sequence that is a sSPE path must set a price p lower the SR monopoly price $p^m(r_1)$.

We determine $\bar{w}(r_1,\delta)$ separately for two regions of $r_1 > \bar{r}$. First we look at r_1 with $r_1 > \bar{r} \geq p^m(r_1)$. That is, the LR monopolist policy cannot be sustained starting from r_1, but it can be sustained starting from any reference lower than the SR monopolist price induced by reference r_1. Second, we extend the result for this region to values of r_1 with $r_1 > p^m(r_1) > \bar{r}$.

Note that we have $r > p^m(r) = \frac{v+\lambda r}{2(1+\lambda)}$ for any $r > \frac{v}{2+\lambda}$ and $\bar{r} \geq r^{ss} = \frac{v}{2+\lambda(1-\delta)} > \frac{v}{2+\lambda}$. So if r_1 is greater than $\bar{r}$, it is also greater than $p^m(r_1)$. Thus, the two regions above (r_1 such that $r_1 > \bar{r} \geq p^m(r)$ and r_1 such that $r_1 > p^m(r_1) > \bar{r}$) span $(\bar{r}, v]$.

Region 1: r_1 such that $r_1 > \bar{r} \geq p^m(r_1)$

Under any sSPE price path, the price p in the initial round has to be below the SR monopoly price, which in this region is lower than $\bar{r}$. Thus, a necessary condition for a price path to be sSPE path is $p < \bar{r}$ in the initial round. Since $r_2 = p < \bar{r}$, the highest sSPE continuation payoff (in the game starting from Period 2) upon setting p is $\frac{V(p)}{2}$. As $p < p^m(r_1)$, the optimal Period 1 deviation is to undercut p by an arbitrarily small amount. So under grim trigger strategies, the Period 1 incentive constraint for a price path that initially sets price p and yields the highest sSPE payoff in the game starting in Period 2 is given by

$$\frac{\pi(p|r_1) + \delta V(p)}{2} \geq \pi(p|r_1) \Leftrightarrow h(p|r_1) := \delta V(p) - \pi(p|r) \geq 0$$

where $p \in [0, p^m(r_1))$. Since the left-hand side of the first inequality is the equilibrium path payoff, and is strictly increasing in p at any $p \in [0, p^m(r_1))$, the highest symmetric sSPE payoff is obtained by setting $p^*(r_1) := \max\{p \in [0, p^m(r_1)) : h(p|r_1) \geq 0\}$ in the initial stage, and following the LR monopolist policy $\{f_t(p^*(r_1))\}_{t=1}^{\infty}$ from the second period onward.

That is, the highest sSPE payoff under $\delta \in [\underline{\delta}, \bar{\delta})$ and $r_1 > \bar{r}(\delta) \geq p^m(r_1)$ is given by:

$$\bar{w}(r_1,\delta) = \frac{\pi(p^*(r_1)|r_1) + \delta V(p^*(r_1))}{2}$$

with $p^*(r_1)$ as defined above. Next, we show that in the region $p \in [0, p^m(r_1))$, function $h(p|r_1)$ crosses zero exactly once, and it is from above. That is, for each r_1 with $r_1 > \bar{r} \geq p^m(r_1)$, there exists a unique $p^*(r_1) \in [0, p^m(r_1))$ that satisfies $h(p^*(r_1)|r_1) = 0$, $h(p|r_1) > 0$ for all $p \in [0, p^*(r_1))$ and $h(p|r_1) < 0$ for all $p \in (p^*(r_1), p^m(r_1))$.

Since both $V(p)$ and $\pi(p|r_1)$ are continuously differentaible (Lemma $1(d)$), $h(p|r_1)$ is continuously differentiable in p at any $p \in (0, p^m(r_1))$. Furthermore, for all $r_1 \in [0, v]$, we have:

$$\delta V(0) = \delta\pi(f(0)|0) + \delta^2 V(f(0)) > 0 = \pi(0|r_1) \Rightarrow h(0|r_1) > 0$$

Next, note that for all $r_1 > \bar{r}$:

$$\begin{aligned}
&\frac{\pi(p^m(r_1)|r_1) + \delta V(p^m(r_1))}{2} \le \frac{V(r_1)}{2} < \pi(p^m(r_1)|r_1) \\
&\Rightarrow \delta V(p^m(r_1)) < \pi(p^m(r_1)|r_1) \\
&\Rightarrow h(p^m(r_1)|r_1) < 0 \qquad \text{(B.10)}
\end{aligned}$$

The first inequality holds because under r_1, both sides are feasible and the right-hand side is the highest feasible payoff by construction. The second inequality holds because $r_1 > \bar{r}$ implies $g(r_1) = V(r_1) - 2\pi(p^m(r_1)|r_1) < 0$ by definition of $\bar{r}$. Since $h(p|r_1)$ is continuous and we have $h(0|r_1) > 0$ and $h(p^m(r_1)|r_1) < 0$, we know that there is at least one $p \in (0, p^m(r_1))$ that yields $h(p|r_1) = 0$.

Differentiating $h(p|r_1)$ with respect to p, we obtain

$$\frac{dh(p|r_1)}{dp} = \delta\frac{dV(p)}{dp} - \frac{d\pi(p|r_1)}{dp} = \delta\lambda\left(f(p) + \frac{\mu(p)}{1+\lambda}\right) - \frac{d\pi(p|r_1)}{dp}$$

where as before, $\mu(p)$ is the shadow value of the choke price constraint ($r' \le p^c(r)$) under state p and optimal policy $f(p)$. Recall that $\pi(p|r_1)$ is strictly concave in p at all $p \in (0, p^m(r_1))$. If the choke price constraint is non-binding in the LR problem under state p, we have $\mu(p) = 0$ and $\delta\frac{dV(p)}{dp} = \delta\lambda f(p)$, which is strictly increasing. As a result, $h(p|r_1)$ is strictly convex whenever the choke price constraint $r' \le p^c(r)$ is non-binding under state $r = p$ in the LR monopolist problem. By Lemma $1(c)$, this implies that if the continuously differentiable function $h(p|r_1)$ is convex at some $p = p_1$, it is also convex at any greater value $p = p_2 > p_1$. As a result, $h(p|r_1)$ can cut zero from above for at most one p. This, together with the two observations $h(0|r_1) > 0$ and $h(p^m(r_1)|r_1) < 0$ implies that there exists exactly one value $p^*(r_1)$ in $(0, p^m(r_1))$ that satisfies $h(p^*(r_1)|r_1) = 0$, and it yields $h(p|r_1) > 0$ for all $p \in [0, p^*(r_1))$ and $h(p|r_1) < 0$ for all $p \in (p^*(r_1), p^m(r_1)]$.

