



**Department of Economics**

# **Coordination, Expectations and Crises**

**Sanne Zwart**

Thesis submitted for assessment with a view to obtaining the degree of  
Doctor of Economics of the European University Institute

Florence, October 2007

EUROPEAN UNIVERSITY INSTITUTE  
**Department of Economics**

# **Coordination, Expectations and Crises**

**Sanne Zwart**

Thesis submitted for assessment with a view to obtaining the degree of  
Doctor of Economics of the European University Institute

**Examining Board:**

Prof Giancarlo Corsetti, EUI, Supervisor  
Prof. Bernardo Guimaraes, London school of Economics  
Prof. Karl Schlag, EUI and Universitat Pompeu Fabra  
Prof. Eric Van Damme, Tilburg University

© 2007, Sanne Zwart

No part of this thesis may be copied, reproduced or  
transmitted without prior permission of the author

*To Arenda, Peter, Marijn, Jeroen and Aysen*

## Acknowledgements

I am grateful to my supervisors Giancarlo Corsetti and Karl Schlag for advice, stimulating discussions and support. I learnt a lot from their different ways of approaching economic topics and their gentle prodding sharpened my way of argumentation and exposition. I also benefitted from comments by Aitor Erce, Karel Mertens, Markus Poschke and Pontus Rendahl made over countless coffees and lunches. Javier Rivas and Joël van der Weele, with whom I shared the office, also provided constructive remarks. In addition I would like to thank Gara Minguez-Afonso, Nicolas Sauger, Massimo Sbracia, Jochen Schanz, Rick van der Ploeg and seminar participants of the Global Finance Conference 2005, the Banca D'Italia, the London School of Economics (Financial Markets Group), the London Business School and the European Summer Symposium in International Macroeconomics 2006. The comments of two anonymous referees of the Journal of Financial Stability significantly contributed to the paper which constitutes the first chapter of this thesis.

A four year scholarship by the Dutch Ministry of Education is gratefully acknowledged. The MAPMU network of the European Union provided financial support for a visit to the London Business School.

I also want to thank Manuela Bastianelli, Thomas Bourke, Marcia Gastaldo, Nicki Hargreaves, Martin Legner, Michela Pistolozzi, Jessica Spataro, Julia Valerio and Lucia Vigna for their invaluable work and logistic support during all these years.

Two other persons deserve special mentioning. Stefaan Marteel who, as a flat-mate for three years, had to listen occasionally - admittedly much too often - to "economics talk", but was always in for a talk about everything else. Ayşen İsaoglu did not only provide valuable comments on my research, but most importantly was always there for me to share the good and the bad moments.

Sanne Zwart

Florence, April 2007

## Contents

Acknowledgements	iii
Introduction	1
Chapter 1. The Mixed Blessing of IMF Intervention:	
Signalling versus Liquidity Support	7
1.1. Introduction	8
1.2. The Model	13
1.3. The Signalling Effect	20
1.4. Signalling and the IMF Funds	28
1.5. The Importance of Timing	36
1.6. Conclusion	38
Bibliography	39
Appendix 1.A. Proofs	41
Chapter 2. Liquidity Runs with Endogenous Information Acquisition	47
2.1. Introduction	48
2.2. The Liquidity Run Model with Fixed Information Structures	51
2.3. The Liquidity Run Model with Endogenous Information Acquisition	53
2.4. The Value of Information	54
2.5. The Value of Information and the Fundamentals	57
2.6. Equilibrium Implications of Endogenous Information Acquisition	60
2.7. Information with Endogenous Precision	65
2.8. Conclusion	69
Bibliography	69
Appendix 2.A. Non-Symmetrical Equilibria	71
Appendix 2.B. Proofs	74

Chapter 3. Fixing the Quorum: Representation versus Abstention	81
3.1. Introduction	82
3.2. The Referendum Model	86
3.3. The Quorum and the Population Majority Outcome	93
3.4. A Non-Optimal Quorum	102
3.5. Conclusion	111
Bibliography	112
Appendix 3.A. Different Intensities of Voters' Preferences	114
Appendix 3.B. A Non-Optimal Quorum when $\gamma \neq 0$	115
Appendix 3.C. Proofs	118

## Introduction

Our choices and decisions do not only affect ourselves. Whatever we do has an impact on the well-being of others. Someone can thus take a decision that is best for her/himself, but is very bad for everyone else. To make things even more complicated, sometimes the result of our decisions depends on the behavior of others. The expectations we have about their behavior are then important for our decisions. For example, why do we wait for a red traffic light and cross the road when it turns to green? We expect that following this rule is safest because we expect that others will follow this rule as well.

The traffic light example has two features that help guaranteeing a desirable overall outcome. Firstly, the objectives of individuals are in line with general well-being. For an individual it is safest to wait for the red light and doing so increases overall safety. Secondly, although traffic is potentially very dangerous, the behavior of its participants can be coordinated in an easy way. The traffic lights inform everyone whether to cross the road or not. Since everyone expects everyone else to obey the rules, traffic lights help to avoid dangerous situations.

However, decisions that are optimal from an individual's perspective do not necessarily lead to desirable overall outcomes. In this thesis I analyze two of these situations in more detail: bank runs and referenda. In bank runs, withdrawing deposits from a bank reduces the risks of an individual's portfolio, but increases the risk of the bank's default. I first analyze whether these individuals can be coordinated to reach a more desirable overall outcome. I then investigate in what way the expectations held by individuals about the behavior of others matter for the outcome. In a referendum, someone can decide not to participate because of the involved efforts, but then the participating voters do not necessarily represent the population anymore. Here, I address the question of whether, and if so how, the referendum can be designed in such a way that the individual decisions do lead to the outcome preferred by the majority of the population.

The first two chapters deal with bank runs. A bank run occurs when customers of a bank fear that it has insufficient money to honor its debts and therefore withdraw their deposits in large numbers before it is too late. The bank has to call in on loans it made, which in turn implies closing businesses. When it still does not have enough cash, it might limit withdrawals or may have to default. In either way people and businesses will have problems making their payments, which causes problems for their suppliers and so on. An economic crisis can be the result. One of the most recent bank runs occurred in Argentina in 2001. The Argentine economy was in trouble and people fearing the worst began withdrawing large sums of money from their bank accounts. Banks saw their reserves shrink very quickly and became reluctant to honor more requests for withdrawals. People became frustrated since they could not access their own money. Scenes of Argentines banging pots and pans in front of their banks spread around the world.

Bank runs can affect individual banks, but also countries. Argentina had borrowed heavily from foreign investors. For years, those investors were willing to renew the loans at the end of the contractual period. However, this automatic renewal was not a given. In 2001, Argentina's economic situation worried investors. Fears of a possible default made investors reluctant to renew their loans: they wanted to be repaid before it was too late. Investors massively fled out of the country, and a severe crisis was the result. Although in Argentina's case most scholars agree that the economy was in serious trouble, the banking system did not necessarily have to face a run. People were fleeing because they feared that a bank run would occur, which then indeed happened because they fled. Likewise, defaulting countries might have had enough money to repay the investors who run if most of the investors had renewed their loans. In these situations, better coordination among investors could have avoided the country's default. The difficult thing is that it is easy to write a postmortem about whether a run could have been avoided or not, but at the time when investors make their decisions the situation may not be so clear.

In the first chapter I analyze how coordination among investors can be improved. More precisely, the International Monetary Fund (IMF) can help a country in trouble. It does so by granting a loan. By helping the country to meet its repayment obligations, investors might be convinced that the country has enough money to repay its debt. In this way the IMF can persuade investors not to ask for repayment



but instead renew their loans. When fewer investors need to be repaid, the country's default might be avoided. The IMF loan thus helps the investors to coordinate on making a decision that is not only best for themselves, but also for everyone else, including the country.

