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Imperfect Financial Markets and Investment Dynamics

THOMAS STEINBERGER

BADIA FIESOLANA, SAN DOMENICO (FI)

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European University Institute
Badia Fiesolana
I-50016 San Domenico (FI)
Italy

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Thomas Steinberger[2]

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[2]European University Institute, Badia Fiesolana, Via dei Roccettini 9, 50016 San Domenico di Fiesole, Italy.
Webpage: http://www.iue.it/Personal/Researchers/Steinberger/
E-mail: thomas.steinberger@iue.it

Abstract

This paper studies optimal dynamic investment and financial policy of the firm, if the interest rate on the firm's debt depends on its capital structure. We characterize the optimal investment and financing decisions and show how the incentive to invest and the market value of the firm are affected by financial considerations. Conditions are derived under which *average* Q remains a sufficient statistic for investment. Our results imply that empirical work must simultaneously analyze investment and financial policy of the firm in order to determine whether financial market imperfections affect investment. We also derive an estimable investment equation that is valid even if financial markets are imperfect. Our empirical results suggest that investment is indeed affected by financial market imperfections.

Keywords: Investment, Capital structure, Tobin's Q

JEL classification: D92, E22

1 Introduction

The standard theory of investment assumes that firms operate in perfect capital markets and face convex costs of adjusting capital. Despite its popularity, this approach has faced considerable difficulties in empirical implementation. Hayashi (1982) shows that a strong set of additional assumptions is necessary to derive an estimable investment equation from the model. The main empirical problem is that *average* Q does not contain all the relevant information about investment, although under these additional assumptions it should be a sufficient statistic. Instead, coefficients on variables like current income, cash-flow or sales turn out to be estimated significantly different from zero when these variables are added to Q-regressions (see Hubbard (1998) for a recent survey on panel-data studies and Caballero (1999) for aggregate investment data).

In particular, there is a large literature sparked by the work of Fazzari, Hubbard & Petersen (1988), which uses firm-level data partitioned into subgroups according to some firm characteristic and finds that the size of the coefficient on the current cash-flow variable varies systematically among subgroups. Since the characteristics are chosen such as to select groups of firms facing different degrees of imperfections in financial markets, this finding is often interpreted as evidence for the importance of financial market imperfections for investment on the firm level. Examples of firm characteristics which have been used to partition the sample include: dividend/income ratios (Fazzari, Hubbard & Petersen (1988)), firm age and size (Gilchrist & Himmelberg (1995)), membership in a "keiretsu" (Hoshi, Kashyap & Scharfstein (1991)),.... Even after more than ten years however, there is still considerable debate within the profession about the conclusiveness of the empirical evidence brought forward in this regard (see Kaplan & Zingales (1997)). The main question obviously is whether the correlation between current income and current investment after controlling for *average* Q is actually caused by frictions in financial markets, or whether this is the result of some other departure from the assumptions of Q-theory. The question has been addressed by researchers in various ways. Some have based their estimation on the Euler equation rather than the investment-Q relationship (Whited (1992)), others have reconstructed *marginal* Q from different measures of investment opportunities rather than using *average* Q (Gilchrist & Himmelberg (1995)). Cooper & Ejarque (2001) have recently argued that strict concavity of the profit function of the firm generates the observed correlation. They present numerical simulations which reproduce the estimation results obtained by Gilchrist & Himmelberg (1995) in a model with perfect financial markets. Erickson & Whited (2000) on the other hand argue that biased estimates due to measurement error in *average* Q rather than financing frictions cause this empirical phenomenon.

In this paper, we tackle the question by explicitly introducing financial market imperfections into the Q-model and allowing for imperfect product markets. We characterize the optimal investment and financial policy of the firm and derive an estimable investment equation, which takes into account potential discrepancies between *average* and *marginal* Q. By proceeding in this structural way, we can gain some understanding of how the incentive to invest may differ from the market valuation of the installed capital and what other variables beyond *average* Q should be expected to help explaining investment. After all, the true incentive to invest, *marginal* Q, adjusts endogenously to the existence of financial and product market imperfections and therefore also carries information about the financing options and the market power of the firm. Some of this information will also be reflected in *average* Q. We believe that this approach successfully addresses some of the concerns about the methodology used by Fazzari, Hubbard & Petersen (1988). First, by using a dynamic model, we are taking into account the forward-looking nature of investment decisions and market valuation. Further, our structural equation is valid for all firms and is not subject to the critique that sample splits are to a large degree arbitrary. Finally, allowing for both, imperfect product markets and imperfect financial markets, we address the concerns of Cooper & Ejarque (2001), who find that imperfect product markets alone can explain the observed correlation between investment and current income.

Previously, Hayashi (1985) has provided an analysis of optimal investment with adjustment costs and imperfect financial markets, in which the cost of debt finance is increasing in the debt-capital ratio. In his model, different tax rates on dividends and capital gains also drive a wedge between the cost of retained earnings and the cost of new equity issue, and he distinguishes three different financing regimes of investment. He shows that under a specific assumption on bankruptcy costs, Q-theory holds in two of the three regimes. However, he does not provide a general specification for an investment equation. The model we present below is simple enough to allow us to derive an estimable investment equation taking into account financial market and product market imperfections. The empirical evidence we find shows that financial market imperfections do have a significant effect on firm investment, in particular for small firms. We find no evidence of a strictly concave operating income function. In the following section we present the model, characterize analytically the optimal policy and provide some numerical examples. In section 3 we derive the model's implications for empirical research and provide empirical evidence for the importance of financial market imperfections for the explanation of firm investment. We conclude by discussing possible limitations of the analysis and by indicating questions to be answered by future research.

2 Introducing financial variables into the Q-model

Perfect financial markets imply that there is a single interest rate in the economy which is applied to all financial transactions. For the individual firm this means that the three principal modes of financing, i.e. debt finance, retained earnings, and equity issues, have the same opportunity cost. In such a setting, the Modigliani-Miller theorem (Modigliani & Miller (1958)) holds and financial policy is irrelevant for the value of the firm[1]. The path of optimal investment is determined without regard of financing decisions. In particular, the path of dividend payments, and the capital structure are indeterminate and do not affect the evolution of the capital stock. In a world where financial markets do not work frictionlessly however, the opportunity costs of the three modes of financing may be quite different. In their seminal article from 1958, Modigliani and Miller state that

> „...Economic theory and market experience both suggest that the yields demanded by lenders tend to increase with the debt/equity ratio of the borrowing firm (or individual)...."

Our way of introducing financial market imperfections is to follow the suggestion of Modigliani and Miller and allow the interest rate on the firm's debt to depend on the capital structure of the firm. In the microeconomic literature one can find many models of fundamental contractual problems in financial markets, but most of them are static and do not match well with the explicitly dynamic analysis we want to provide here. For this reason and to keep the model simple we therefore do not dig deeper into the microeconomic foundations and take the dependence of the interest rate on the capital structure as given. It should be noted though that the most popular microeconomic approach to contractual problems in credit markets - the costly state verification model due to Townsend (1979) and Gale & Hellwig (1985) - also implies that the interest rate charged by the lender depends on the capital structure of the firm (project)[2].

The model is set in a dynamic deterministic framework in order to focus on the interaction between the optimal investment and financial policy and

[1] Differential tax treatment of interest payments, dividends and retained earnings may be a reason for the Modigliani-Miller theorem to break down. I am not considering tax issues here, since I am focussing on the imperfect financial markets hypothesis.

[2] Another more recent approach to contractual problems in credit markets is based on work by Hart & Moore (1994) and assumes imperfect enforceability of debt contracts. The equilibrium interest rate in this model is the safe interest rate r, but the size of the loan is restricted to be smaller than some multiple of the collateral supplied by the entrepreneur. In the appendix we explore the (similar) implications of this alternative way to model financial market imperfections.

the endogenous adjustment of *marginal* and *average* Q. A companion paper (Steinberger (2001)) studies the stochastic version of the problem. It can be written as

$$\max_{\{I(t),X(t)\}_0^\infty} \int_0^\infty \Gamma(t)\left(\Pi[.] - \Psi[.] - \rho[.]\, B(t) + X(t)\right) dt \tag{1}$$

subject to

$$\begin{aligned} \dot{K}(t) &= I(t) - \delta . K(t) \\ \dot{B}(t) &= X(t) \end{aligned}$$

where $\Gamma(t) = \int_{\tau=0}^{t} \exp(-\gamma(\tau))\, d\tau$.