Thus, the highest Period 1 price p in $[0, p^m(r_1))$ that satisfies incentive constraint $h(p|r_1) \ge 0$ is the unique value $p^*(r_1)$ that solves $\pi(p^*(r_1)|r_1) = \delta V(p^*(r_1))$. Plugging this equality into the lifetime profit, we obtain

$$\bar{w}(r_1,\delta) = \frac{\pi(p^*(r_1)|r_1) + \delta V(p^*(r_1))}{2} = \delta V(p^*(r_1))$$

Region 2: r_1 such that $r_1 > p^m(r_1) > \bar{r}$

We extend the above result to all $r_1 > \bar{r}$. As shown above, this corresponds to deriving $\bar{w}(r_1,\delta)$ for r_1 such that $r_1 > p^m(r_1) > \bar{r}$. We show by induction that if $p^m(r_1) > \bar{r}$, then $\bar{w}(r_1,\delta) = \delta V(p^*(r_1))$ where $p^*(r_1)$ is the unique value in $(0,\bar{r})$ that solves $\delta V(p^*(r_1)) = \pi(p^*(r_1)|r_1)$

Denote by $p^m_n(r_1)$ the n^{th} iterate of the SR monopolist price function on reference r_1. Note that $\lim_{n\to\infty} p^m_n(r_1) = \frac{v}{2+\lambda} < r^{ss} \leq \bar{r}$ for all $r_1 \in [0,v]$. Then for any $r_1 \in (\bar{r},v]$, there exists a lowest $\tilde{n} \in \mathbb{N}$ such that $p^m_n(r) \leq \bar{r}$ for all $n > \tilde{n}$.

Base Case: Let $\tilde{n} = 1$. That is, $p^m(r_1) > \bar{r} \geq p^m(p^m(r_1))$. As before, since no symmetric initial price above $p^m(r_1)$ can deter deviation, we only need to look at initial prices $p \in [0,p^m(r_1))$. Given $p^m(r_1) > \bar{r}$ we can divide this interval in two: $p \in [0,\bar{r}]$ and $p \in (\bar{r},p^m(r_1))$. The first period incentive constraint for setting $p \in [0,\bar{r}]$ is as before. When initially setting $p \in (\bar{r},p^m(r_1))$, the highest sSPE continuation payoff in the game starting tomorrow is $\delta V(p^*(p))$ with $p^*(p) \in (0,p^m(p))$. This is because we have $p^m(p) < p^m(p^m(r_1)) \leq \bar{r}$ and thus, any $p \in (\bar{r},p^m(r_1))$ as an initial reference is in "Region 1" discussed above.

We can write the Period 1 incentive constraints under the highest sSPE continuation payoffs when setting initial price $p \in [0,p^m(r_1))$ as:

$$\begin{aligned} &h(p|r_1) = \delta V(p) - \pi(p|r_1) \geq 0 \;\text{ if }\; p \in [0,\bar{r}] \\ &\frac{\pi(p|r_1)}{2} + \delta\bar{w}(p,\delta) \geq \pi(p|r_1) \Leftrightarrow \delta^2 V(p^*(p)) - \frac{\pi(p|r_1)}{2} \geq 0 \;\text{ if }\; p \in (\bar{r},p^m(r_1)) \end{aligned} \tag{B.11}$$

Recall from the proof for Region 1 that for all $r_1 > \bar{r}$, there is a unique $p^*(r_1) \in (0,p^m(r_1))$ that satisfies $h(p^*(r_1)|r_1) = 0$ with $h(p|r_1) > 0$ for all $p \in [0,p^*(r_1))$ and $h(p|r_1) < 0$ for all $p \in (p^*(r_1),p^m(r_1)]$.

Next, we show that if $\bar{r} < p^m(r_1)$, we have $h(\bar{r}|r_1) < 0$: Let $\tilde{r} \in (\bar{r},v)$ be given by $p^m(\tilde{r}) = \bar{r}$. By Inequality (10), we know that $h(\bar{r}|\tilde{r}) < 0$. Then since $\pi(\bar{r}|r)$ is strictly increasing in r, we have $\delta V(\bar{r}) < \pi(\bar{r}|\tilde{r}) < \pi(\bar{r}|r_1)$ for all $r_1 > \tilde{r}$. Thus, $h(\bar{r}|r_1) = \delta V(\bar{r}) - \pi(\bar{r}|r_1) < 0$ for all r_1 such that $p^m(r_1) > \bar{r}$.

For any r_1 with $p^m(r_1) > \bar{r}$, we now know that $p^*(r_1) \in (0,\bar{r})$ and $h(p|r_1) < 0$ for all $p \in (\bar{r},p^m(r_1))$. We can use this observation to show that the incentive constraint for setting initial price

$p \in (\bar{r}, p^m(r_1))$ (i.e. Inequality (11)) can never hold: For any $p \in (\bar{r}, p^m(r_1))$, we have:

$$\frac{\pi(p|r_1)}{2} > \frac{\delta V(p)}{2} > \delta^2 V(p^*(p))$$

which contradicts Inequality (11). The first inequality follows from $h(p|r_1) = \delta V(p) - \pi(p|r_1) < 0$ for all $p \in (\bar{r}, p^m(r_1))$. The second inequality holds because under initial reference p, $\frac{V(p)}{2}$ is the highest feasible payoff and $\delta V(p^*(p))$ is the highest sSPE payoff (since p is in Region 1).