However, there is a problem: the fact that the IMF grants a loan can also be bad news for investors. Apparently the country is in such a bad condition that the IMF finds it necessary to intervene. I build a model to assess the impact of IMF loans, taking into account both the loan's positive liquidity effect and its negative informational effect. When the IMF has a large budget available for loans, it indeed succeeds in convincing investors to renew their debt and thereby it helps to avoid the country's default. The IMF makes an assessment of the country's situation and due to its large budget it can grant a loan that is likely to tackle the problems. Although investors are alarmed when the IMF deems a loan necessary, they are thus convinced that the loan is sufficiently large to avoid default. On the other hand, when the IMF has a limited budget for granting loans, it makes things worse for the country. When observing an IMF loan, investors not only realize that the country might be in trouble, but also that the loan is probably too small to be of any help. Instead of being persuaded to renew their loans, investors want to have their money back before the country defaults. Countries that borrow money from the IMF have been studied extensively. One of the findings is that an IMF loan has a more positive effect when it is below the limit set for that country. This is in line with a key result of my model. The IMF helps the country and meanwhile conveys the message that a larger loan is not necessary to avoid the country's default. Investors are assured that default will be avoided and are willing to renew their loans.

In the second chapter I focus on the role of information in bank runs. Investors run because they *expect* the country to default. They base their expectations on the information they have. But they also choose the quality of their information. They could decide to spend a lot of time and money collecting information from various sources in order to construct a more reliable picture of the country's situation. But they could also decide that this is a waste of resources. Especially when on the basis of some readily available information they think that the country is in a very bad or in a very good condition, they do not want to spend money on getting more precise information. In the former case they are rather sure that the country will

have problems and they want to be repaid, in the latter case they do not see any potential problems with renewing their loans. Information is most valuable when it is not clear in what condition the country is.

Since the investors' decisions depend on their information, it is clear that the occurrence of bank runs depends on the availability of reliable information. But in turn, the probability of a bank run is important for the decision whether or not to acquire more accurate information. I analyze these mutual dependencies in a theoretical model. The main contribution is for the case when the decision whether or not to acquire additional information is a real decision, so when the cost of this information is neither too high nor too low. In models where investors rely on the same readily available information, they have the same expectations and take the same decisions. The two possible outcomes are extreme cases: either all investors renew their loans, or all investors demand repayment. But one of these outcomes is eliminated when additional information is not too costly. Suppose for example that the country's situation is rather bad but that default will probably be avoided if all investors renew their loans. But even then, it can still happen that the country's situation turns out to be so bad that it has to default anyway. Although without additional information staying would be optimal, an investor can make a better decision when s/he has more accurate information about the country's situation: depending on whether this information is good or bad s/he can either stay or run. By helping investors to make the right decision, additional information thus leads to a higher investment return. Hence, investors will acquire additional information when it is not too costly. But then the outcome where all investors renew their loans cannot occur anymore: it is destabilized by the possibility to acquire additional information.

The third chapter deals with another setting in which individual decisions do not necessarily lead to desirable overall outcomes: referenda. The aim of a referendum is to consult the population about a specific proposal or law, but often the outcome is valid only if a minimum participation rate, a quorum, is met. When voters decide whether or not to cast their votes, they weigh the moral duty they feel for fulfilling their task as citizens in a democracy, the efforts of going to the polling booths and the benefits from the possible outcomes. Some will then decide to stay at home rather than going to the polling booth. Again, although this can be optimal for an

individual, it will distort the outcome of the referendum. When people can abstain from voting, it can occur that the majority of the participating voters is in favor while the majority of the population is against. This can happen when for example the proposal affects the proponents more than the opponents. Proponents are then more inclined to vote than opponents and thus can be the majority among the participating voters, but a minority in the population. To guarantee a certain degree of representativeness, several referenda have a quorum. But a quorum can have unintended consequences, as shown by a recent referendum in Italy. In this country, a referendum was held about lifting stringent requirements on test-tube pregnancies and on research using stem cells from embryos. One of the main opponents was the Roman Church. However, instead of encouraging their followers to vote against the proposal, they had a different strategy. They urged people to go to the beach on the referendum day instead of to the polling booths. By discouraging people from voting, they hoped that the quorum would not be met and that the referendum would be invalid. And their strategy worked out: the participation rate was only 26%, far below the required level of 50%.

The Italian example indicates that imposing a quorum does not necessarily improve representation. I build a model to better understand the effect of a quorum on the referendum outcome. A positive result is that when the quorum is appropriately set, it can guarantee that the outcome of the referendum is in line with the population majority. The intuition is as follows. Suppose proponents are more likely to participate. The proposal can thus be accepted although the majority of the population is against. When the quorum is set in such a way that in these cases it is not met, the referendum outcome will represent the preference of the population majority. However, there is also bad news for the advocates of referenda. Its intended effect is not very robust. Especially in any of the following three cases rejection is more likely: when going to the ballot box makes voters to care more about the referendum outcome; when insufficient knowledge or a lack of political power makes it impossible to set the quorum at the appropriate level; or when, as in the Italian referendum, pressure groups strategically use the quorum. Although difficult to compare, the shortcomings of a referendum with a quorum seem to be more serious than the problems with representation that motivated imposing the

quorum. Hence, it is probably optimal to have a referendum without a quorum, in particular when both opponents and proponents have high participation rates.

In this thesis I analyze situations in which decisions that are optimal from an individual's perspective can lead to undesirable overall outcomes. In the case of bank runs, the IMF can enhance coordination among investors by granting a loan. However, whether or not this is beneficial crucially depends on the IMF's budget for loans. I also show that the information structure should be taken into account when analyzing bank runs. Due to its role in forming expectations, the availability of not too costly additional information can lead to runs that otherwise might have been avoided, or vice versa. Finally, I show that when voters can abstain in referenda, a quorum can be used to guarantee representativeness of the referendum outcome. However, this result is not very robust; in practice it might be better not to hold the referendum or not to impose a quorum.

## CHAPTER 1

# The Mixed Blessing of IMF Intervention: Signalling versus Liquidity Support<sup>1</sup>

**ABSTRACT.** Although IMF support is supposed to benefit a country, it might be bad news that the IMF believes intervention is necessary. This paper analyzes a bank run model in which both the liquidity effect and the signalling effect of the intervention occur. The IMF strategically provides liquidity support to facilitate market functioning. When the IMF intervenes and has large resources, it uses the signalling to aim for a “half run” and off-sets the negative consequences with the liquidity support. For small IMF resources, the negative signalling effect might not be off-set and the IMF presence can be distorting.

**KEYWORDS.** Bank runs, catalytic finance, coordination problems, strategic signalling.

**JEL CLASSIFICATION.** C73, D82, F33, F34.

---

<sup>1</sup>Published in the *Journal of Financial Stability* (2007), **3**(2), 149-174.

### 1.1. Introduction

When the IMF decides to provide liquidity support to a country, this is good news: the country has a larger budget to tackle any liquidity problems it might face. However, it can also be bad news: the IMF makes it apparent that in its assessment the country may not be sufficiently sound to deal with its own problems. The impact of the IMF support on investors crucially depends on the relative importance of these interpretations. This paper analyzes these two mechanisms through which the IMF affects the behavior of the investors.

A core element of bank run models is the coordination problem among investors: when an investor withdraws her money she (potentially) lowers the investment return of other investors. In bank runs models following the seminal paper of Diamond and Dybvig (1983), this coordination problem results in multiple equilibria. In one equilibrium all investors run, in another all stay. Inspired by the two-player model of Carlsson and van Damme (1993), Morris and Shin (1998, 2001) eliminate this multiplicity by introducing noisy private information about the determinants of the investment return. This reduces the reliance on public information. When the precision of the private information is sufficiently high, there is a unique hybrid equilibrium in which some investors run and others stay. Although the unique equilibrium seems to be convenient for policy evaluation, Angeletos, Hellwig and Pavan (2006) show that this can be misleading. The policy choice itself most probably conveys information that affects the investors' decisions. As in their model, this can again lead to multiple equilibria. In our paper, however, the information signalled by the policy choice facilitates coordination and leads to a unique equilibrium.