In the appendix this formulation is derived from a model in which the objective of the firm is to maximize the value of the firm's capital at time 0. This value is defined as the sum of $B(0)$, the market value of debt held by the firm at time 0, and the market valuation of the firm's equity at time 0. The capital accumulation equation is the usual one, with δ denoting the rate at which capital goods are assumed to decay. The debt accumulation equation simply defines $X(t)$ as the rate of change of debt at time t, which implies that there are no costs of adjusting the capital structure. We allow for an infinite rate of change in the state variables, so that at the initial point in time the firm could discretely adjust the stock of debt[3]. The main differences to the standard model are the appearance of another state variable and the statement of the objective function of the firm. The operating income function $\Pi[K(t)]$ captures any profits the firm generates from normal operations. It depends on the size of capital stock $K(t)$ only and is assumed to be twice continuously differentiable and concave in $K(t)$, $\Pi_K[.] > 0$, $\Pi_{KK}[.] \leq 0$[4]. Operating income is zero, if the firm does not have any capital: $\Pi[0] = 0$. The standard adjustment cost function $\Psi[I(t), K(t)]$ is assumed to be twice continuously differentiable with respect to both arguments, strictly convex in $I(t)$, $\Psi_I[.] > 0$, $\Psi_{II}[.] > 0$ and $K(t)$,.$\Psi_K[.] < 0$, $\Psi_{KK}[.] \geq 0$. If investment is zero, adjustment costs are zero as well: $\Psi[0, K(t)] = 0$.

The next two terms represent the non-standard part of the model. The cost of debt finance is captured by the term $\rho[r(t), B(t), K(t)]\, B(t)$, which

[3] Arrow & Kurz (1970) show that in a deterministic, concave, infinite horizon problem with continuous dynamics of the exogenous variables discrete adjustments (jumps) in state variables only accour in the initial period.

[4] A well-defined and concave operating income function follows for example from assuming a Cobb-Douglas technology $Q(t) = \left(K(t)^\alpha (A(t)\, L(t))^{1-\alpha}\right)^\varphi$, with $0 < \alpha \leq 1$, $\varphi > 0$ and a constant elasticity demand function $p(t) = D(t)\, Q(t)^{\mu - 1}$, with $0 < \mu\sigma < 1$ and the following specification of the firm's static profit maximization problem $\Pi[K(t)] = \max_{L(t)} p(t)\, Q(t) - w(t)\, L(t)$.

represents the interest payments on the stock of debt $B(t)$. The interest rate on the firm's debt $\rho[.] > 0$ depends on the „riskless" interest rate $r(t)$ and the size of debt and capital of the firm. As mentioned before, such an interest rate function can be rationalized by the existence of informational asymmetries between the firm and the lender. Since very low levels of debt are essentially riskless for the lenders, it is assumed that $\rho[r(t), K(t), 0] = r(t)$. Further, the interest rate depends per assumption continuously on its arguments, with continuous second order partial derivatives and $\rho_B[.] \geq 0$, $\rho_K[.] \leq 0$ and $\rho_{KK}[.] \geq 0$. To see that these assumptions are consistent with the view of Modigliani and Miller, we have to recall the balance sheet of the firm, which states that the book value[5] of assets $K(t)$ equals the book value of debt $B(t)$ plus the book value of equity $E(t)$, $K(t) = B(t) + E(t)$. An increase in the debt-equity-ratio clearly implies an increase in the debt-capital-ratio and vice versa. Keeping capital fixed, the interest rate increases with an increase in debt, which explains the sign of the first partial derivative. Keeping debt fixed, the interest rate decreases with an increase in the amount of capital, determining the sign of the second partial derivative. Rather than imposing homogeneity on the interest rate function from the beginning (as suggested by Modigliani and Miller), we allow the interest rate to depend in a more general way on its arguments.

Net debt repayments are denoted by $X(t)$. It is important to mention that in the model, $\gamma(t) > r(t)$, i.e. the required rate of return on the firm's equity, $\gamma(t)$, is assumed to be bigger than the „riskless" rate, $r(t)$. In fact, the difference between these rates of return is an essential element of the analysis. If $r(t) = \gamma(t) = \rho[.]$ the model collapses to the standard perfect capital markets Q-model. We believe that a good argument for this assumption can be made by appealing to the well-known „equity premium puzzle", which states that the spread between the rate of return on equity and the rate of return on riskless bonds is much higher than would be predicted by a standard representative agent capital asset pricing model. From here the analysis will proceed in two steps. First we characterize the solution of the model and present some numerical examples of optimal policies. Second, we will derive the implications of our model for the specification of investment equations and clarify the relationship between *marginal* and *average* Q in the presence of financial market imperfections.

[5] We are fixing the price of capital at 1, such that $K(t)$ equals the book value of capital and assume that the book value of debt is equal to its market value.

2.1 The optimal investment and financial policy

The necessary and sufficient FOC of the problem are given by:

$$\gamma(t) - \rho[r(t), K(t), B(t)] = \rho_B[r(t), K(t), B(t)]\, B(t) \tag{2}$$

$$q(t) = \Psi_I[I(t), K(t)] \tag{3}$$

$$\dot{q}(t) = (\gamma(t) + \delta)\, q(t) - (O_K[.] - \Psi_K[.] - \rho_K[.]\, B(t)) \tag{4}$$

the transversality conditions

$$\begin{aligned} \lim_{t\to\infty} q(t) K(t) \Gamma(t) &= 0 \\ \lim_{t\to\infty} B(t) \Gamma(t) &= 0 \end{aligned} \tag{5}$$

and the two dynamic constraints of the problem.

The state-space of the model is given by debt, capital and *marginal* Q, $\{B, K, q\}$. If the operating income function is strictly concave and the effect of the capital stock on adjustment costs and the interest rate is not too strong, a steady-state exists at which $q^* = 1$, and K^* and B^* are finite. If these conditions do not hold, it is still possible to determine the optimal investment rates and the optimal financial policy of the firm, but $q(t)$ does not converge to 1 and the capital stock and the debt level go to infinity as t goes to infinity. Importantly $B(t)$, the firm's stock of debt, is a jump variable. This is a result of assuming no adjustment costs of debt and frictionless access to equity finance. This reduces the dimension of the state space and equation (2) to a static relationship. The optimal amount of leverage is found at the point at which the marginal benefit of more debt finance $\gamma(t) - \rho[r(t), K(t), B(t)]$ equals its marginal cost $\rho_B[r(t), K(t), B(t)]\, B(t)$. Since $\rho[r(t), K(t), 0] = r(t)$ and $\gamma(t) > r(t)$, the optimal amount of debt is positive at any time. We can use the implicit function theorem to see how $B(t)$ changes in response to changes in $K(t)$. We obtain the following expression for the derivative of $B(t)$ with respect to $K(t)$:

$$\frac{\partial B(t)}{\partial K(t)} = -\frac{\rho_K[.] + \rho_{BK}[.]\, B(t)}{2\rho_B[.] + \rho_{BB}[.]\, B(t)} \tag{6}$$

By using this expression, we can prove the following proposition:

Proposition 1 *If the discount rate $\gamma(t)$ and the „riskless" rate $r(t)$ are constant over time, the debt-capital ratio is also constant if the interest rate function is homogeneous of degree zero in $B(t)$ and $K(t)$.*

Proof. Homogeneity of degree zero implies: $-\rho_K[.]\,K(t) = \rho_B[.]\,B(t)$. Taking the derivative of this expression with respect to $B(t)$ yields $-\rho_{KB}[.]\,K(t) = \rho_{BB}[.]\,B(t) + \rho_B[.]$. Multiplying both sides of the equation by $B(t)$ and adding the resulting expression to the initial one we obtain

$$-\rho_K[.]\,K(t) - \rho_{KB}[.]\,K(t)\,B(t) = 2\rho_B[.]\,B(t) + \rho_{BB}[.]\,B(t)^2$$

By comparing the above equation with (6) we can see that it implies $\frac{\partial B(t)}{\partial K(t)} = \frac{B(t)}{K(t)}$, which is equivalent to the definition of a constant debt-capital ratio $\frac{\partial B(t)}{B(t)} = \frac{\partial K(t)}{K(t)}$. ■

This first result essentially means that if the interest rate only depends on the debt-capital-ratio, optimal leverage does not depend on operating income, adjustment costs or the size of the firm, but only on the financial parameters $r(t)$, $\gamma(t)$, and $\rho[.]$. In general however, we expect optimal leverage to depend on fundamentals[6].