So if the firms are initially setting $p \in (\bar{r}, p^m(r_1))$ the Period 1 incentive constraint is violated even under the highest sSPE continuation payoff. Thus, there is no symmetric sSPE following $p \in (\bar{r}, p^m(r_1))$ that is good enough to deter the first period deviation available. Then any sSPE strategy must initially set price $p \in [0, \bar{r}]$. In this interval, the highest price that satisfies the incentive constraint $h(p|r_1) \geq 0$ is the unique value $p^*(r_1)$ that solves $\delta V(p^*(r_1)) = \pi(p^*(r_1)|r_1)$. Thus, we have $w(r_1, \delta) = \delta V(p^*(r_1))$ with $p^*(r_1) \in (0, \bar{r})$ for all r_1 with $p^m(r_1) > \bar{r} \geq p^m(p^m(r_1))$.

Inductive Step: Suppose we have some $\tilde{n} \in \mathbb{N}_+$ such that for any $n \in \{1, ..., \tilde{n}\}$, $\bar{r} \in [p^m_{n+1}(r_1), p^m_n(r_1))$ implies $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ where $p^*(r_1) \in (0, \bar{r})$ is defined as before.

Now let initial reference $r_1 \in (\bar{r}, v]$ be such that $\bar{r} \in [p^m_{\tilde{n}+2}(r_1), p^m_{\tilde{n}+1}(r_1))$. We need to show that this implies $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$. The first period incentive constraint under the highest sSPE continuation payoff for initial price $p \in [0, \bar{r}]$ is once again $h(p|r_1) \geq 0$. For any $p \in (\bar{r}, p^m(r_1))$ we have $\bar{r} \in [p^m_{n+1}(p), p^m_n(p))$ for some $n \in \{1, ..., \tilde{n}\}$. Therefore by our induction hypothesis we have $\bar{w}(p, \delta) = \delta V(p^*(p))$. The Period 1 incentive constraint for setting initial price $p \in (\bar{r}, p^m(r_1))$ under the highest sSPE continuation payoff is then given by

$$\delta^2 V(p^*(p)) \geq \frac{\pi(p|r_1)}{2}$$

which is identical to Inequality (11) and is shown in the base case to be violated for all $p \in (\bar{r}, p^m(r_1))$. Therefore in any sSPE, the initial price must be in the interval $[0, \bar{r}]$. As we have shown above, the highest sSPE payoff in this interval is obtained by setting the unique $p^*(r_1) \in (0, \bar{r})$ that satisfies $\pi(p^*(r_1)|r_1) = \delta V(p^*(r_1))$ which yields $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$.

We have shown the induction hypothesis to be true for $\tilde{n} = 1$. Thus we have $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ with $p^*(r_1) \in (0, \bar{r})$ for any initial reference r_1 with $\bar{r} \in [p^m_{n+1}(r_1), p^m_n(r_1))$ for some $n \in \mathbb{N}_+$. The values of r_1 that satisfy $\bar{r} \in [p^m_{n+1}(r_1), p^m_n(r_1))$ for some $n \in \mathbb{N}_+$ span the entire Region 2 (r_1 such that $r_1 > p^m(r_1) > \bar{r}$).

Combining the results for Region 1 and Region 2, we can summarize the highest sSPE payoffs as follows: For $\delta \in [\underline{\delta}, \bar{\delta})$ and $r_1 > r(\bar{\delta})$, we have $\bar{w}(r_1, \delta) = \delta V(p^*(r_1))$ with $p^*(r_1)$ the unique value in the interval $(0, \min\{p^m(r_1), \bar{r}(\delta)\})$ that solves $\delta V(p^*(r_1)) = \pi(p^*(r_1)|r_1)$.

□

C

Appendix to Chapter 3

Appendix

Lemma 1. *For every player, each rationalizable strategy rejects with certainty following a low signal.*

Proof. We prove this using iterated elimination of strictly dominated strategies. First note that rejecting always yields a non-negative expected payoff. In particular, if Player i plays $a_i = R$, her expected payoff is given by $u_i = p(a_j = A \text{ for some } j \in \{1, ..., N\}|s = H)p(s = H) \geq 0$.

When Player 1 receives a low signal, her Bayesian posterior that it is the high state is $p(s = H|m_1 = l) = \frac{0.5(1-p)}{0.5(1-p)+0.5p} = 1 - p$. Thus, adopting yields an expected payoff of $1 - p - c < 0$. So for Player 1, any strategy that adopts with positive probability after a low signal is strictly dominated.

Next, suppose by induction that players $\{1, ..., i-1\}$ all reject with certainty under the low signal. Then Player i's highest possible Bayesian posterior bekief that $s = H$ is obtained when players $\{1, ..., i-1\}$ all play a pooling strategy (i.e. they also reject for certain under the high signal). In that case, the high state posterior of Player i is once again $1 - p$ and she rejects following a low signal as well. Other high signal actions by previous players lead to an an even lower high state posterior $\frac{p(a_1=...=a_{i-1}=R|s=H)(1-p)}{p(a_1=...=a_{i-1}=R|s=H)(1-p)+p(a_1=...=a_{i-1}=R|s=L)p} < 1 - p$ for Player i. Thus, if all previous players reject for certain under the low signal, it is strictly preferred for Player i to reject under low signal as well. Together with the above base case for $i = 2$, this implies that any rationalizable strategy by any player rejects for certain under the low signal. $\square$

Equilibria

The derivation of the equilibria will proceed as follows. First, some notation is introduced for conciseness. Then, the best response function of each player is stated. Using this, the unique pure strategy equilibrium of the game (i.e., Proposition 1) is derived. Finally all mixed equilibria of the game and the conditions for their existence are shown, and the mixed equilibrium of interest σ^N is indicated.

As stated above, let σ_i denote the probability that player $i \in \{1,...,N\}$ adopts following a high signal $m_i = h$. From Lemma 1, we know that if every other player is playing a rationalizable strategy, Player i's Bayesian posterior belief that the state is H can be written as a function of the other players' actions σ_{-i} following a high signal. In particular, after she receives a high signal herself, we can define Player i's belief function $\mu_i : [0,1]^{N-1} \mapsto [0,1]$ where

$$\mu_1(\sigma_{-1}) := p \;\; \wedge \;\; \mu_i(\sigma_{-i}) := \frac{p\prod_{j=1}^{i-1}(1-p\sigma_j)}{p\prod_{j=1}^{i-1}(1-p\sigma_j)+(1-p)\prod_{j=1}^{i-1}(1-(1-p)\sigma_j)}, \forall i \in \{2,...,N\}$$

The gain from adopting is equal to the probability that state is high and none of the future players will adopt. We can define Player i's gain function $g_i : [0,1]^{N-1} \mapsto [0,1]$ with $g_i(\sigma_{-i}) := \mu_i(\sigma_{-i})\prod_{j=i+1}^{N}(1-p\sigma_j)$. Since the cost of adopting is c, the best response correspondence ρ_i of Player i is given by

$$\rho_i(\sigma_{-i}) := \begin{cases} 1; & g_i(\sigma_{-i}) > c \\ [0,1]; & g_i(\sigma_{-i}) = c \\ 0; & g_i(\sigma_{-i}) < c \end{cases}$$

where $\rho_i(\sigma_{-i}) = 1 (= 0)$ means it is strictly optimal to adopt (reject) for certain and $\rho_i(\sigma_{-i}) = [0,1]$ means indifference.