By adding the IMF as a player, we extend the bank run model of Morris and Shin (2001).<sup>2</sup> Investors with noisy private information simultaneously have to decide on rolling-over their investments in a country. This coordination problem can result in the country being solvent but illiquid. When the IMF expects this to be the case, it is willing to approve the country's request for liquidity support. It sets the size of a loan for the country before the investors take their decisions. The loan size fully or partially reveals the IMF's private information to investors. Since this effect is

---

<sup>2</sup>The paper is phrased in terms of international finance like most papers using related techniques; however, the main findings hold *mutatis mutandis* for a setting in which a large bank considers providing liquidity support to a private company financed by investors.

understood by the IMF, it uses the signalling strategically. In choosing the loan size, the IMF can be constrained by the resources available for supporting the country. This constraint influences the extent of the signalling and in turn the effectiveness of the intervention.

The main findings of this paper are the following. Firstly, when the IMF resources are sufficiently large, the signalling effect is a useful tool for coordinating investors. When a loan is granted, the IMF not only conveys the message that the country is not sufficiently sound to deal with its problems, but also that the IMF is confident that its involvement will be effective. The IMF succeeds in reducing the probability of the country being solvent but illiquid. Secondly, when the IMF resources are small, the IMF presence can be distorting. The IMF can no longer intervene convincingly. Although the liquidity effect of the loan is positive, the main effect of the loan is signalling that the private information of the IMF indicates the country to be solvent but illiquid. Despite its good intentions, the IMF might thus aggravate the country's problems. This is in sharp contrast with the simultaneous model in which the signalling effect is absent and the IMF is necessarily successful, even for small resources. Thirdly, explicit expressions for the main equilibrium variables are derived. Very few models built around global games allow for explicit solutions when private information is not arbitrarily precise. For large resources any non-zero loan is associated with a unique assessment. In other words, a non-zero loan fully reveals the IMF's private information. In contrast, when the IMF has small resources, the maximum loan is granted for various values of the private information. The maximum loan now only partially reveals the IMF's private information.

Interestingly, when the IMF grants a non-zero loan smaller than its resources, its equilibrium behavior coincides with making the "median" investor indifferent between running and staying. To see why this is the case, first note that the IMF sets the loan size such that the expected return of staying investors is zero. Due to the noisy information, it expects that half of the investors have lower expectations about the return than it has itself. These investors withdraw their money and run. So, the IMF chooses the loan that neutralizes the country's condition: whenever a loan is granted, a "half run" is expected. Without the IMF the expected run would

have been larger. The model thus exhibits catalytic finance: the IMF provides liquidity and the smaller run lowers the need for it.<sup>3</sup>

Empirical evidence of how IMF crises lending affects investors is in general not conclusive, see the overview of Cottarelli and Giannini (2006). The results of our paper suggest that the effect of the IMF intervention depends on the total available resources. Empirical evidence discussed by Mody and Saravia (2006) and Eichengreen, Kletzer and Mody (2006) supports these theoretical findings. In their analysis of how IMF programmes affect investors, Mody and Saravia (2006) include programmes that “turned precautionary”, i.e. programmes of which the country makes initial drawings, but then voluntarily halts disbursements.<sup>4</sup> For these programs it is clear that the IMF resources are not critical. They conclude that “Precautionary programmes are catalytic. ... In contrast, the amount of Fund lending does not appear to robustly catalyse capital flows.” Both findings are predicted by the model of this paper. When the IMF resources are not binding, catalytic finance occurs because the signalling tool can be effectively used. Since the IMF loan is tailored to neutralize the country’s fundamentals, conditional on granting a loan the IMF expects the same reaction of the investors. The size of the IMF loan is thus redundant in explaining any catalytic effect.

The findings of Eichengreen et al. (2006) provide a way to analyze how the IMF’s effectiveness depends on the available resources for the country. In general this point is difficult to address since there is not a clear upper bound for the resources, as recent exceedings of the 300% quota limit by e.g., Brazil and Turkey show. However, external debt/GDP ratios can be interpreted as indications of how large available IMF resources are relative to the debt. Eichengreen, Kletzer and

---

<sup>3</sup>Cottarelli and Giannini (2006) give the following definition of catalysis: “the IMF’s involvement in a country has a catalytic effect to the extent that the announcement of an economic program backed up by a limited amount of IMF resources (compared with the size of the potential capital outflow) increases the propensity of private investors to lend to the country concerned, thereby reducing the adjustment burden falling on the debtor country with respect to the no-catalysis scenario”. They identify 5 channels through which the IMF can (theoretically) catalyze private capital flows, namely policy design, information, commitment, screening and insurance/liquidity. This paper focusses on the information and liquidity channels.

<sup>4</sup>Although in this case it is the country that chooses a loan below its maximum to signal its credibility, the effects are similar to the model of this paper where the IMF decides on the loan size and thereby gives a signal about the fundamentals of the country.



Mody (2006) find that for countries with external debt/GDP ratios below 60%, the IMF presence matters for spreads but the size of the loan does not, while for higher ratios the effect of the IMF disappears. This suggests that for large resources there is a positive effect which is reduced for smaller resources. Since the IMF does have a positive affect for countries with high external debt/GDP ratios that turn precautionary, this reduction might be due to a lower signalling ability.

In our model it is possible to analyze the effectiveness of the IMF since there is a unique equilibrium. Angeletos et al. (2006) also discuss signalling in global games and show how the endogenous information provision can lead to multiple equilibria. The different results stem from the way the institutional player is modelled. In their model it maximizes its utility, which depends on the investors' behavior. There is thus scope for large feedback effects which invite multiplicity. In our model instead, the IMF aims for a smooth market functioning closely in line with the IMF's Articles of Agreement (1990). The freedom of choosing its actions in pursuit of a single overall objective is restricted by principles that should be followed. This limitation of possible actions reduces the scope for feedback effects to such extent that there is a unique equilibrium.

The IMF provides liquidity support when it expects the country to be solvent but illiquid without intervention. In this case the coordination problem among investors turns their (net) return of staying from a profit into a loss. Since for investors a zero return in expectation is the dividing line between investing and not investing, the IMF wants to raise the expected return by granting a loan. Concerns about its monetary balances lead the IMF to grant the smallest loan sufficient for reaching its objective. When its resources are not sufficient, the IMF provides the maximum loan. As in the real world, the IMF does not possess perfect information about the fundamentals of the country. It has to base its decision about the loan size on noisy private information. Numerical analysis suggests that when the IMF has large resources an increase in the precision of its private information has minor impact on its effectiveness. This is the result of two opposite effects: the IMF intervenes less often but upon intervening its loan is larger. Since it knows better when loans are appropriate, the main consequence of an increased precision is lowering the expected loan size without substantially affecting its effectiveness. However, for an observer

who only considers the cases where the IMF is involved with granting a loan, the IMF is perceived as becoming less effective.

The effect of an institutional player on bank runs is analyzed in a similar setting by Morris and Shin (2006) and Corsetti, Guimarães and Roubini (2006). Both papers focus on the behavior of the country's government and particularly on the role of moral hazard. Absence of private information for the institutional player in the former and simultaneous actions in the latter rule out signalling. Compared to these two models, however, we abstract from the conditionality of loans on a change in government behavior. This is a considerable simplification. Although explicitly modelling the country would affect the information structure of the model, the main mechanisms would be qualitatively identical. The benefit of not including the country is that many of the other involved issues can be analyzed more clearly due to the higher tractability. The model thus provides an interesting and relevant starting point for analysis. By abstaining from country policy that affects the fundamentals, there is no potential conflict of interest between the IMF and the country. This situation describes the renewal of short to intermediate term debt, which, although partially influenced by long-term expectations about the country, will be more dependent on expectations of short-term returns. Interestingly, the role of moral hazard is downplayed by recent literature suggesting that governments will be punished by national lenders via a bank run. Together with the fact that recent IMF loans to Brazil and Turkey did not demand policy adjustments, this suggests that the model might also be relevant for the renewal of long-term debt.