We see from (3) that the standard Q-investment relationship continues to hold. Investment is positive if *marginal* Q is bigger than 1 and negative if it is below 1. The pace of investment in the imperfect financial markets case is still governed by the nature of adjustment costs. But what determines the incentive to invest? It is in the determination of *marginal* Q, where the interaction between optimal financial and investment policy is most important. Integrating the dynamic equation for *marginal* Q from 0 to ∞ assuming constant γ, we obtain:

$$q(0) = \int_0^\infty \exp\left[-(\gamma+\delta)\,t\right]\left(O_K[.] - \Psi_K[.] - \rho_K[.]\,B(t)\right)dt \qquad (7)$$

This equation shows that the incentive to invest is unambiguously increased by the possibility of using debt. This follows from the assumptions for $\rho_K[.]$ and the result above, which found that optimal debt is always positive. The financial factor increasing the incentive to invest is the fact that holding the amount of debt constant, installing more capital decreases the cost of debt finance. The strength of this effect is captured by $\rho_K[.]$ and it is stronger the more debt the firm uses. At the same time, the optimal amount of debt depends on the size of the capital stock. Both quantities therefore endogenously adjust to each other and their optimal paths must be determined jointly.

It should also be pointed out that there is another interesting innovation in the expression for *marginal* Q, which establishes a potentially important

[6] It would be a straightforward extension to let the interest rate on the firm's debt also depend explicitly on time. Then optimal leverage would also change with the age of the firm.

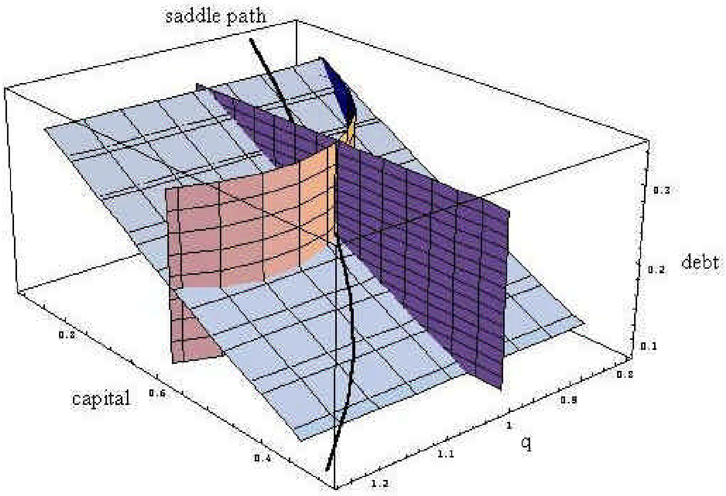

Figure 1: A numerical example of the saddlepath

channel between the financial sector and the real sector of the economy. This is the fact that any change in the exogenous factor $r(t)$ or the parameters of the relationship between debt, the capital stock and the interest rate affects $\rho_K[.]$ and therefore also directly the incentive to invest. The desired capital stock will adjust accordingly and there will be a significant effect on future investment rates. It is quite plausible, that this effect is one of the channels through which monetary policy and developments in the financial sector affect the real sector of the economy. Rather than following up on this remark, we will present in the section some numerical examples which illustrate the way financial policy and the incentive to invest affect each other.

2.2 A numerical analysis

Figure 1 shows a numerical example of a solution path in the 3-dimensional state space $\{B, K, q\}$ for a parameter specification for which the model has a determinate steady-state and saddle-path-stable dynamics.

In the picture, three two-dimensional surfaces are drawn which intersect at the steady state. These correspond to the $\dot{q} = 0$-surface, the (vertically drawn) $\dot{K} = 0$-plane, and the surface representing all optimal combinations of $K(t)$ and $B(t)$. The saddle path lies on the latter surface and selects among all possible solution paths the unique solution path which does not violate the transversality condition. It is clear that the optimal paths are

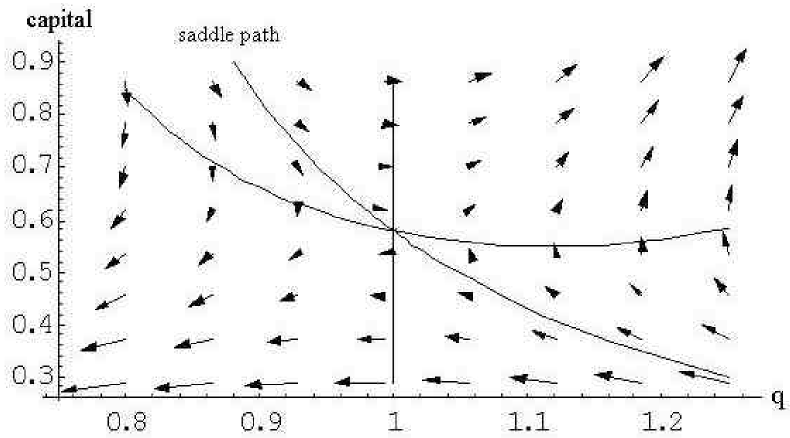

Figure 2: A numerical example of the vectorfield

restricted to lie on the two-dimensional surface representing the static relationship between $K(t)$ and $B(t)$. Hence, a two-dimensional picture is sufficient to illustrate the dynamics of the system and this picture can be obtained by projecting the solution onto the $K-q$-space. Such a projection is displayed below in Figure 2. Again, we see the saddle path solution of the system and the arrows indicate the direction of movement of points away from the saddle path.

It is also instructive to show that imperfections in financial markets have considerable effects on investment. Below, there is a plot of the debt/capital ratio paths of 3 firms characterized by the same production and adjustment technology, but facing different interest rate functions. We also simulate the optimal investment policy of a firm operating in perfect capital markets. For this firm optimal financial policy is of course indeterminate. In our simulations all firms face the same cost of debt finance, $r = 0.04$, at a debt/capital ratio equal to 0. Also the initial capital stock, the operating income function and the adjustment cost function are equal in all 4 simulations. In particular, we used a simple power specification for the operating income function $\Pi[.] = 0.5K(t)^{0.5}$ and a standard quadratic adjustment cost function, $\Psi[.] = 0.5\frac{I(t)^2}{K(t)} + (1-0.1)I(t)$. Any difference in investment behavior must therefore originate from the financing aspect of the problem. The rate of return on equity is chosen to be $\gamma = 0.09$ in the imperfect financial market cases. The interest rate function is represented by a power function of the form $\rho[.] = r + 0.04B(t)^{\mu} K(t)^{-\beta}$ as well.

A first result is that there is quite a large difference between the perfect

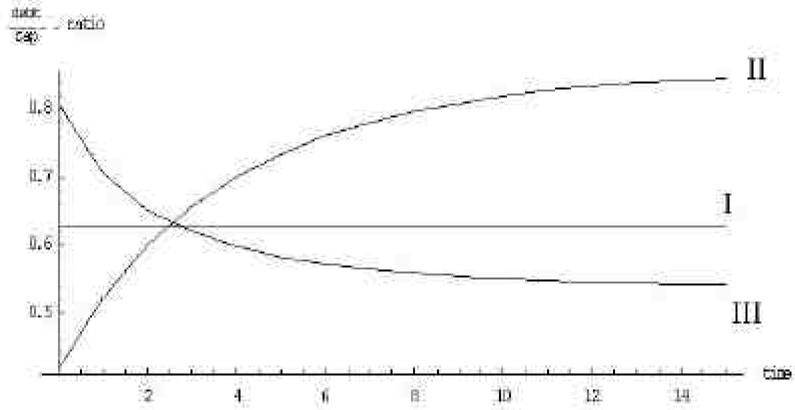

Figure 3: The dynamics of the debt-capital-ratio

financial markets case and the imperfect financial markets cases. The steady-state level of the capital stock in the former is 3.429, while in the latter case, the steady-state capital stocks lie in range from 2.088 to 2.427. Hence, investment is considerably lower if financial markets are imperfect. The structure of the financial market imperfection itself does not lead to such large differences among the firms. This is true, although the financial policies associated with each of these investment paths vary widely. The steady-state debt-capital ratios range from 0.539 to 0.858. Also, the paths of the debt/capital-ratio maybe increasing, decreasing or flat depending on the relative effects of the stocks of debt and capital on the interest rate. The picture below shows the results for the debt-capital ratio.