Pure Strategy Equilibrium

Proposition 1. *The game yields a unique pure strategy equilibrium σ^P. The strategy profile is given by $\sigma^P_N = 1$ and $\sigma^P_i = 0$ for all $i \in \{1,...,N-1\}$.*

Proof. The following observations regarding the best response function ρ are useful for the derivation.

Observation (i): If $\sigma_j = 0$ for all $j \neq i$, then $\rho_i(\sigma_{-i}) = 1$. This is because

$$\sigma_j = 0, \forall j \neq i \Rightarrow \quad \mu_i(\sigma_{-i}) = p \wedge \prod_{j=i+1}^{N}(1-p\sigma_j) = 1$$

$$\Rightarrow \quad g_i(\sigma_{-i}) = p > c$$

Observation (ii): If $\sigma_j = 1$ for some $j > i$, then $\rho_i(\sigma_{-i}) = 0$. This is because

$$\sigma_j = 1 \text{ for some } j > i \Rightarrow \prod_{j=i+1}^{N}(1-p\sigma_j) \leq 1-p$$
$$\Rightarrow \quad g_i(\sigma_{-i}) \leq \mu_i(\sigma_{-i})(1-p) < 1-p < c$$

Observation (iii): If $\sigma_j = 1$ for some $j < i$ and $\sigma_k = 0$ for all $k \notin \{i,j\}$, then $\rho_i(\sigma_{-i}) = 1$. This is because

$$\sigma_j = 1 \text{ for some } j < i \ \ \wedge \ \ \sigma_k = 0, \forall k \notin \{i,j\}$$
$$\Rightarrow \quad \mu_i(\sigma_{-i}) = \frac{1}{2} \wedge \prod_{j=i+1}^{N}(1-p\sigma_j) = 1$$
$$\Rightarrow \quad g_i(\sigma_{-i}) = \frac{1}{2} > c$$

From Observation *(i)*, it follows that $\sigma_i = 0$ for all $i \in \{1,...,N\}$ cannot be an equilibrium. This is because in that case, every Player i is strictly better off deviating to $\sigma_i' = 1$. Therefore, in any pure strategy equilibrium there must be at least one player who adopts.

From Observation *(ii)*, it follows that there is no equilibrium where $|\{i \in \{1,...,N\} : \sigma_i = 1\}| > 1$. This is because under such a profile, the earliest player $j = \min\{i \in \{1,...,N\} : \sigma_i = 1\}$ is strictly better off deviating to $\sigma_j' = 0$. So in any pure strategy equilibrium, there can be at most one player who adopts.

Observations *(i)* and *(ii)* together imply that for any $c \in (1-p, 0.5)$, the profile σ^P with $\sigma_N^P = 1$ and $\sigma_i^P = 0$ for all $i < N$ is an equilibrium.

The remaining pure candidate profiles are those where $\sigma_i = 1$ for some $i \in \{1,...,N-1\}$ and $\sigma_j = 0$ for all $j \neq i$. Observation *(iii)* implies that if $\sigma_i = 1$, Player $i+1$ strictly prefers deviating to $\sigma_{i+1}' = 1$. Thus, the player who adopts in a pure strategy equilibrium can only be the final mover (i.e., Player N).

We have thus exhausted all pure candidates. The unique pure strategy equilibrium of the game is σ^P as described above.

□

Mixed Strategy Equilibria

Lemma 2 states all mixed strategy equilibria of the game for any combination of $c \in (1-p, 0.5)$ and $N \geq 2$. This result is then used to prove Proposition 2.

Lemma 2. *There exists a cost level $\underline{c} \in (1-p, 0.5)$ that satisfies the following.*

1. *If $c \leq \underline{c}$, then the game yields no mixed strategy equilibria.*
2. *For all $n > 2$, there exists a cost level $c_n \in (\underline{c}, 0.5)$ such that;*

   a) If $c > c_n$, then for each subset $\tilde{N} \subseteq \{1,...,N\}$ of players with $|\tilde{N}| \geq n$ there exists a mixed equilibrium $\sigma^{\tilde{N},a}$ of the following form: $\sigma_i^{\tilde{N},a} \in (0,1)$ for all $i \in \tilde{N}$ and $\sigma_i^{\tilde{N},a} = 0$ for all $i \notin \tilde{N}$.

   b) If $c \geq c_n$, then for each subset $\tilde{N} \subseteq \{1,...,N\}$ of players with $|\tilde{N}| \geq n$ there exists a mixed equilibrium $\sigma^{\tilde{N},b}$ of the following form: $\sigma_{\min\{\tilde{N}\}}^{\tilde{N},b} = 1$, $\sigma_i^{\tilde{N},b} \in (0,1)$ if $i \in \tilde{N} \backslash \min\{\tilde{N}\}$, and $\sigma_i^{\tilde{N},b} = 0$ if $i \notin \tilde{N}$.

   c) For all $n > 2$, we have $c_n > c_{n+1}$.

Proof. We derived all pure strategy equilibria in Proposition 1. This, together with Observation (ii) means that the only remaining candidates for equilibrium are strategy profiles σ such that there is a subset $\tilde{N}$ of players with $\sigma_{\min\{\tilde{N}\}} \in (0,1]$, $\sigma_i \in (0,1)$ for all $i \in \tilde{N} \backslash \min\{\tilde{N}\}$, and $\sigma_j = 0$ for all $j \notin \tilde{N}$.