Several features of the model also appear in related models. Rochet and Vives (2004) add a central bank to the bank run model of Morris and Shin (1998) to cope with the coordination problem, but simultaneous actions rule out signalling. Corsetti, Dasgupta, Morris and Shin (2004) discuss the presence of a large trader in a currency attack model. Compared to the static model, they find a significant magnified influence when the large trader moves first. They also find that the size of the large trader (for a fixed market size) is not important if he is arbitrarily better informed. Since the big trader will also invest when the fundamental is very strong, the investors will follow him blindly, which is a key difference from our model. How a large trader affects market sentiment is also analyzed by Bannier (2005). Atkeson (2001) has argued that equilibrium prices would restore perfect information

in global games. Hellwig, Mukherji and Tsyvinski (2006) make this concern precise. However, the unique equilibrium result is preserved in other papers. For example, Tarashev (2007) discusses a currency crisis model in which the interest rate informs the investors about the actions (and thus about the information) of other investors without leading to common perfect information and multiple equilibria. Similarly, Angeletos and Werning (2006) show for a crisis model that if asset prices imperfectly aggregate private information in a secondary market, a unique equilibrium can also exist. These findings suggest that including asset prices in our model would not substantially alter the results.

The paper is organized as follows: Section 1.2 introduces the model; Section 1.3 discusses the signalling effect of IMF loans; the relation between the signalling effect, the resources of the IMF and its effectiveness are discussed in Section 1.4; Section 1.5 analyzes the importance of timing by comparing the sequential and simultaneous models and Section 1.6 concludes. Proofs are deferred to the appendix.

## 1.2. The Model

**1.2.1. The Time Line.** The model analyzes the interaction between investors and the IMF when new information about a country's market becomes available. The information revision can be the result of, for example, a shock to the fundamentals of the country, deteriorated forecasts or a sudden withdrawal of investors. The model setting is taken from Morris and Shin (2001),<sup>5</sup> which in turn builds on the model of Diamond and Dybvig (1983), and is extended by adding the IMF. The time line of the model is as follows:

**Period 0:** investors have allocated money in a country.

**Period 1:** revised information about the return of the investment in the country replaces all previous information. This new public knowledge also accounts for the request of the country for IMF support. It causes the investors to reconsider their presence in the country, a decision that can be affected by IMF policy.

**Period 1a:** the IMF receives noisy private information about the fundamentals of the country. It then decides on the loan size for the country.

---

<sup>5</sup>The model is also discussed in Morris and Shin (2003).

**Period 1b:** the investors receive noisy private information about the fundamentals of the country. They then decide either to withdraw their money and run, or to stay.

**Period 2:** the return of the investment is realized.

In the main model of this paper, period 1a takes place *before* period 1b and the IMF decision is public information. This allows investors to condition their decisions on the behavior of the IMF. In other words, it allows the IMF to give a signal about its private information. In Section 1.5, the importance of timing is analyzed by assuming that period 1a and 1b take place *simultaneously*. Although there are other plausible orders of moves to describe the real world, this model is an important benchmark: it is the simplest model in which the signalling effect can be analyzed. Following the reasoning of Corsetti et al. (2004), investors can receive their information at the same time as the IMF when there are no costs associated with waiting from period 1a to 1b. Since the IMF is known to choose the loan in period 1a, investors will then wait until period 1b with making their decision in order to take advantage of any information revealed by the IMF's actions.

**1.2.2. The Investors.** In period 1 there is a continuum of identical investors with a total measure equal to 1. Investors only derive utility from monetary holdings in the second period and money is mapped into utils by the logarithmic transformation. All investors have invested one unit in the country. When an investor withdraws her money and runs in period 1, the investment will be fully refunded. A risk-free investment alternative with a standardized net return of zero then implies that the utility of running is  $\log 1 = 0$ .

When an investor stays until period 2, her gross return equals  $Re^{-\ell+L}$ , where  $R$  is a random variable summarizing the fundamentals of the country,  $\ell \in [0, 1]$  is the fraction of investors who run in period 1 and  $L$  is the size of the IMF loan. The idea is that if investors withdraw their money, investment projects have to be downsized or even cancelled, which negatively affects the return for the staying investors. Moreover, when the outflow of investors causes a devaluation of the exchange rate, the return will be lowered even further. The return function combines the fundamental and these negative effects of premature withdrawals. By defining  $\theta = \log R$ , the rate of return can be written as  $e^{\theta-\ell+L}$ . The utility of an investor who stays is then given by  $\theta - \ell + L$ . To simplify the discussion,  $\theta - \ell + L$  is referred

to as the *return* of the investment instead of the “return of the investment in utils”. The fundamental  $\theta$  is normally distributed with mean  $\hat{\theta} \in \mathbb{R}$  and precision  $\alpha > 0$ . These parameters are public knowledge.

The decision of an investor in period 1 to run or to stay is based on noisy private information about the true value of  $\theta$  and, if the IMF acts before the investors, on the size of the IMF loan. The private information of an investor is a noisy signal about the fundamental. Investor  $i$  receives the signal  $x_i = \theta + \varepsilon_i$ , where the noise  $\varepsilon_i$  is independent across investors and drawn from the normal distribution with mean 0 and precision  $\beta > 0$  (both the mean and the precision of the noise are public knowledge). The private information reflects the high degree of complexity of real-world economies where even publicly known statistics need interpretation.

An investor stays if conditional on her signal she expects  $\theta - \ell + L$  to be positive, while she runs if she expects it to be negative. Since the measure of indifferent investors is zero, it can be assumed that they run. Let  $\ell_i$  be 1 if investor  $i$  runs, and 0 if she stays. The decision  $\ell_i$  is a function of investor  $i$ 's private information  $x_i$ . If the IMF chooses the loan size before the investors decide, investors observe the loan size  $L$  without noise. The decision  $\ell_i$  is then also a function of the loan size  $L$ . The fraction of running investors is given by  $\ell = \int_0^1 \ell_i di$ .

As already noted, this paper extends the model of Morris and Shin (2001) by including the IMF. In the absence of the IMF, conditional on her signal  $x_i$ , investor  $i$  expects a return  $\mathbb{E}[\theta - \ell | x_i]$  when staying. To measure the informativeness of the public information relative to the private information define  $c^{MS} = (\alpha^2/\beta)(\alpha + \beta)/(\alpha + 2\beta)$ . Morris and Shin (2001) then show that when  $c^{MS} \leq 2\pi$ , so when the public information has a relatively low informativeness, there is a unique equilibrium in which all investors have the same switching point strategy. More precisely, investor  $i$  runs if and only if  $x_i \leq x^{MS}$  where  $x^{MS}$  is the unique solution to  $\mathbb{E}[\theta - \ell | x^{MS}] = 0$ . If the investors had been able to cooperate, they could have derived the true value of the fundamental  $\theta$  since this would have been the mean of their signals. Cooperation would thus imply that all investors stay if  $\theta$  is positive and run if  $\theta$  is negative. When the investors cannot cooperate, they face a coordination problem. Uncertainty about the true value of the fundamental implies that there are always some investors who run while others stay. Since  $\ell$  is strictly greater than 0 in all cases, the return of the investment is strictly lower than the true value

of the fundamental. Hence, there are positive values of the true fundamental for which the return nevertheless will be negative. This coordination problem is at the heart of the model.