The steady-state capital stock is increasing in the debt-capital ratio in our simulations, because the interest rate increases at different speeds with debt. Firms adjust to a quickly increasing interest rate by choosing both a lower debt-capital ratio and a lower capital stock. In all three cases, we set $\beta = -1$, and the speed at which the interest arte increases is therefore governed by μ. Also the dynamics of the debt/capital ratio are governed by the shape of the interest rate function. Firm I ($\mu = 1$) chooses a constant debt/capital ratio because of the homogeneity of the interest rate function, its steady-state capital stock is 2.178 and its steady state debt-capital ratio is 0.625. Firm II ($\mu = 0.75$) chooses an increasing debt-capital ratio because for a constant debt/capital ratio, the interest rate decreases with capital. Its steady-state debt-capital ratio is 0.858. The lower financing costs make it optimal for the firm to invest at a higher rate and achieve a steady-state capital stock of 2.427, about 11% bigger than Firm I. Firm III ($\mu = 1.25$) instead chooses a decreasing path because for a constant debt-capital ratio, its interest rate increases with capital and selects a steady-state debt-capital ratio of 0.539. The firm therefore has a lower incentive to invest and its steady-state capital stock of 2.088 is about 14% below the capital stock of Firm II . After having illustrated how imperfections in financial markets affect the incentive to invest, we show in the next section how the existence of these imperfections affects the most widely used measure of this incentive, the market valuation of the firm's capital.

2.3 The relationship between marginal and average Q

Hayashi (1982) shows that if the profit function and the adjustment cost function are homogeneous of degree 1, then the incentive to invest, *marginal* Q, equals the market valuation of the firm's capital, *average* Q. This result is the basis of most empirical work on investment, but it is unclear whether this result still holds in the presence of financial market imperfections. Hayashi (1985) argues that the result continues to hold, since bankruptcy costs are likely to be constant per unit of capital. We have taken a different approach of modelling financial market imperfections, focusing instead on differences in the opportunity costs of different methods of financing. We have shown above that even with imperfect financial markets, the incentive to invest is still captured by *marginal* Q. In fact, equation (3) shows that *marginal* Q is still a sufficient statistic for investment. We would now like to know to what extent the true incentive to invest is reflected in *average* Q.

Average Q, $Q(0)$, is defined as the sum of market capitalization $V(0)$ and total debt $B(0)$ divided by the book value of capital $K(0)$

$$Q(0) = \frac{V(0) + B(0)}{K(0)}$$

Unlike *marginal* Q, *average* Q is an observable quantity that can be used as an explanatory variable in empirical research. It is shown in the appendix that in the presence of financial market imperfections Hayashi's assumptions of constant returns to scale of the operating income and adjustment cost functions do not imply that *marginal* Q equals *average* Q. Instead, *marginal* Q equals:

$$q(0) = \frac{V(0) + B(0)}{K(0)} + PDV \tag{8}$$

where

$$PDV = -\frac{1}{K(0)} \int_0^{\infty} \exp\left[-\gamma(t)\right] \left\{ \left(\frac{\rho_B[.]\,B(t)}{\rho[.]} + \frac{\rho_K[.]\,K(t)}{\rho[.]} \right) \rho[.]\,B(t) \right\} dt \tag{9}$$

Expression (9) shows that *average Q* needs to be adjusted by a part of the market value derived from the use of debt in the future to arrive at the incentive to invest. When we are willing to assume that the interest rate function $\rho[.]$ is of the constant elasticity form, then we can use (9) to determine the sign of the correction term and we see that correcting for financial market imperfections will decrease the value of *average Q*, if and only if

$$-\frac{\rho_K[.]\,K(t)}{\rho[.]} < \frac{\rho_B[.]\,B(t)}{\rho[.]}$$

or equivalently that the negative of the elasticity of the interest-rate function with respect to capital is lower than the elasticity with respect to debt. In fact, we can prove the following proposition:

Proposition 2 *Firms operating in imperfect financial markets with profit and adjustment cost functions which are homogeneous of degree 1 exhibit marginal Q smaller than average Q, if the elasticity of the interest-rate with respect to capital is lower in absolute value than the elasticity with respect to debt for all values of $B(t)$ and $K(t)$.*

Proof. The sign of PDV is determined by an integral over sums of future elasticities of the interest rate function. If $-\frac{\rho_K[.]K(t)}{\rho[.]} < \frac{\rho_B[.]B(t)}{\rho[.]}$ holds for all $B(t)$ and $K(t)$, then all elements of the integral are positive and the integral itself is positive as well. Positivity of the integral implies that *marginal Q* is smaller than *average Q*. ■

The intuition for this result is that as the firm accumulates capital, it will also increase the stock of its debt. If the interest rate reacts more strongly to the increase in debt than to the increase in the capital stock, the average financing cost of capital will increase if investment is financed in the same way as the existing capital stock. The financial value of the average unit of capital will therefore decrease over time. It is this change in the average financing cost of capital which decreases the incentive to invest relative to the market value of the average unit of capital. If the interest rate only depends on the debt/capital-ratio, optimal leverage is constant and no correction is necessary:

Proposition 3 *If the interest rate function $\rho[r(t), K(t), B(t)]$ is homogeneous of degree zero in $K(t)$ and $B(t)$, then $PDV = 0$ and if also the profit function and the adjustment cost function are homogeneous of degree 1, average Q is equivalent to marginal Q.*

Proof. In the appendix. ■

In the appendix we show that a similar result holds for a model in which financial market imperfections are modeled by collateral constraints. In the usual case of a linear collateral constraint of the form $B(t) \leq \alpha K(t)$, *marginal* Q still equals *average* Q under Hayashi's homogeneity assumptions. If the collateral constraint is non-linear however, an adjustment of *average* Q is necessary to capture the true incentive to invest. The next section will use the result obtained in this section to derive an estimable investment equation, which explicitly takes into account the possibility that *marginal* Q is different from *average* Q. It is shown that, under some specific assumptions, we can identify the source of the wedge between the incentive to invest and the market valuation of capital by estimating a structural equation for the investment rate.

3 Implications for empirical work

The results above imply that without some guidance from a structural model it is difficult to detect empirically whether financial market imperfections are important determinants of investment. In particular, the procedure used by Fazzari, Hubbard & Petersen (1988) is not well-suited to test for the importance of financial market imperfections. It is obvious that by adding a current cash-flow or current operating income variable to the Q-equation one cannot test for financial market imperfections, because the assumptions of Q-theory may not be satisfied, but we want to stress another point. More interestingly, even if financial market imperfections were important determinants of investment at the firm level, the empirical researcher who simply adds current cash-flow or current operating income to the regression might wrongly conclude that they do not matter. This is because, as we have shown above, the market value of equity not only carries information about the operating profitability of the capital stock, but also about the cost of financing the capital. The market value therefore endogenously adjusts to the financing frictions that some firms may face. In some cases, e.g. when collateral constraints are linear or when the interest rate depends only on the debt-capital-ratio, the adjustment is such that it fully captures the effects of financing frictions on investment. One then observes that the standard *average* Q-investment relationship continues to hold, but as we showed above, this does not imply that financial factors do not matter for investment. In order to make sure that this is the case one also needs to study the relationship between the financial and investment choices of the firms.

What would be the case, if *average* Q turned out not to be a sufficient statistic of investment? Then the theoretical argument on which Q-regressions are based is no longer valid and we need to include additional explanatory variables to the equation. Building a structural model provides some guidance on what variables should be included. Our model does not suggest that current cash-flow or operating income should be one of them. The reason for this is that with frictionsless access to equity finance the financing cost structure of the firm does not change with additional profits. The cost of equity finance is fixed at γ and additional operating income is equivalent to paying lower dividends or raising more equity finance. We therefore expect current operating income to be driven out of the investment equation when the relevant explanatory variables are added. We now show which variables should be expected to contribute to explaining investment rates, if *marginal* Q is different from *average* Q.