First, we check the existence of a profile with the above structure that satisfies $g_i(\sigma_{-i}) = c$ (indifference) for all $i \in \tilde{N}$. Since adding (past or future) players j with $\sigma_j = 0$ does not change Player i's gain $g_i(\sigma_{-i})$ from adopting, we can assume without loss of generality that $\tilde{N} = \{1,...,N\}$ for this step. Upon establishing the conditions for existence of such a profile, we show that $\sigma_j = 0$ is indeed the best response for all $j \notin \tilde{N}$.

We are looking for a profile σ that satisfies

$$g_i(\sigma_{-i}) = c, \quad \text{for all} \quad i \in \{1,...,N\} \tag{C.1}$$

This implies $g_1(\sigma_{-(1)}) = g_i(\sigma_{-i})$ for all $i \geq 2$. Or equivalently

$$p \prod_{j=2}^{N} (1 - p\sigma_j) = \mu_i(\sigma_{-i}) \prod_{j=i+1}^{N} (1 - p\sigma_j)$$

which can be simplified as follows.

$$\begin{aligned} p \prod_{j=2}^{N} (1 - p\sigma_j) &= \frac{p \prod_{j=1}^{i-1} (1 - p\sigma_j)}{p \prod_{j=1}^{i-1} (1 - p\sigma_j) + (1-p) \prod_{j=1}^{i-1} (1 - (1-p)\sigma_j)} \prod_{j=i+1}^{N} (1 - p\sigma_j) \\ \Leftrightarrow 1 - p\sigma_i &= \frac{(1 - p\sigma_1)}{p \prod_{j=1}^{i-1} (1 - p\sigma_j) + (1-p) \prod_{j=1}^{i-1} (1 - (1-p)\sigma_j)} \end{aligned} \tag{C.2}$$

Thus, given that a profile satisfies Condition (1), we can write σ_1 as a function only of the strategies $\{\sigma_j\}_{j=1}^{i-1}$ of earlier players. For instance, setting $i = 1$ in Equation (2), we obtain

$$\sigma_2 = \frac{1}{p}\left(1 - \frac{1-p\sigma_1}{p(1-p\sigma_1)+(1-p)(1-(1-p)\sigma_1)}\right) \tag{C.3}$$

where the right hand side is a function of σ_1 that is continuous and takes values in $[0,1]$ over $\sigma_1 \in [0,1]$. Let $\sigma_2^a : [0,1] \mapsto [0,1]$ denote the value of σ_2 that satisfies Equation (3), as a function of σ_1. We can then plug in $\sigma_2^a(\sigma_1)$ for σ_2 and obtain the unique value $\sigma_3^a(\sigma_1) \in [0,1]$ that satisfies Equation (2) for $i = 3$, as a continuous function of only σ_1. If we iterate this process, Equation (2) for any i can be written as follows.

$$1 - p\sigma_i^a(\sigma_1) = \frac{1-p\sigma_1}{p\prod_{j=1}^{i-1}(1-p\sigma_j^a(\sigma_1)) + (1-p)\prod_{j=1}^{i-1}(1-(1-p)\sigma_j^a(\sigma_1))} \tag{C.4}$$

where $\sigma_j^a : [0,1] \mapsto [0,1]$ is as defined above for all $i \in \{2,..,i\}$ and $\sigma_1^a(\sigma_1) = \sigma_1$. That is, for any given σ_1, there is a unique sequence of strategies $\{\sigma_i^a(\sigma_1)\}_{i=2}^{N} \in [0,1]^{N-1}$ that satisfies equation (2) for all $i \geq 2$. Then indifference Condition (1) is satisfied by profile σ if and only if σ_1 is such that

$$g_1(\{\sigma_i^a(\sigma_1)\}_{i=2}^{N}) = p\prod_{i=2}^{N}(1-p\sigma_i^a(\sigma_1)) = c$$

Define function $h^N : [0,1] \mapsto [0,1]$ as $h^N(\sigma_1) := \prod_{i=2}^{N}(1-p\sigma_i^a(\sigma_1))$. Then condition (1) is satisfied if and only if $h^N(\sigma_1) = \frac{c}{p}$. We now show derive the conditions under which a value of σ_1 that satisfies this equality exists and is unique. We do so using the following observations.

Observation (a): $h^N(0) = 1$ for all $N \geq 2$.

To see this, first note that Equation (3) yields $\sigma_2^a(0) = 0$ and thus, $h^2(0) = 1$. By induction, suppose that for some $i \in \{3,...,N\}$, we have $\sigma_j^a(0) = 0$ for all $2 \leq j < i$. Then from Equation (4), we have $1 - p\sigma_i^a(0) = 1$ and by the induction hypothesis (which is shown above to hold for $i = 3$), we have $h^i(0) = 1$.

Observation (b): $h^N(\sigma_1)$ is strictly decreasing at any $\sigma_1 \in (0,1)$, for all $N \geq 2$.

To see this, recall that we defined $\{\sigma_i^a(\sigma_1)\}_{i=2}^{N}$ as the unique sequence $\{\sigma_i\}_{i=2}^{N}$ which for a given σ_1 satisfies $g_1(\sigma_{-(1)}) = g_i(\sigma_{-i})$ for all $i \in \{2,...,N\}$. This sequence can also be described as the unique solution (for given σ_1) to the following system of $N-1$ equations: $g_i(\sigma_{-i}) - g_{i+1}(\sigma_{-(i+1)}) = 0$ for $i \in \{1, ..., N-1\}$. Then we can differentiate $\{\sigma_i^a(\sigma_1)\}_{i=2}^N$ as implicit functions in the above system of $N-1$ equations and obtain (suppressing the arguments of functions σ_i^a and g_i for conciseness):

$$\begin{bmatrix} \frac{d\sigma_2^a}{d\sigma_1} \\ \vdots \\ \vdots \\ \frac{d\sigma_N^a}{d\sigma_1} \end{bmatrix} = - \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ b_1 & a_2 & 0 & \dots & 0 \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b_{N-2} & a_{N-1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{d(g_1-g_2)}{d\sigma_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $a_i := \frac{d(g_i - g_{i+1})}{d\sigma_{i+1}}$ and $b_i := \frac{d(g_{i+1} - g_{i+2})}{d\sigma_{i+1}}$. Note that $a_i < 0$ and $b_i > 0$ for all $i \in \{1, ..., N-1\}$. Denote the row i, column j entry in the inverse of the above bidiagonal matrix by ν_{ij}. The first column entries of the inverse are as follows.

$$\nu_{11} = \frac{1}{a_1} < 0 \quad \wedge \quad \nu_{i1} = (-1)^{i+1} \frac{\prod_{j=1}^{i-1} b_i}{\prod_{j=1}^{i} a_i} < 0, \ \ \forall i \geq 2$$

Since the first column of the inverse consists of strictly negative numbers and $\frac{d(g_1-g_2)}{d\sigma_1} > 0$, we can conclude that $\frac{d\sigma_i^a}{d\sigma_1} > 0$ for all $i \in \{2, ..., N\}$. Thus, $h^N(\sigma_1) = \prod_{i=2}^N (1 - p\sigma_i^a(\sigma_1))$ is strictly decreasing for all $N \geq 2$.