**1.2.3. The IMF.** The IMF has to decide on the size of the loan  $L$ . Let  $\bar{L}$  be the resources of the IMF, or alternatively, the maximum loan that the IMF can grant ( $\bar{L}$  is public knowledge). The IMF then chooses a loan  $L \in [0, \bar{L}]$ , where a loan of size zero indicates that the IMF does not grant a loan. A loan of size  $L$  is equal to the investments of a measure  $L$  of investors. When the IMF grants a loan of size  $L$  it will thus *ceteris paribus* raise the return of staying investors by  $L$ .

The IMF bases its decision whether or not to grant a loan on noisy private information about the true fundamental  $\theta$ . This information is a signal  $x_I = \theta + \varepsilon_I$ , where the noise  $\varepsilon_I$  is independent of the investors' noises and drawn from the normal distribution with mean 0 and precision  $\gamma > 0$  (both the mean and the precision of the noise are public knowledge). Whether the real-world IMF has private information and if so whether its information is more or less precise than investors' information is controversial. Although it can be argued that some investors are so well informed that no action of the IMF reveals new information about the fundamental to them, it can also be argued that when making their decision, the majority of investors will take into account any information that can be derived from the IMF's behavior. Moreover, the IMF has internal and published forecasts which suggests that the IMF is not completely revealing its information. For example, Artis (1996) and the United States General Accounting Office (2003) evaluate the economic forecasts the IMF publishes in the World Economic Outlook and find that for developing countries GDP growth forecasts are generally upwards biased, while inflation forecasts are downwards biased. Also, the IMF's internal current account forecasts do have explanatory power, while the published current account forecasts do not improve on a random walk model. Similarly, on the national level Romer and Romer (2000) show that the Federal Reserve has considerable information about inflation beyond what is known to commercial forecasters. Here, we only assume that the IMF information is different from the information investors have, and its quality (measured by the precision  $\gamma$ ) is left as a parameter.

Since investors have noisy private information, the IMF's behavior, regardless of the quality of the information it is based on, contains information for investors.

Given the sequentiality of the moves, this immediately implies a role for signalling. Note that, like in the real world, reliance on noisy private information can cause the IMF to make an *ex post* non-optimal decision.

The objective of the IMF's behavior is inspired by the Articles of Agreement of the IMF (1990), especially by Article 1 (v), which says that it is a purpose of the IMF:

“To give confidence to members by making the general resources of the Fund temporarily available to them under adequate safeguards, thus providing them with opportunity to correct maladjustments in their balance of payments without resorting to measures destructive of national or international prosperity.”

In the model, the maladjustment in the balance of payments is a run when the country is solvent but illiquid. In this case, the coordination problem leads to a run caused by self-fulfilling prophecies. To avoid these runs, the IMF should thus address the coordination problem among the investors. The noisy private information of the investors, however, ensures that there will always exist a coordination problem in the sense that some investors run while they should have stayed or vice-versa. When the remaining investors make a positive profit, the coordination problem, although present, is not very serious (the risk free rate of return is zero). However, when the remaining investors make a loss while the fundamental has a strictly positive value, the coordination problem is more detrimental. The IMF, being aware that it cannot succeed in solving the coordination problem entirely, wants to eliminate the cases where remaining investors make a loss due to the coordination problem. This is the paradigm of IMF intervention for solvent but illiquid countries translated to the context of the model. It shows that the IMF is concerned about the functioning of the market, since it aims at providing an investment environment that reflects the underlying fundamentals.

The IMF is also concerned about its monetary balances. It wants to grant the smallest loan that will lead to a zero expected return for the staying investors. When its resources are not sufficiently large to accomplish this, it is obliged for public or political reasons to provide assistance in the best possible way by granting the maximum loan. The IMF's utility when it grants loan  $L$  for signal  $x_I$  is then

given by

$$U(L; x_I) = -L - \phi \mathbf{1}_{\{\mathbb{E}[\theta - \ell(0)|x_I] \leq 0 \leq \mathbb{E}[\theta|x_I]\}} \mathbf{1}_{\{\mathbb{E}[\theta - \ell(L)|x_I] + L < 0\}} \mathbf{1}_{\{L < \bar{L}\}},$$

where  $\phi > \bar{L}$ . Note that since the investors observe the size of the IMF loan, the IMF needs to take into account that different loan sizes might *ceteris paribus* lead to different expected values of  $\ell$ . The first term represents the IMF's concerns about its monetary reserves: the utility is decreasing in the granted loan. The second term is a punishment if the IMF does not act according to its principles. The IMF can only be punished if it expects the country to be solvent,  $\mathbb{E}[\theta|x_I] \geq 0$ , but illiquid,  $\mathbb{E}[\theta - \ell(0)|x_I] \leq 0$ .<sup>6</sup> In this case it is punished if the granted loan  $L$  does not lead to an expected return of at least zero, unless the IMF grants the maximum possible loan  $\bar{L}$ .

Since the IMF maximizes its utility and the maximum loan  $\bar{L}$  is smaller than the punishment  $\phi$ , it always prefers to grant a loan to avoid the punishment. But when the IMF is never punished, the utility is just the negative of the granted loan. By maximizing its utility, the IMF thus minimizes the granted loan while being constrained by its principles.

Suppose the IMF expects the country to be solvent but illiquid. When it can grant a loan that makes the expected return non-negative, it will do so because this implies a higher utility than avoiding the punishment by granting the maximum loan. The three principles defining the IMF's behavior are thus: i) only grant a loan when the country is expected to be solvent; ii) only grant a loan when the country is expected to be illiquid; iii) grant the smallest admissible loan that will make the expected return for staying investors zero, and grant the maximum loan only in the case no loan size is sufficiently large to accomplish this. The IMF decision is then described by

$$L(x_I) = \begin{cases} \min\{L \in [0, \bar{L}] \mid \mathbb{E}[\theta - \ell(L)|x_I] + L \geq 0\} \cup \{\bar{L}\} & 0 \leq \mathbb{E}[\theta|x_I] \leq \mathbb{E}[\ell(0)|x_I], \\ 0 & \text{otherwise.} \end{cases}$$

Granting the loan  $L(x_I)$  maximizes the IMF's utility when it receives signal  $x_I$ .<sup>7</sup>

<sup>6</sup>The weak inequalities are for notational convenience only and are not driving the results.

<sup>7</sup>Since the IMF chooses the size of the loan before or at the same time as the investors decide whether or not to stay, it is possible that the IMF loan more than offsets the negative effect of the



It is interesting to compare the decision rule of the IMF with the utility function of the institutional player in Angeletos, Hellwig, and Pavan (2006). In their paper, a policy maker can affect the investors' decision whether to run or to stay by setting the opportunity cost of attacking. Higher opportunity costs are more expensive for the policy maker. The policy maker thus faces a trade-off between the cost of policy intervention and the net benefit of maintaining the status quo. Due to the sequential setting, the interest of the policy maker in maintaining the status quo is partially reflected by its choice of the investors' opportunity cost. The IMF maximizes its utility, which depends on the behavior of the investors, which in turn depends on the IMF's choice of the opportunity cost. Hence, there is scope for large feedback effects which lead to multiple equilibria. In our model, the utility maximization gives back the principles that define the IMF's behavior. These principles reduce the scope for feedback effects. For example, there is no inactive equilibrium since the IMF is bound to undertake action when it expects the country to be solvent but illiquid. Similarly, explicit concerns about its monetary balances restrict the loan choice of the IMF more directly than utility maximization. In other words, the IMF cannot freely choose its loan in order to pursue a single overall objective, but is restricted by principles. Basing the behavior of the IMF on its Articles of Agreement through principles thus limits its actions and therefore considerably reduces the scope for multiple equilibria.