In order to do this however, we must make some assumptions on functional forms. We will from now on assume that the adjustment cost function takes a simple quadratic form, homogeneous of degree 1 in its arguments

$$\Psi\left[I\left(t\right),K\left(t\right)\right]=I\left(t\right)\left(1-\delta\theta\right)+\frac{\theta}{2}\frac{I\left(t\right)^{2}}{K\left(t\right)}$$

with $\theta > 0$ and $\delta\theta < 1$. The first-order condition for investment is linear in this case and can be written in discrete time as

$$\left(\frac{I}{K}\right)_{it}=\left(\frac{\delta\theta-1}{\theta}\right)+\frac{1}{\theta}q_{it} \tag{10}$$

A similar equation would hold, if the adjustment cost function took a different form and we used a linear approximation around the steady-state. We would like to allow the operating income function to be strictly concave and will only impose that it is a simple power function, possibly non-homogeneous ($0 < \alpha \leq 1$)

$$O[K\left(t\right),Z(t)]=AK(t)^{\alpha}$$

Further, we assume that the interest rate function can be approximated by an affine power function of the form.

$$\rho\left[K\left(t\right),B\left(t\right)\right]=r+\eta\left(B\left(t\right)^{\beta+\lambda}K\left(t\right)^{-\beta}\right)$$

where $\eta > 0$.

If $\alpha = 1$ and $\lambda = 0$, the model behaves like the standard Q-model. The homogeneity assumptions are satisfied and *average* Q is a sufficient statistic for investment. The debt-capital ratio is constant. A departure of any of the two parameters from this value will break the relationship and while *marginal* Q still remains a sufficient statistic, *average* Q must be adjusted in order to arrive at the true incentive to invest. It is shown in the appendix that the relation between *marginal* and *average* Q in our case takes the following form

$$q(0) = \frac{V(0) + B(0)}{K(0)} + PDV_{\alpha} + PDV_{\lambda} \tag{11}$$

where PDV_{α} is the correction due to the fact that $\alpha < 1$ or that the operating income function is strictly concave. Similarly, PDV_{λ} reflects the correction term due to the non-homogeneity of the interest rate function. Analytical expressions can be derived for each of these corrections necessary. They take the following form

$$\begin{aligned} PDV_{\alpha} &= -\frac{1}{K(0)} \int_0^{\infty} \exp[-\gamma t] \{(1-\alpha) AK(t)^{\alpha}\} dt \\ PDV_{\lambda} &= -\frac{1}{K(0)} \int_0^{\infty} \exp[-\gamma t] \left\{ \lambda \eta \left(\frac{B(t)}{K(t)} \right)^{\beta} B(t)^{\lambda} \right\} dt \end{aligned}$$

The sign of these correction terms only depend on α and λ. $PDV_{\alpha} < 0$, if $0 < \alpha < 1$ because a strictly concave profit function implies that the marginal unit of capital earns less than the previously installed units. Hence, *average* Q needs to be corrected downwards in order to arrive at the true incentive to invest. $PDV_{\lambda} < 0$, if $\lambda > 0$ and $PDV_{\lambda} > 0$, if $\lambda < 0$. The argument here rests on the fact that with $\lambda > 0$ financing large capital stocks is costlier than financing small capital stocks for any given debt-capital ratio. The financial gain from financing capital with debt is largest for the first units of capital installed. Hence, *average* Q must be corrected downward to arrive at the true incentive to invest. The inverse argument holds, if $\lambda < 0$. In this case, the financial advantage increases with the number of capital goods installed and the marginal unit of capital increases the profitability of all previously installed units. *Average* Q must be adjusted upward.

All of these corrections involve future variables and are therefore unobservable like *marginal* Q. In order to use this informatin in empirical work, it is necessary to approximate these expressions. One way of doing this is to evaluate the integral at the steady state and take a first-order Taylor-expansion of the resulting expression around the steady state. The expressions for the correction term then are linear functions of the current variables with the sign of the coefficients determined by the parameters of the relevant functions.

$$\begin{aligned} P\tilde{D}V_{\alpha} &= \alpha_1 + \alpha_2 K_{it} \\ P\tilde{D}V_{\lambda} &= \lambda_1 + \lambda_2 \left(\frac{B}{K}\right)_{it} + \lambda_3 B_{it} \end{aligned}$$

These approximations can be used for substitution into (10) which then takes the form

$$\left(\frac{I}{K}\right)_{it} = \left(\frac{\delta\theta - 1 + \alpha_1 + \lambda_1}{\theta}\right) + \frac{1}{\theta}Q_{it} + \frac{\alpha_2}{\theta}K_{it} + \frac{\lambda_2}{\theta}\left(\frac{B}{K}\right)_{it} + \frac{\lambda_3}{\theta}B_{it} + \varepsilon_{it} \quad (12)$$

where $\varepsilon_{it} = \tau_i + \upsilon_{it}$ is an error term with a random firm-specific and time-invariant component τ_i and an i.i.d. component $\upsilon_{it} \sim N(0, \sigma)$. We assume that the τ_i, the υ_{it} and the regressors are independent of each other.

If the profit function was strictly concave, we should observe a positive coefficient on the scale variable K, since $\alpha_2 = 0$ if $\alpha = 1$ and $\alpha_2 > 0$ if $0 < \alpha < 1$. This result might be surprising, since the true incentive to invest is lower than *average* Q in the case of a concave profit function, but becomes clearer if one recognizes that the we use linear approximations here and while the total correction to *average* Q is negative for all values of K, the correction term becomes smaller in absolute size the bigger the installed capital stock is. The intuition for this is that a strictly concave profit function implies that the marginal unit of capital earns less than the previously installed units. Hence, *average* Q needs to be corrected downwards in order to arrive at the true incentive to invest. This downward adjustment is smaller the bigger the capital stock is, because the discrepancy between average and marginal profitability is decreasing in the capital stock for the simple power function we have assumed[7]. In this case, the difference between *average* and *marginal* Q is given by $(1-\alpha)AK(t)^{\alpha-1}$, which is positive, but decreasing in K, for $0 < \alpha < 1$.

Similarly, if the interest rate function was non-homogeneous, we should observe a significant coefficient for the debt-capital ratio $\left(\frac{B}{K}\right)$ and the debt variable B. The financing cost can be written as $\gamma(K - B) + \rho[B, K]B$. After substituting from (2), we find that the derivative of this expression with respect to K is $\left(\frac{\rho_B[.]B + \rho_K[.]K}{K}\right)\frac{B}{K}$. If $\lambda = 0$, this derivative is 0. In this case, the optimal debt-capital ratio is constant and newly installed units of capital financed in the same way as the existing capital stock generate the same financial profit as the already installed units. Hence, if $\lambda = 0$

[7] This is not a general property of increasing and strictly concave functions because over some range, strong curvature can lead to an increasing difference between the average and the marginal increase.

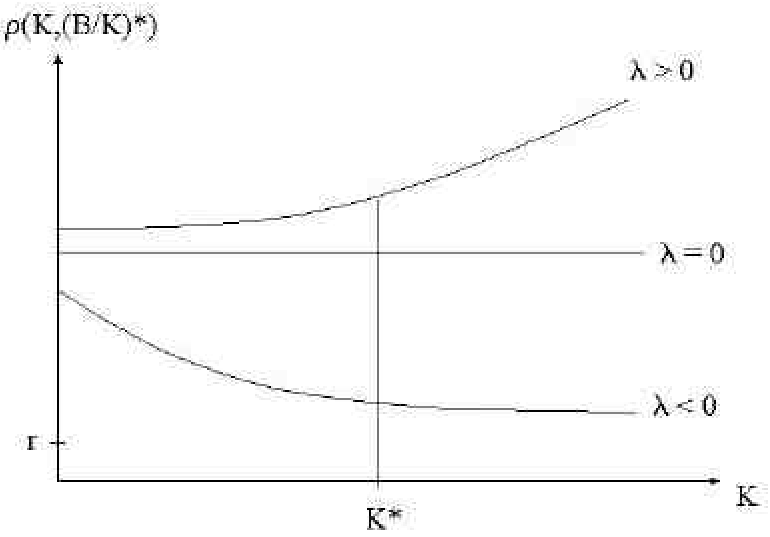

Figure 4: Financing costs and capital stock

then $\lambda_2 = \lambda_3 = 0$ and the effect is fully accounted for by *average* Q. If instead $\lambda > 0$ financing costs are increased for all of the existing capital stock, if the debt-capital ratio was held constant and therefore *average* Q must be adjusted downward. Why does the adjustment also depend on the debt-capital ratio? If the firm does not use leverage, there is no financial disadvantage of increasing the interest rate, so the incentive to invest remains unchanged. If instead the firm uses high leverage, the financial disadvantage is large and the incentive to invest is significantly lower Therefore we have $\lambda_2 < 0$ if $\lambda > 0$. The reverse argument holds, if $\lambda < 0$, in this case, $\lambda_2 > 0$, and *average* Q of course must be adjusted upward.