Observations (a) and (b), together with the continuity of h^N imply that there exists a value $\sigma_1^a \in (0, 1]$ such that $h^N(\sigma_1^a) = \frac{c}{p}$ if and only if $h^N(1) \leq \frac{c}{p}$. If such a value exists, it is unique.

The final step in deriving the existence of equilibrium $\sigma^{N,a}$ of the form $\sigma_i^{N,a} \in (0, 1)$ for all $i \in \{1, ..., N\}$ is to obtain the conditions on c and N such that $h^N(1) < \frac{c}{p}$.

First note that for all N, we have $h^N(1) > h^{N+1}(1) = (1 - p\sigma_{N+1}^a(1)) \prod_{i=2}^N (1 - p\sigma_i^a(1))$. That is, $h^N(1)$ is strictly decreasing in N. Furthermore, since $h^2(1) = \frac{1}{2p} > \frac{c}{p}$ (from Equation (4)), we know that there cannot be a strategy profile that satisfies Condition (1) when $N = 2$.

To see when $h^N(1) \leq \frac{c}{p}$, we check $\lim_{N \to \infty} h^N(1)$. In doing so, it is useful to write $h^N(1)$ as a recursive sequence. That is,

$$h^N(1) = h^{N-1}(1)(1 - p\sigma_N^a(1)) = \frac{h^{N-1}(1)}{p\left(h^{N-1}(1) + \prod_{j=2}^{N-1}(1 - (1-p)\sigma_j^a(1))\right)}$$

Next, suppose that $h^N(1) > \frac{1-p}{p}$. Then,

$$
\begin{aligned}
h^{N+1}(1) &= \frac{h^N(1)}{p\left(h^N(1) + \prod_{j=2}^N (1-(1-p)\sigma_j^a(1))\right)} \\
&> \frac{\frac{1-p}{p}}{p\left(\frac{1-p}{p} + \prod_{j=2}^N (1-(1-p)\sigma_j^a(1))\right)} \\
&= \frac{\frac{1-p}{p}}{1-p\left(1-\prod_{j=2}^N (1-(1-p)\sigma_j^a(1))\right)} \\
&> \frac{1-p}{p}
\end{aligned}
$$

So since $h^N(1) > \frac{1-p}{p}$ implies $h^{N+1}(1) > \frac{1-p}{p}$ and $h^2(1) = \frac{1}{2p} > \frac{1-p}{p}$, we know that $h^N(1)$ is bounded below by $\frac{1-p}{p}$. This, together with the observation that $h^N(1)$ is decreasing with N implies $\lim_{N\to\infty} h^N(1) \in [\frac{1-p}{p}, \frac{1}{2p})$.

Next, we show that $h^N(1)$ is bounded away from $\frac{1-p}{p}$. First note that with the same reasoning as we applied to $h^N(1)$, we have that $\prod_{j=2}^N (1-(1-p)\sigma_j^a(1))$ is also strictly decreasing in N. Since $p > \frac{1}{2}$, we have $\prod_{j=2}^N (1-(1-p)\sigma_j^a(1)) > h^N(1)$ for all N. Then $\lim_{N\to\infty} h^N(1) \geq \frac{1-p}{p}$ implies that $\lim_{N\to\infty} \prod_{j=2}^N (1-(1-p)\sigma_j^a(1)) \geq \frac{1-p}{p} > 0$. This allows us to state the following inequality:

$$
\begin{aligned}
\lim_{N\to\infty} h^N(1) &= \frac{\lim_{N\to\infty} h^N(1)}{p\left(\lim_{N\to\infty} h^N(1) + \lim_{N\to\infty} \prod_{j=2}^N (1-(1-p)\sigma_j^a(1))\right)} \\
&= \frac{1-p\left(\lim_{N\to\infty} \prod_{j=2}^N (1-(1-p)\sigma_j^a(1))\right)}{p} \\
&> \frac{1-p}{p}
\end{aligned}
$$

We can summarize as follows: First, $h^N(1)$ is strictly decreasing with respect to N. Second, $h^2(1) = \frac{1}{2p} > \frac{1-p}{p}$. Third, $h^N(1)$ is bounded away from $\frac{1-p}{p}$ as N tends to infinity.

Defining $\underline{c} := p(\lim_{N\to\infty} h^N(1))$, we conclude the following. If $c \in (1-p, \underline{c})$, then $h^N(1) > \frac{c}{p}$ for all $N \geq 2$. Thus, there is no strategy profile that satisfies Condition (1) (i.e., makes all players indifferent).

For every $N \geq 3$, there exists a cost $c_N \in (\underline{c}, \frac{1}{2p})$ such that $h^N(1) = \frac{c_N}{p}$. Note that since $h_N(1)$ is decreasing with respect to N, we have $c_N > c_{N+1}$ for all $N \geq 3$. This corresponds to the statement of Lemma 2.2(c).

When $c > c_N$, there is a (unique) interior value $\sigma_1^{N,a} \in (0,1)$ which satisfies $h^N(\sigma_1^{N,a}) = \frac{c}{p}$ and therefore indifference Condition (1). Hence, the strategy profile $\{\sigma_i^a(\sigma_1^{N,a})\}_{i=1}^N$ is an equilibrium of the form stated in Lemma 2.2(a). If $c = c_N$, then we have $h^N(1) = \frac{c}{p}$ and the value of σ_1 that satisfies Condition (1) is $\sigma_1^{N,b} = 1$. Since $\sigma_i^a(1) \in (0,1)$ for all $i \geq 2$, this gives us an equilibrium of the form described in Lemma 2.2(b).