A few more words are in order on what the IMF tries to achieve by granting a loan. In the real world, three parties with different interests are involved: the IMF, the investors and the country. The IMF is expected to balance the interests of all these parties. In the model, the IMF only cares about the coordination problem. However, this indirectly affects the interests of all parties. The IMF cares about the consequences for its own monetary reserves or its net worth when granting a loan. Nevertheless, by not completely ruling out problems regarding the future repayment of the loan, it is taking a risk. An adequate safeguard as demanded in the Articles is provided for. Since  $\mathbb{E}[\theta - \ell + L|x_I] \geq 0$  implies  $\mathbb{E}[e^{\theta - \ell + L}|x_I] > 1$  by Jensen's inequality, the IMF expects the investment to generate a positive net return when

---

running investors. This results in a return that is higher than the fundamental. To avoid this, the return could be modelled as  $\theta - \ell + \max\{\ell, L\}$ . However, this is in fact an *ex post* adjustment of the loan and allowing for this in particular cases while not in others would be inconsistent.

the loan is expected to fix the coordination problem. This implies that it expects the country to be able to repay the loan without any problems in the majority of the cases.<sup>8</sup> Concerning the payoff of the investors, it would be perverse to let the IMF only maximize the return of the investors. However, the investors' interests are taken care of since the very aim of the IMF is to reduce the coordination problem. This increases the fraction of investors making the right decision, which in turn increases the expected return of staying investors. Finally, the country's payoffs are not explicitly defined in this model. It could be argued that it is in the country's interest to have the projects succeed. In this case  $\theta - \ell + L$  could be seen as a proxy of the success. Again, this is exactly what the IMF is concerned about.

### 1.3. The Signalling Effect

**1.3.1. Strategies.** Before deciding whether to run or to stay, the investors observe the loan size chosen by the IMF. This loan size will affect their decisions in two ways. Firstly, there is the direct effect of the liquidity support. Secondly, there is the indirect signalling effect. This effect occurs since the IMF uses its private information when choosing the loan size. The size of the loan thus provides a signal about the private information of the IMF. The IMF uses these two effects to strategically influence the behavior of the investors. The analysis is confined to interval strategies for the IMF and conditional switching point strategies for the investors. We will show that in equilibrium only these strategies can occur.

Suppose that the investors observe an IMF loan of size  $L \in [0, \bar{L}]$ . An investor conditions her expectations and decisions on both her private information and the observed loan size. A strategy for the investors is then characterized by a conditional switching point  $x(L)$  such that investors with information below the switching point run and the others stay. Let  $\ell(x(L))$  denote the fraction of running investors under this strategy. An investor who receives the switching point as her private information is indifferent between running and staying. The conditional switching point  $x(L)$  is a best-reply to the loan  $L$  if it satisfies

$$\mathbb{E}\left[\theta - \ell(x(L)) \mid x(L), L\right] + L = 0. \quad (1)$$

---

<sup>8</sup>Repayment is not discussed by Corsetti et al. (2006) and only briefly Morris and Shin (2006) in the case of infinitely precise information. The guaranteed repayment rate of our model is broadly in line with empirical evidence described in Jeanne and Zettelmeyer (2001).

From the discussion of the IMF's behavior it is clear that the IMF will only grant a non-zero loan for intermediate private information: for very high signals the IMF expects the country to be liquid, while for very low signals it expects the country to be insolvent. Let  $[\underline{x}_I, \bar{x}_I]$  denote the interval of signals for which the IMF will grant a non-zero loan.

**1.3.2. Conditional Expectations.** Both the investors and the IMF base their decisions on the expected value of the return. When forming their expectations, they will use their information. In other words, the investors and the IMF condition their expectations on their information. In this subsection, mathematical expressions are derived first for the conditional expectation of the fundamental and then for the conditional expectation of the fraction of running investors. These expressions are used in the next subsections when discussing the behavior of the investors and the IMF.

1.3.2.1. *Fundamental.* Suppose that investor  $i$  forms her expectation of the fundamental using both the public information and her private information. The public information has precision  $\alpha$  and her private information precision  $\beta$ . Bayesian updating can be used to show that her expected value of the fundamental is just a weighted average of the public information  $\hat{\theta}$  and the private information  $x_i$ , with the precisions as weights. So, investor  $i$ 's expected value of the fundamental conditional on her signal is

$$\mathbb{E}[\theta|x_i] = \frac{\alpha\hat{\theta} + \beta x_i}{\alpha + \beta}. \quad (2)$$

If the private information is very precise, so if  $\beta$  is high,  $x_i$  has a large impact on the conditional expectation, which is in line with intuition. In fact,  $\theta|x_i$ , the fundamental conditional on  $x_i$ , has a normal distribution with the stated mean and precision  $\alpha + \beta$ .

The IMF's expectation of the fundamental given its information can be found in the same way. Replacing  $\beta$  by  $\gamma$  and  $x_i$  by  $x_I$  in Equation (2) directly gives the expressions.

When investor  $i$  would also know the private information of the IMF, she can condition on both  $x_i$  and  $x_I$ . Repeated application of the above expression then gives that  $\theta|(x_i, x_I)$ , the fundamental conditional on both  $x_i$  and  $x_I$ , has a normal

distribution with mean  $(\alpha\hat{\theta} + \beta x_i + \gamma x_I)/(\alpha + \beta + \gamma)$  and precision  $\alpha + \beta + \gamma$ . Again, the conditional expectation is a weighted average of all the information.

1.3.2.2. *Fraction of Running Investors.* Now suppose that investor  $i$  uses the public information about  $\theta$  and her private information to form expectations about the fraction of running investors. Suppose that investors use a switching point strategy that is characterized by  $x$ . To find an expression for the expected fraction of running investors conditional on  $x_i$ , the law of large numbers can be applied (see Judd (1985)). The fraction of running investors is equal to the probability that investor  $j$  receives a signal that is below the switching point  $x$ . Investor  $i$  thus needs the probability distribution of investor  $j$ 's signal conditional on her signal  $x_i$ . Since the noises are independent, the conditional signal of investor  $j$  equals  $(\theta|x_i) + \varepsilon_j$ . The conditional signal of investor  $j$  thus has a normal distribution with mean  $(\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta)$  and precision  $(1/(\alpha + \beta) + 1/\beta)^{-1} = \beta(\alpha + \beta)/(\alpha + 2\beta)$ . Conditional on her private information and the IMF's private information, investor  $i$  then expects the fraction of running investors to be

$$\mathbb{E}[\ell(x)|x_i] = \Phi\left(\sqrt{\frac{\beta(\alpha + \beta)}{\alpha + 2\beta}}\left(x - \frac{\alpha\hat{\theta} + \beta x_i}{\alpha + \beta}\right)\right),$$

where  $\Phi$  denotes the standard normal distribution function.

The IMF's conditional expected fraction of running investors is obtained in the same way by using the mean and precision of  $\theta|x_I$  which are derived above. Similarly, if investor  $i$  would also know the private signal of the IMF, the expected fraction of running investors conditional on  $x_i$  and  $x_I$  is found in the same way by using the mean and precision of  $\theta|(x_i, x_I)$ .

**1.3.3. A Fully-Revealing Loan.** The IMF does not grant a loan when its signal is very low or very high: in the former case it expects the country to be insolvent so that it does not qualify for liquidity support, in the latter case it expects the country to be liquid so that it does not need a loan. Only when its signal is contained in the intervention interval, the IMF will grant a loan. Intuitively, on this intervention interval the loan size will be weakly decreasing in the signal: for higher signals the IMF expects the fundamental to be higher so that a smaller loan suffices. The only reason the loan size is not necessarily *strictly* decreasing is that the IMF resources impose an upper limit on the loan size. The IMF can thus be constrained by its resources for signals on the lower part of its intervention interval. In the next

section we will show that this intuition is correct and that there can exist a value  $\hat{x}_I$  such that on  $[\underline{x}_I, \hat{x}_I]$  the maximum loan  $\bar{L}$  is granted, while on  $[\hat{x}_I, \bar{x}_I]$  the loan size is strictly decreasing. The direct consequence of a strictly decreasing loan size as function of the private information is that each interior loan  $L \in (0, \bar{L})$  is associated with, at most, a single IMF signal. This in turn implies that by granting an interior loan, the IMF fully reveals its private information  $x_I$ . When investors observe an interior loan, they thus know the IMF signal. Investors can thus condition their expectations and decisions on both their own private information and the IMF's private information.