In addition to leverage, also the scale of debt affects the incentive to invest, if the interest rate function is non-homogeneous. The reason is that the optimal capital structure changes with the installation of new capital goods. The newly installed capital will be financed with a different capital structure. The coefficient on the debt level, λ_3, is positive only if $0 < \lambda < 1$. For the case of $\lambda < 0$, the reasoning here is analogous to the one for the concavity of the profit function. The total correction to *average* Q must be positive, but it is at the same time decreasing in the debt level because the per unit cost savings are declining with an increase in the level of debt. If $\lambda > 0$, we must distinguish three cases. $\lambda = 1$ means that there is no additional adjustment of the incentive to invest due to scale. The interest rate effect is proportional to the debt-capital ratio. In the case of $\lambda > 1$, the incentive to invest is decreased further by an increase in the scale of debt, since the cost of outside finance is increasing at a faster rate than is captured by the debt-capital ratio. In the last case $0 < \lambda < 1$ however, the correction of *average* Q through the debt-capital ratio overstates the interest rate effect on the incentive to invest and the correction through the scale variable is positive.

4 Empirical evidence

The data we use for empirical work is constructed from an 11-year balanced panel of balance sheet and income statement data for 87 listed Italian companies with a total of 870 observations[8] from the Worldscope financial database. We have standardized information on most items of the balance sheet and income statement, including gross capital expenditures, fixed capital, long and short-term debt, book value of equity, market capitalization,... The sum of long-term and short-term debt at the beginning of the period corresponds to our definition of the variable $B(t)$. We take capital expenditures to correspond to our $I(t)$-variable and the beginning of period net value of property, plant and equipment to correspond to $K(t)$. Unfortunately, for some firms and years information on capital expenditures or market capitalization is missing and we therefore exclude these observations (136 observations lost). Also, some of the companies contained in the dataset are holding companies with consolidated balance sheets only for some years[9]. We exclude the observations of such companies for the years in which their balance sheets are unconsolidated. Further, companies that have made large acquisitions within the period covered are excluded as well. At the end, we are left with an unbalanced sample of 649 observations on 78 companies with the number of observations ranging from 1 to 10.

As a first pass toward determining whether financial market imperfections are important determinants of investment, we run a Q-regression on the full dataset and two subsamples with the operating income rate added as an additional explanatory variable as is standard in the financial market imperfections literature. The size of the firm (beginning of period net value of property, plant and equipment) is chosen as the exogenous variable determining the sample split. The subsamples hence represent the big and small

[8] We lose one year of observation for each company because we are interested in beginning of period stocks and investment in a given period, but companies report end-of-period stocks simultaneously with investment in a given period.

[9] These companies are Ifil, Fin. Part. spa, Schiapparelli spa, Finmeccanica spa, Pirelli&C. sapa, Camfin spa, Simint spa, Parmalat Finanziaria spa, Bulgari spa, Mediaset spa, Istituto Finaziario Industriale spa, Cofide spa.

firms in the dataset. The cutoff used is of course to a large extent arbitrary and we have chosen a value of approx. 35 mil Euros for the capital stock. We have experimented with different thresholds and other cutoff criteria (number of employees, level of sales), but results were qualitatively similar to the ones we report. We include year and sector dummies in our econometric specification to capture general business cycle and sector-specific effects and use a standard random-effects model for estimation[10].

Table 1: Augmented Q-regression

Dependent variable: $(I/K)_{it}$

	Full Sample	Small Firms	Big Firms
Explanatory variables[1]			
Q_{it}	0.023* (0.003)	0.017* (0.005)	0.035* (0.006)
$(\Pi/K)_{it}$	0.048* (0.023)	0.079* (0.031)	-0.038 (0.036)
Constant	0.139* (0.062)	0.021 (0.190)	0.119 (0.078)
Wald[2]	88.39	39.75	58.2
Hausman[3]	25.55	8.98	13.6

[1] a set of time and industry dummies was included; k=17; we report coefficients with their standard errors in parentheses; stars denote coefficients which are significant at the 95%-level

[2] Wald is the statistic of a Wald-test on the joint significance of the coefficients distributed as a Chi-squared(k)

[3] Hausman is the test-statistic of the Hausman misspecification test distributed as a Chi-squared(k-6)

The results of the estimation shown in Table I confirm what is known from the literature on Q-regressions. *Average* Q and the operating income rate are both significant in the small firm sample and the overall sample, but insignificant in the sample of big firms. Further the coefficient on the operating income rate is biggest for the sample of small firms and decreases as the sample contains less small firms. This suggests that *average* Q is not a sufficient statistic for investment, at least for the subset of small firms.

We argued above that theoretically the significant coefficient on the operating income rate could be due to two separate effects: strict concavity of the operating income function or non-homogeneity of the interest rate function. Equation (12) captures the empirical implications of our theory.

[10] Hayashi & Inoue (1991) have pointed out that if the error term in the investment-Q regression originates from shocks to the profit or adjustment cost function with a specific correlation structure, then an instrumental variable GMM-estimation of a differenced version of the investment-Q relation is the appropriate estimation method. We are aware of these endogeneity issues, but take a simpler econometric approach, hoping to gain more insight into the structure of empirical data this way.

Adding the three additional explanatory variables, capital stock K_t, debt-capital-ratio $\left(\frac{B}{K}\right)_t$, and debt level B_t, we expect the operating income rate to become insignificant. A priori, we cannot say however, whether the contribution of the operating income function points to non-homogeneity of the operating income function or non-homogeneity of the interest rate function. The estimation results will shed some light on this since, according to our theoretical model, the sign of the coefficients of the additional explanatory variables K_t, $\left(\frac{B}{K}\right)_t$, and B_t are directly related to properties of the operating income and interest rate function. Table 2 presents the results of the estimation.

Table 2: Q-regression taking into account financial factors and non-homogeneity

Dependent variable: $(I/K)_{it}$

	Full Sample	Small Firms	Big Firms
Explanatory variables[1]			
Q_{it}	0.033* (0.004)	0.031* (0.007)	0.039* (0.007)
$(\Pi/K)_{it}$	0.022 (0.024)	0.046 (0.032)	-0.049 (0.036)
$(B/K)_{it}$	-0.046* (0.011)	-0.077* (0.024)	-0.029* (0.016)
K_{it}	-0.021 (0.055)	-17.77 (11.91)	-0.005 (0.060)
B_{it}	0.004 (0.069)	8.166 (5.637)	-0.041 (0.074)
Constant	0.177* (0.063)	0.130 (0.198)	0.157* (0.081)
Wald[2]	109.62	63.43	63.51
Hausman[3]	29.71	16.46	18.71

[1] a set of time and industry dummies was included; k=20; we report coefficients with their standard errors in parentheses; stars denote coefficients which are significant at the 95%-level

[2] Wald is the statistic of a Wald-test on the joint significance of the coefficients distributed as a Chi-squared(k)

[3] Hausman is the test-statistic of the Hausman misspecification test distributed as a Chi-squared(k-6)

In fact, as predicted by the theory, the scale of capital and debt and the debt-capital ratio jointly drive out the operating income rate as an explanatory variable in all three samples. The coefficient on the scale of capital is negative and insignificant, suggesting that the operating income function indeed is homogeneous of degree one. In both the small firm sample and the full sample, the coefficient on the debt-capital ratio is highly significant and negative, suggesting that financial market imperfections do matter for investment. Since the scale of debt does not have an additional significant effect on investment, we conclude that the impact of financial market imperfections on investment is fully captured by an adjustment through the debt-capital ratio. In Table 3 we report for comparison the estimation results for some alternative econometric specifications for our structural equation (12). All three estimation methods, random-effects estimation, fixed effects estimation and estimation in differences give similar results, demonstrating some robustness of the results.