Next, we relax the assumption $\tilde{N} = \{1,...,N\}$ and show that for all players outside $\tilde{N}$ the best response is indeed to reject. That is, we need to show that $g_k(\sigma_{-k}) < c$ for all $k \in \{1,...,N\}\backslash\tilde{N}$ where σ satisfies Condition (1) for all $i \in \tilde{N}$ and $\sigma_j = 0$ for all $j \in \{1,...,N\}\backslash\tilde{N}$.

Consider some $k \notin \tilde{N}$ and let $i := argmin_{j\in\tilde{N}}|j-k|$. That is, i is the member of $\tilde{N}$ that is closest in order of move to k. By the definition of $\tilde{N}$, we have $\sigma_i > 0$ and $\sigma_j = 0$ for all $j \in \{\min\{i,k\} + 1,...,\max\{i,k\} - 1\}$. Then,

$$
\begin{aligned}
&c - g_k(\sigma_{-k}) = g_i(\sigma_{-i}) - g_k(\sigma_{-k}) \\
&= \mu_i(\sigma_{-i}) \prod_{j=i+1}^{N} (1 - p\sigma_j) - \mu_k(\sigma_{-k}) \prod_{j=k+1}^{N} (1 - p\sigma_j) > 0
\end{aligned}
$$

The first equality holds from Condition (1) (i.e., the indifference of all players in $\tilde{N}$). The second equality is the definition of the function g. The inequality holds because if $k < i$, we have $\mu_i(\sigma_{-i}) = \mu_k(\sigma_{-k})$ and

$$
\prod_{j=k+1}^{N} (1 - p\sigma_j) = (1 - p\sigma_i) \prod_{j=i+1}^{N} (1 - p\sigma_j) < \prod_{j=i+1}^{N} (1 - p\sigma_j)
$$

If $k > i$, we have $\prod_{j=k+1}^{N}(1 - p\sigma_j) = \prod_{j=i+1}^{N}(1 - p\sigma_j)$ and

$$
\mu_k(\sigma_{-k}) = \frac{p\prod_{j=1}^{i}(1 - p\sigma_j)}{p\prod_{j=1}^{i}(1 - p\sigma_j) + (1 - p)\prod_{j=1}^{i}(1 - (1 - p)\sigma_j)} < \mu_i(\sigma_{-i})
$$

In either case, we have $c = g_i(\sigma_{-i}) > g_k(\sigma_{-k})$, and the best response of player $k \notin \tilde{N}$ to σ_{-k} is indeed $\rho_k(\sigma_{-k}) = 0$, Hence, we can conclude that a profile σ such that there is a subset $\tilde{N}$ of players where Condition (1) is satisfied for all $i \in \tilde{N}$ and $\sigma_j = 0$ for all $j \notin \tilde{N}$ is indeed an equilibrium.

Since the remaining candidate strategy profiles allowed for $\sigma_1 = 1$, we need to check one last case where Player $\min\{\tilde{N}\}$ strictly prefers to play A and all other players in $\tilde{N}$ are indifferent. Once again, assume $\tilde{N} = \{1,...,N\}$ without loss of generality (the extension of the following equilibrium to proper subsets are analogous to the case above).

Since Player 1 strictly prefers adopting and all others are indifferent, we now replace Condition (1) with the following condition on profile σ.

$$g_1(\sigma_1) > c \qquad \wedge \qquad g_i(\sigma_i) = c, \forall i \geq 2 \tag{C.5}$$

First note that if $\sigma_1 = 1$, then we have $\mu_2(\sigma_{-2}) = \frac{1}{2}$. From $g_2(\sigma_{-(2)}) = g_i(\sigma_i) = c, \forall i \geq 3$ and $\sigma_1 = 1$, we can write

$$1 - p\sigma_i^b(\sigma_2) = \frac{2(1-p\sigma_2)}{\prod_{j=2}^{i-1}(1-p\sigma_j^b(\sigma_2)) + \prod_{j=2}^{i-1}(1-(1-p)\sigma_j^b(\sigma_2))}$$

where $\sigma_2^b(\sigma_2) = \sigma_2$ and $\sigma_i^b : [0,1] \mapsto [0,1]$ yields the value of σ_i as a function of σ_2 in the unique sequence $\{\sigma_j\}_{j=3}^N$ that satisfies $g_2(\sigma_{-(2)}) = g_i(\sigma_i)$ for all $i \geq 3$, given $\sigma_1 = 1$. Next, define function $f^N(\sigma_2) := \prod_{j=3}^N (1 - p\sigma_j^b(\sigma_2))$ for all $N \geq 3$. Then we are looking for the value of σ_2 that satisfies:

$$f^N(\sigma_2) = 2c \tag{C.6}$$

and

$$(1-p\sigma_2)f^N(\sigma_2) = \frac{c}{p} \tag{C.7}$$

Condition (6) implies the indifference of all players $\{2, .., N\}$ and Condition (7) implies that Player 1 strictly prefers adopting. Given that Condition (6) holds, Condition (7) is equivalent to

$$(1-p\sigma_2)f^N(\sigma_2) > \frac{f^N(\sigma_2)}{2p}$$

which simplifies to

$$\sigma_2 < \frac{2p-1}{2p^2} \tag{C.8}$$

Then we are looking for a value of $\sigma_2 \in (0,1)$ such that Condition (6) and Condition (8) hold simultaneously. Recall that $\frac{dg_i(\sigma_{-i})}{d\sigma_k} = \frac{dg_j(\sigma_{-j})}{d\sigma_k} < 0$ for all $k < \min\{i,j\}$ and all $k > \max\{i,j\}$. Then setting up $\{\sigma_i^b(\sigma_2)\}_{i=2}^N$ as implicit functions in a system of $N-2$ equations of the form $\{g_{i-1}(\sigma_{-(i-1)}) - g_i(\sigma_{-i}) = 0\}_{i=3}^N$, and differentiating with respect to σ_2 yields $\frac{df^N(\sigma_2)}{d\sigma_2} < 0$ for all $\sigma_2 \in (0,1)$ (following the same steps we followed for showing that $h^N(\sigma_1)$ is decreasing).