The insight that interior loans reveal the IMF's private information simplifies the definition of the investors' best-reply switching point given in Equation (1). In the previous subsection it was found that conditional on her private information and the IMF's private information, investor  $i$  expects the fundamental to be

$$\mathbb{E}[\theta | x_i, x_I] = \frac{\alpha \hat{\theta} + \beta x_i + \gamma x_I}{\alpha + \beta + \gamma}. \quad (3)$$

Without knowing the private signal of the IMF, the conditional expectation of the fundamental would have been  $(\alpha \hat{\theta} + \beta x_i)/(\alpha + \beta)$ . Now, an investor with a very low private signal will have a low expected value of the fundamental. When the observed loan size reveals a high private signal of the IMF, her expected value of the fundamental is increased. This happens when  $(\alpha \hat{\theta} + \beta x_i)/(\alpha + \beta) < x_I$ . When the reverse inequality holds, the low IMF signal decreases her expected value of the fundamental. In this case the IMF loan is a mixed blessing: although the liquidity effect of the loan increases her expected return, the signalling effect decreases it.

Let  $L \in (0, \bar{L})$  be the (interior) loan associated with the IMF signal  $x_I$ . The fraction of running investors is equal to the probability that investor  $j$  receives a signal that is below the switching point  $x(L)$ . Conditional on her private information and the IMF's private information, investor  $i$  then expects the fraction of running investors to be

$$\mathbb{E}[\ell(x(L)) | x_i, x_I] = \Phi \left( \sqrt{\frac{\beta(\alpha + \beta + \gamma)}{\alpha + 2\beta + \gamma}} \left( x(L) - \frac{\alpha \hat{\theta} + \beta x_i + \gamma x_I}{\alpha + \beta + \gamma} \right) \right). \quad (4)$$

When the switching point  $x(L)$  is fixed, a higher IMF signal reduces the expected fraction of running investors. Compared to not knowing the IMF signal, this knowledge decreases the expected fraction of running investors when  $(\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta) < x_I$  and increases it otherwise.

Abstaining from the liquidity effect, the IMF thus has opposite effects on the expected return of investors with  $(\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta)$  above or below  $x_I$ . In equilibrium the effectiveness of the IMF depends on whether an investor who would be indifferent in the absence of the IMF, will now stay or run.

**1.3.4. A Partially-Revealing Loan.** The IMF reveals its private information only partially when the same loan is granted for various signals. Instead of knowing the value of the IMF's private information, investors now have a probability density over the IMF signals. The only non-zero loan that can be granted for various signals is the maximum loan  $\bar{L}$ . This loan is granted on the interval  $[\underline{x}_I, \hat{x}_I]$  when the IMF resources are small. When investors observe the loan size  $\bar{L}$  they cannot infer the value of the IMF signal, but only that it is contained in the interval  $[\underline{x}_I, \hat{x}_I]$ .

Since investors are aware that the IMF received revised private information, signalling also occurs when no loan is granted. When investors observe that the IMF is not granting a loan, they know that the private information of the IMF is not contained in the intervention interval. The only loan sizes that can partially reveal the IMF's private information are thus 0 and  $\bar{L}$ .

Using Equation (3), the expectation of the fundamental conditional on the private information  $x_i$  and the loan size  $L \in \{0, \bar{L}\}$  is found to be

$$\mathbb{E}[\theta|x_i, L] = \mathbb{E}\left[\mathbb{E}[\theta|x_i, x_I] \middle| x_i, L\right] = \frac{\alpha\hat{\theta} + \beta x_i + \gamma\mathbb{E}[x_I|x_i, L]}{\alpha + \beta + \gamma}. \quad (5)$$

Comparing this equation with Equation (3) shows that the difference between the fully- and partially-revealing loan cases is that  $x_I$  is replaced by its conditional expectation. The conditional expectation of the fundamental can be rewritten as

$$\mathbb{E}[\theta|x_i, L] = \frac{(\alpha + \beta)\mathbb{E}[\theta|x_i] + \gamma\mathbb{E}[x_I|x_i, L]}{\alpha + \beta + \gamma}.$$

The mean of  $\theta$  conditional on  $x_i$  and  $L$  is thus a weighted average of the mean conditional on  $x_i$  and the expectation of  $x_I$  conditional on  $x_i$  and  $L$ . To analyze the latter conditional expectation the conditional density of  $x_I$  is needed.

In a similar way as the distribution of investor  $j$ 's signal conditional on investor  $i$ 's signal was found in Subsection 1.3.2, it follows that the IMF's private signal conditional on the signal of investor  $i$  has a normal distribution with mean  $(\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta)$  and precision  $\gamma(\alpha + \beta)/(\alpha + \beta + \gamma)$ . When an investor observes that the IMF grants a loan of size  $\bar{L}$ , she knows that the IMF signal is contained in  $[\underline{x}_I, \hat{x}_I]$ . The density  $f_{\bar{L}}(\cdot|x_i)$  of the IMF signal conditional on both the private information of investor  $i$  and the observed loan size  $\bar{L}$  then satisfies

$$f_{\bar{L}}(x_I|x_i) = \begin{cases} C_{\bar{L}}(x_i)e^{-\frac{1}{2}\frac{\gamma(\alpha+\beta)}{\alpha+\beta+\gamma}\left(x_I - \frac{\alpha\hat{\theta} + \beta x_i}{\alpha + \beta}\right)^2} & x_I \in [\underline{x}_I, \hat{x}_I], \\ 0 & x_I \notin [\underline{x}_I, \hat{x}_I], \end{cases} \quad (6)$$

where  $C_{\bar{L}}(x_i)$  makes the density function integrate to 1. Since the conditional density is only positive for signals contained in  $[\underline{x}_I, \hat{x}_I]$ , the conditional expectation of  $x_I$  is also contained in  $[\underline{x}_I, \hat{x}_I]$ .

In Figure 1 the conditional expectation of the IMF signal as a function of  $x_i$  is depicted when the observed loan equals  $\bar{L}$ . For higher private signals  $x_i$ , higher IMF signals  $x_I$  have a higher relative density. The conditional expectation is thus increasing in  $x_i$ , but the slope is very small since the range is limited to  $[\underline{x}_I, \hat{x}_I]$ . Let  $x^c$  be the signal for which  $(\alpha\hat{\theta} + \beta x^c)/(\alpha + \beta) = \frac{1}{2}(\underline{x}_I + \hat{x}_I)$ . The density function  $f_{\bar{L}}(\cdot|x^c)$  is symmetrical around  $\frac{1}{2}(\underline{x}_I + \hat{x}_I)$ , see Equation (6). When an investor receives the signal  $x^c$ , both the expected value of the IMF signal and the expected value of the fundamental equal  $\frac{1}{2}(\underline{x}_I + \hat{x}_I)$ . Now suppose an investor receives a signal below  $x^c$ . When she observes that the IMF grants a loan, apparently the fundamental is not that bad after all, and her expectation of the fundamental increases. Conversely, suppose an investor receives a signal above  $x^c$ . When she now observes the IMF granting a loan, this is bad news that lowers the expected value of the fundamental. Similarly to the fully-revealing loans, in this case the IMF loan is a mixed blessing.

Since the investors know about the country's request for IMF support, signalling also occurs when no loan is granted. The density  $f_0(\cdot|x_i)$  of  $x_I$  given  $x_i$  and no IMF loan can be derived in a similar way as  $f_{\bar{L}}(\cdot|x_i)$ . The signalling effects of no loan and a loan of size  $\bar{L}$  are opposite; see Figure 1. Let  $x^d$  be the signal for which  $(\alpha\hat{\theta} + \beta x^d)/(\alpha + \beta) = \frac{1}{2}(\underline{x}_I + \bar{x}_I)$ . When an investor receives a signal  $x_i < x^d$ , she expects a low fundamental and her pessimism is strengthened when the IMF does not grant a loan. For higher signals instead her optimistic expectations are strengthened.