Table 3: Alternative econometric specifications

Full sample results			
Dependent variable:	$(I/K)_{it}$	$(I/K)_{it}$	$D(I/K)_{it}$
Specification	Random effects	Fixed effects	Random effects
Explanatory variables[1]			
Q_{it}	0.035* (0.004)	0.050* (0.006)	-
DQ_{it}	-	-	0.055* (0.008)
$(B/K)_{it}$	-0.049* (0.011)	-0.062* (0.013)	-
$D(B/K)_{it}$	-	-	-0.062* (0.020)
K_{it}	-0.022 (0.056)	-0.143 (0.099)	-
DK_{it}	-	-	-0.308 (0.241)
B_{it}	0.005 (0.070)	-0.032 (0.111)	-
DB_{it}	-	-	-0.004 (0.182)
Constant	0.172* (0.064)	0.135* (0.029)	0.068 (0.051)
Wald[2]	108.76	-	68.19
Hausman[3]	18.55	-	n.a.[4]

[1] a set of time and industry dummies was included; k=19; we report coefficients with their standard errors in parentheses; stars denote coefficients which are significant at the 95%-level

[2] Wald is the statistic of a Wald-test on the joint significance of the coefficients distributed as a Chi-squared(k)

[3] Hausman is the test-statistic of the Hausman misspecification test distributed as a Chi-squared(k-6)

[4] the estimated variance of the individual effects is 0, invalidating the Hausman-test statistic

Apart from the implications for investment equations our theory also carries implications for the financial choices of the firms. In a perfect capital markets world, the financial policy of the firm is undetermined and we therefore expect that firm's choices are widely dispersed. In principle any debt-capital combination could be optimal in such a setting and a cross-plot of the logarithm of debt against the logarithm of capital should not have a particular structure. A priori we would expect a "cloud" of points. Our imperfect capital markets model instead predicts a particular shape for this plot. Given our power function assumption and the empirical results above, the model implies that the points are dispersed along a straight line with a positive slope that is smaller than 1 since for the case of $\lambda > 0$, the debt-capital ratio should be decreasing with the capital stock of the firm.

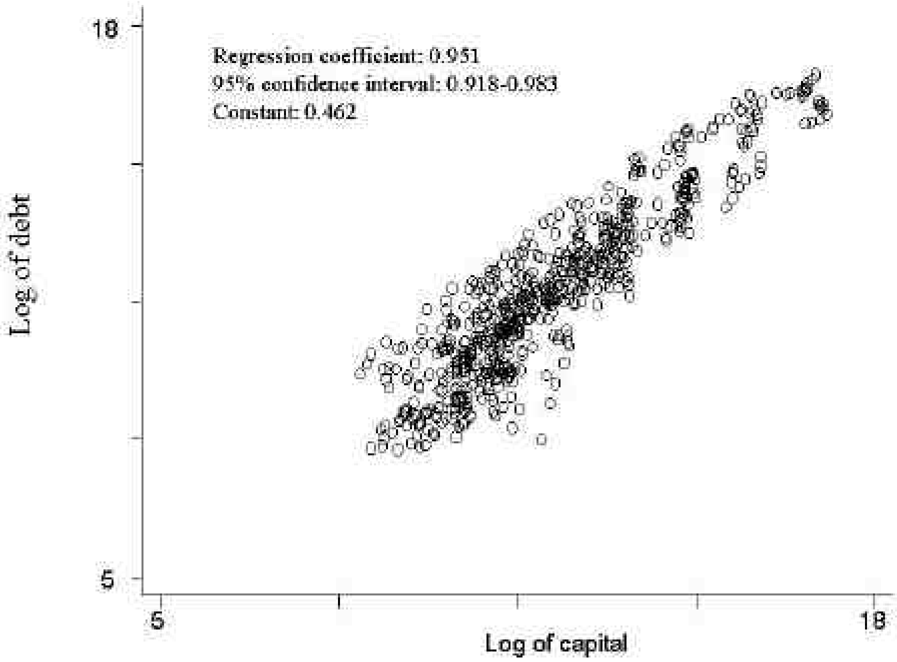

Figure 5: Debt and Capital across Firms

One can see, that the picture is in line with the predictions of the model and the empirical results from the investment regression. Firms with more capital do tend to have lower debt-capital ratios because the interest rate reacts more strongly to an increase in debt than to an increase in capital. A regression of the log of capital on the log of debt gives as a result that the slope of the regression line is significantly lower than 1, although only slightly so. Results for the partitioned samples give similar point estimates, but the slope coefficient is estimated less precisely, such that it is not significantly different from 1.

5 Conclusions

Firms operating in imperfect financial markets simultaneously choose both an optimal financial and an optimal investment policy. We have shown above that there is a direct relationship between the firm's capital structure, the firm's investment policy, and the extent of the financing friction. Firms for which access to financial markets is difficult, face interest rates that increase quickly with the amount of debt finance the firms use. Such firms will find it optimal to have relatively low levels of debt and capital and will mostly rely on equity to finance their investments. Firms with easy access to financial markets will tend to have higher capital stocks and use more leverage. The character of the financial market imperfection also has implications for the dynamics of the capital structure and the market evaluation of the firm. If the interest rate only depends on the debt/capital ratio, the optimal debt/capital ratio is constant. If also the operating income and adjustment cost functions are homogeneous of degree 1, then *average* Q equals *marginal* Q even if financial markets are imperfect. Non-homogeneity of any of these functions breaks the equivalence between *marginal* and *average* Q.

It is an important question for economic theory and policy whether the existence of financing frictions significantly affects the investment choices of firms. Our results imply that in order to determine whether financial market imperfections are important, one needs to analyze jointly the investment and financial decisions of the firms. Our empirical results suggest that in Italy imperfections in financial markets do affect firm investment. We find evidence that the debt/capital ratio decreases with the size of the firm suggesting that the elasticity of the interest rate is greater with respect to debt than with respect to capital. The non-homogeneity of the interest rate function also breaks the equivalence between *average* and *marginal* Q and seems to cause the correlation between investment and current operating income that is documented by previous empirical work. We do not find evidence that the adjustment cost function or the operating income function are non-homogeneous.

It would be interesting to find out in future research whether these results can be confirmed for other countries and other datasets. A possible limitation of our analysis is that we have not considered taxes and transaction costs in equity markets. These issues were considered by Hayashi (1985) and he finds that optimal policies are much more complex in this case. Another issue we have not considered is irreversibility of investment. If investment was to a large extent irreversible even at the firm level our results would no longer hold. Whether any of these issues are important for the study of firm investment must at this point be answered by future research.

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A Appendix

A.1 Deriving the objective function

The optimal investment problem of the firm is stated as maximizing the market value of the equity of the firm

$$V(0) = \max_{\{I(t),X(t)\}_0^\infty} p^S(0)\, N(0) \tag{13}$$

subject to

the capital and debt accumulation equations

$$\begin{aligned} dK(t) &= [I(t) - \delta.K(t)]\, dt & (14)\\ dB(t) &= X(t)\, dt & (15) \end{aligned}$$

the flow of funds constraint

$$D(t) = \Pi[.] - \rho[r(t), K(t), B(t)]\, B(t) - \Psi[.] + X(t) + p^S(t)\, \dot{N}(t) \tag{16}$$

and the pricing equation for the shares of the firm

$$\gamma(t) = \frac{\dot{p}^S(t)}{p^S(t)} + \frac{D(t)}{p^S(t).N(t)} \tag{17}$$

Here $D(t)$ is the dividend rate, $N(t)$ is the number of stocks issued at time t and $p^S(t)$ is the market price of a share at time t.