Furthermore, we have $f^3(0) = 1$ and $\sigma_3^b(0) = 0$. By induction, suppose for some i that $\sigma_j^b(0) = 0$ for all $j \in \{2,...,i-1\}$. Then we have $1 - p\sigma_i^b(0) = \frac{2}{2} = 1$ which means $\sigma_i^b(0) = 0$. The base case is shown to hold for $i = 3$. Thus, we can conclude that $f^N(0) = \prod_{j=3}^N (1 - p\sigma_j^b(0)) = 1$.

Since $f^N(0) = 1$ and $f^N(\sigma_2)$ is strictly decreasing in σ_2, Condition (6) and Condition (8) are satisfied by some σ_2 if and only if

$$f^N\left(\frac{2p-1}{2p^2}\right) < 2c \tag{C.9}$$

If this is the case, then there is a unique $\sigma_2^{N,b} \in (0, \frac{2p-1}{2p^2})$ that satisfies (6) and (8) and therefore $\sigma_1 = 1$ with $\{\sigma_i^b(\sigma_2^{N,b})\}_{i=2}^N$ is a mixed equilibrium. Now we show for which values of N and c Condition (9) is satisfied. The following observation will be useful:

Observation (c): $\sigma_i^a(1) = \sigma_i^b(\frac{2p-1}{2p^2})$ for all $i \geq 2$, where $\sigma_i^a(\sigma_1)$ is as defined by Equation (4).

To see this, first note that $\sigma_2^a(1) = \frac{2p-1}{2p^2} = \sigma_2^b(\frac{2p-1}{2p^2})$. By induction, suppose for some i we have $\sigma_j^a(1) = \sigma_j^b(\frac{2p-1}{2p^2})$ for all $j \in \{2,...,i\}$ and call this value $\hat{\sigma}_j$. Then:

$$1 - p\sigma_i^b(\frac{2p-1}{2p^2}) = \frac{2(1 - p\frac{2p-1}{2p^2})}{\prod_{j=2}^N (1 - p\hat{\sigma}_j) + \prod_{j=2}^N (1 - (1-p)\hat{\sigma}_j))}$$
$$= \frac{1}{p\left(\prod_{j=2}^N (1 - p\hat{\sigma}_j) + \prod_{j=2}^N (1 - (1-p)\hat{\sigma}_j))\right)} = 1 - p\sigma_j^a(1)$$

where the first and third equalities follow from the definitions of $\sigma_2^b(\sigma_2)$ and $\sigma_j^a(\sigma_1)$ respectively. The induction hypothesis is shown above to hold for $i = 3$. Therefore, we have $\sigma_i^a(1) = \sigma_i^b(\frac{2p-1}{2p^2})$ for all $i \geq 2$.

Using Observation (c), we can relate the values of c and N that satisfy Condition (1) (which is shown above to be equivalent to $h^N(1) \leq \frac{c}{p}$) to those that satisfy Condition (9) as follows.

$$h^N(1) = \prod_{i=2}^N (1 - p\sigma_i^a(1)) = \frac{\prod_{i=3}^N (1 - p\sigma_i^a(1))}{2p}$$
$$= \frac{\prod_{i=3}^N (1 - p\sigma_i^b(\frac{2p-1}{2p^2}))}{2p}$$

$$= \frac{f^N(\frac{2p-1}{2p^2})}{2p}$$

where the first equality is the definition of function h^N, the second equality follows from $h^2(1) = \frac{1}{2p}$ as shown above, the third equality is the statement of Observation (c) and the fourth equality is the definition of function f^N. This relation can be rearranged to:

$$2ph^N(1) = f^N(\frac{2p-1}{2p^2}))$$

which means

$$f^N\left(\frac{2p-1}{2p^2}\right) < 2c \Leftrightarrow h^N(1)2p < 2c \Leftrightarrow h^N(1) < \frac{c}{p}$$

This means that Condition (9) holds for exactly the same values of c and N as Condition (1), except for the case $h^N(1) = \frac{c}{p}$ where Condition (1) holds and Condition (9) is violated. Hence, when $c > c_N$, Condition (9) is satisfied and there exists an additional equilibrium $\sigma^{N,b}$ of the form stated in Lemma 2.2(b) where Player 1 strictly prefers to adopt and all later players are indifferent. Condition (9) is not satisfied for any N when $c < \underline{c}$.

This exhausts the candidate profiles for a mixed strategy equilibrium.

□

Proposition 2. *There exists a cost level $\underline{c} \in (1-p, 0.5)$ such that the following hold.*

1. *If $c \leq \underline{c}$, then pure equilibrium σ^P is the unique equilibrium of the game.*
2. *For every $c > \underline{c}$, there is a threshold number of players $\underline{N} > 2$ such that;*
   - For $N < \underline{N}$, pure equilibrium σ^P is the unique equilibrium of the game.
   - For $N \geq \underline{N}$, there exists a mixed equilibrium σ^N where $\sigma_1^N = 1$ and for all $i \in \{2,...,N\}$, we have $\sigma_i^N \in (0,1)$ and $\sigma_i^N < \sigma_{i-1}^N$.

Proof. Proposition 2.1. Follows immediately from Proposition 1 (the unique pure equilibrium is σ^P) and Lemma 2.1 (there are no mixed equilibria for $c \leq \underline{c}$).

For Proposition 2.2., take some $c > \underline{c}$ as given and call $\underline{N} := \min\{n \in \mathbb{N} : c \geq c_n\}$. Recall that c_n is strictly decreasing with respect to n by Lemma 2.2(c). Therefore we have that if $N < \underline{N}$, then $c < c_{|\tilde{N}|}$ for any subset $\tilde{N}$ of players. Since Lemma 2.2 states all mixed equilibria of the game and there is no subset $\tilde{N}$ that satisfies condition $c \geq c_{|\tilde{N}|}$, no mixed equilibria exist when $c > \underline{c}$ and $N < \underline{N}$.

If $c > \underline{c}$ and $N \geq \underline{N}$, then we have $c \geq c_N$, and equilibrium profile $\sigma^{\{1,\ldots,N\},b}$ from Lemma 2.2(b) corresponds to profile σ^N described in Proposition 2.2. $\square$