Taking the time derivative of $V(t) = p^S(t)\, N(t)$, we obtain

$$\dot{V}(t) = \dot{p}^S(t).N(t) + p^S(t).\dot{N}(t) \tag{18}$$

Multiplying (17) by $p^S(t).N(t)$ we obtain

$$\dot{p}^S(t)\, N(t) = \gamma(t)\, V(t) - D(t) \tag{19}$$

Substituting this expression into (18) and substituting from (16), we get

$$\dot{V}(t) = \gamma(t)\, V(t) - \Pi[.] + \Psi[.] + \rho[.]\, B(t) - X(t) \tag{20}$$

Solving this differential equation with random coefficients in $V(t)$, for starting time equal to 0 and horizon s to arrive at

$$\begin{aligned} V_0^s &= C.\exp\left[\int_0^s \gamma(\tau)\,d\tau\right] + \\ &\int_0^s \exp\left[\int_0^t \gamma(\tau)\,d\tau\right] (\Pi[.] - \Psi[.] - \rho[.]\,B(t) + X(t))\,dt \end{aligned}$$

so that maximizing $V(0)$ over an infinite horizon is equivalent to maximizing

$$= \max_{\{I(t),X(t)\}_0^\infty} \int_0^\infty \Gamma(t)\,(\Pi[.] - \Psi[.] - \rho[.]\,B(t) + X(t))\,dt$$

A.2 Hayashi's result for the baseline model

We start from the observation that the following equation holds along any optimal path:

$$\begin{aligned} &\frac{d}{dt}\{[q(t)\,K(t) - B(t)]\,\Gamma(t)\} \qquad (21) \\ = &\left[\dot{q}(t)\,K(t) + q(t)\,\dot{K}(t) - \dot{B}(t) - \gamma(t)\,q(t)\,K(t) + \gamma(t)\,B(t)\right]\Gamma(t) \end{aligned}$$

where $\Gamma(t) = \exp\left(-\int_0^t \gamma(\tau)\,d\tau\right)$. Then substituting from the dynamic equation for $q(t)$, (4), the capital accumulation equation and the optimality condition for $I(t)$, (3),and dropping time indices we obtain

$$= [-\Pi_K[.]\,K + \Psi_K[.]\,K + \rho_K[.]\,BK + \Psi_I[.]\,I - X + \gamma B]\,\Gamma(t) \qquad (22)$$

after cancelling offsetting terms. Now by applying the homogeneity assumptions $\Pi_K[.]\,K = \Pi[.]$ and $\Psi_K[.]\,K + \Psi_I[.]\,I = \Psi[.]$ and adding and subtracting $\rho[.]\,B(t)$ inside the brackets we can write

$$= -[\Pi[.] - \Psi[.] - \rho[.]\,B + X]\,\Gamma + [(\gamma - \rho[.])\,B + \rho_K[.]\,BK]\,\Gamma(t) \qquad (23)$$

which after integrating from 0 to ∞ and using the transversality conditions (5) yields

$$q(0)\,K(0) - B(0) = V(0) - PDV$$

which directly implies (8).

A.3 Collateral constraints

Another well known model of financing frictions due to Hart & Moore (1994) finds that the firm will face a collateral constraint of the form:

$$B(t) \leq \theta K(t) \tag{24}$$

This model is based on symmetric information, but the inability of creditors to punish the defaulting entrepreneur stronger than by taking away his wealth, makes lenders hesitant to lend more than some fraction of his current net worth. The interest rate charged to the firm will be the safe rate of interest $r(t)$. In the present case, the creditor is a firm, its wealth is the capital it owns and we assume that $\sigma, \theta > 0$. Using this approach, but allowing for non-linearity in the collateral constraint, the model reads:

$$\max_{\{I(t),X(t)\}_0^\infty} \left[\int_0^\infty \Gamma(t)\left(\Pi[.] - \Psi[.] - r(t)B(t) + X(t)\right)dt\right]$$

subject to

$$\begin{aligned}
\dot{K}(t) &= (I(t) - \delta.K(t))\,dt \\
\dot{B}(t) &= X(t)\,dt \\
B(t) &\leq \theta K(t)^\sigma
\end{aligned}$$

Facing such a constraint, the firm would still invest according to the Q-rule, but again the value of Q would be somewhat different from the perfect capital markets case. In fact, it can be shown that if the interest rate on debt is lower than the discount rate, the firm would always choose to operate at the collateral constraint, decreasing its cost of capital as much as possible. We then have

$$B(t) = \theta K(t)^\sigma$$

and the deterministic first-order optimality conditions are

$$\begin{aligned}
\dot{K}(t) &= (I(t) - \delta.K(t))\,dt \\
\dot{q}(t) &= (\gamma(t) + \delta)\,q(t) - \left(\Pi_K[.] - \Psi_K[.] + (\gamma(t) - r(t))\,\sigma\theta K(t)^{\sigma-1}\right)
\end{aligned}$$

and the transversality conditions are

$$\begin{aligned}
\lim_{t\to\infty} q(t)\,K(t)\,\Gamma(t) &= 0 \qquad (25) \\
\lim_{t\to\infty} B(t)\,\Gamma(t) &= 0
\end{aligned}$$

With constant γ, *marginal* Q is given by

$$q(0) = E_t \int_0^\infty \exp\left[-(\gamma+\delta)\,t\right]\left(\Pi_K[.] + \sigma\theta(\gamma - r(t))\,K(t)^{\sigma-1} - \Psi_K[.]\right)dt \tag{26}$$

Clearly, the higher are θ and σ, the higher is also the shadow value of capital $q(t)$ and consequently the higher is investment. The standard investment-*marginal* Q relationship still holds

$$q(t) = \Psi_I [I(t), K(t)]$$

Current cash-flow or operating income still do not affect investment because the marginal financing cost to the firm would not change, if cash-flow or profit were higher! It would still be equal to the cost of additional equity finance $\gamma(t)$. The optimal investment and financial policies still must be determined jointly however.

Marginal and *average* Q are equl, if the collateral constraint is linear ($\sigma = 1$) and the homogeneity assumptions apply.

Proposition 4 *If the operating income and adjustment cost functions are homogeneous of degree 1 and the collateral constraint is linear, average Q equals marginal Q*

$$q(0) = \frac{V(0) + B(0)}{K(0)} \tag{27}$$

Proof. Proceeding as above we obtain from (21) if $\sigma = 1$

$$\frac{d}{dt}\{[q(t) K(t) - B(t)] \Gamma(t)\} \tag{28}$$
$$= [-\Pi_K [.] K + \Psi_K [.] K + \theta(\gamma - r) K + \Psi_I [.] I - X + \gamma B] \Gamma(t) \tag{29}$$

and after applying the homogeneity assumptions and once again adding and subtracting $r.B(t)$ inside the brackets we get

$$= -[\Pi[.] - \Psi[.] - rB + X] \Gamma(t) + [(\gamma - r) B - \theta(\gamma - r) K] \Gamma(t) \tag{30}$$

Now realizing that the collateral constraint $B(t) = \theta K(t)$ is always binding along an optimal path and integrating from 0 to ∞ and using the transversality conditions (25) we exactly obtain Hayashi's result:

$$q(0) = \frac{V(0) + B(0)}{K(0)}$$

■

The intuition for this result is based on the fact that the firm in this model always operates at the collateral constraint and the constraint is being relaxed proportionally to the increase in the capital stock. Given homogeneity, the two terms in (9) now exactly offset each other and the market value of the average unit of capital employed by the firm therefore exactly equals the shadow value of the marginal unit of capital.

A.4 The relation between *marginal* and *average* Q

From (21) we know that

$$\begin{aligned} & \frac{d}{dt}\left\{\left[q\left(t\right)K\left(t\right)-B\left(t\right)\right]\Gamma\left(t\right)\right\} \\ = & \left[-\Pi_K\left[.\right]K+\Psi_K\left[.\right]K+\rho_K\left[.\right]BK+\Psi_I\left[.\right]I-X+\gamma B\right]\Gamma\left(t\right) \end{aligned}$$

Adding and subtracting $\Pi\left[.\right]-\Psi\left[.\right]-\rho\left[.\right]B\left(t\right)$ to the right hand side of this equation, we obtain after integrating from 0 to ∞

$$\begin{aligned} q\left(0\right)K\left(0\right) &= V\left(0\right)+B\left(0\right)-\int_0^\infty \Gamma\left(t\right)\left(\Pi\left[.\right]-\Pi_K\left[.\right]K\right)dt \\ &\quad -\int_0^\infty \Gamma\left(t\right)\left(\Psi_K\left[.\right]K+\Psi_I\left[.\right]I-\Psi\left[.\right]\right)dt \\ &\quad -\int_0^\infty \Gamma\left(t\right)\left[\left(\rho_B\left[.\right]B+\rho_K\left[.\right]K\right)B\right]dt \end{aligned}$$

Using the specific functional forms we assume, this confirms our expression (11).