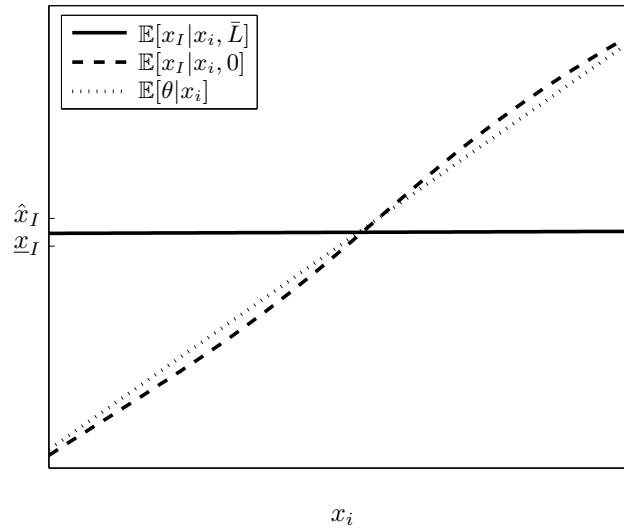


FIGURE 1. Investors base their expectations on their private information  $x_i$  and the observed loan size. When investors observe the maximum loan size  $\bar{L}$ , the expected value of the IMF signal is moderate, which will flatten the investors' expectations about the fundamental. When no loan is observed, the investors' optimism or pessimism is strengthened.

Note, however, that  $\mathbb{E}[x_I | x_i, 0] \rightarrow \mathbb{E}[\theta | x_i]$  when  $x_i \rightarrow \pm\infty$  and equality holds when  $x_i = x^d$ . This shows that the signalling is most informative for an investor who has a low but not very low signal or a high but not very high signal. The former is more likely to run, the latter is more likely to stay.

Obtaining clear expressions for the expected fraction of running investors conditional on  $x_i$  and an observed loan size  $L \in \{0, \bar{L}\}$  is more complicated. In the same way as for the expectation of the fundamental it follows that  $\mathbb{E}[\ell | x_i, L] = \mathbb{E}[\mathbb{E}[\ell | x_i, x_I] | x_i, L]$ . An expression for the inner expectation is already given in Equation (4). The conditional densities of  $x_I$ ,  $f_0(\cdot | x_i)$  and  $f_{\bar{L}}(\cdot | x_i)$ , are also discussed above. It is not possible to simplify this expression for the conditional expected fraction of running investors. However, it is intuitively clear that investors with low signals who observe a non-zero loan, raise their expectation of the fundamental, which decreases the expected fraction of running investors. When investors with high signals observe that the IMF grants a loan, they expect a lower fundamental and thus more running investors. This is again the mixed blessing of IMF loans.



When the IMF does not grant a loan, investors are strengthened in their opinions. To summarize, the signalling effect on the expected return is larger than on the expected fundamental alone.

**1.3.5. The Strategic IMF.** The IMF realizes that the loan size influences the investors' behavior both via the liquidity channel and the signalling channel. By taking account of this, the IMF thus chooses the loan size to strategically affect the investors' behavior.

The IMF only grants a loan when it expects the country to be solvent but illiquid. Based on its information, the IMF's expected value of the fundamental equals  $(\alpha\hat{\theta} + \gamma x_I)/(\alpha + \gamma)$ . The country is solvent when the fundamental is expected to be non-negative. Since the expected fundamental is strictly increasing in the IMF signal, the country is expected to be solvent for sufficiently high signals. Let  $\underline{x}_I$  denote the signal for which the country is expected to be just solvent, so  $\mathbb{E}[\theta|\underline{x}_I] = 0$ . It follows that  $\underline{x}_I = -\alpha\hat{\theta}/\gamma$ . For signals below  $\underline{x}_I$  the IMF expects the country to be insolvent and does not grant a loan.

The loan size affects the investors' behavior, so the liquidity of the country depends on the IMF's decision. Suppose that the conditional switching point  $x(0)$  when the IMF does not grant a loan is given. The IMF expects the country to be illiquid if  $\mathbb{E}[\theta - \ell(x(0))|x_I] < 0$ . An expression for the conditional expectation of the fundamental is already derived; the expected fraction of running investors conditional on the IMF signal equals

$$\mathbb{E}[\ell(x(0))|x_I] = \Phi\left(\sqrt{\frac{\beta(\alpha + \gamma)}{\alpha + \beta + \gamma}}\left(x(0) - \frac{\alpha\hat{\theta} + \gamma x_I}{\alpha + \gamma}\right)\right).$$

This expression shows that  $\mathbb{E}[\ell(x(0))|x_I]$  is strictly positive and strictly decreasing in  $x_I$ . Since  $\mathbb{E}[\theta|x_I] = (\alpha\hat{\theta} + \gamma x_I)/(\alpha + \gamma)$  is strictly increasing in  $x_I$ , the expected return of the staying investors is strictly increasing in  $x_I$  as well. This shows that when the IMF does not intervene, there is a unique IMF signal  $\bar{x}_I$  for which the country is expected to be just liquid. Since the expected fraction of running investors is always positive, it immediately follows that the expected fundamental should be strictly positive. This implies that  $\underline{x}_I < \bar{x}_I$ . Only for signals contained in  $[\underline{x}_I, \bar{x}_I]$  the IMF expects the country to be solvent but illiquid. Note that the upper bound of the intervention interval depends on the investors' conditional switching point  $x(0)$ , but not on  $x(L)$  for  $L > 0$ .

When the IMF grants an interior loan  $L \in (0, \bar{L})$ , the staying investors are expected to make a zero profit. Suppose that the investors use switching point  $x(L)$  when they observe a loan of size  $L$ . Let  $L(x_I)$  denote the best-reply of the IMF when it receives the signal  $x_I \in [\underline{x}_I, \bar{x}_I]$ . When an interior loan is granted, the following equality should hold

$$\mathbb{E}\left[\theta - \ell\left(x(L(x_I))\right)\middle|x_I\right] + L(x_I) = 0. \quad (7)$$

If the maximum loan size  $\bar{L}$  is never granted, this equality holds for all non-zero loans of the IMF. When the IMF is constrained by its resources, the equality cannot be satisfied for low signals in the intervention interval. In this case, let  $\hat{x}_I$  denote the highest signal in the intervention interval for which the IMF is constrained. The IMF then grants the maximum loan  $\bar{L}$  for all signals in  $[\underline{x}_I, \hat{x}_I]$ .

A closer look at Equation (7) gives a key insight: in equilibrium every interior loan size is only chosen for a unique value of the IMF signal. To see this, suppose that there are two distinct signals  $x_{I1} < x_{I2}$  such that  $L(x_{I1}) = L(x_{I2}) = L \in (0, \bar{L})$ . Then  $\mathbb{E}[\theta - \ell(x(L))|x_{I1}] < \mathbb{E}[\theta - \ell(x(L))|x_{I2}]$ , since for  $x_{I1}$  both the expected value of the fundamental is lower and the fraction of running investors is higher (a lower expected fundamental and the same switching point). This implies that  $\mathbb{E}[\theta - \ell(x(L))|x_{I1}] + L < \mathbb{E}[\theta - \ell(x(L))|x_{I2}] + L$  so that Equation (7) cannot hold for both  $x_{I1}$  and  $x_{I2}$ . In equilibrium every interior loan size is thus associated with a unique value of the IMF signal. This formalizes the claim that interior loan sizes fully reveal the IMF's private information.

#### 1.4. Signalling and the IMF Funds

**1.4.1. Equilibrium.** The model has a unique equilibrium for a subset of parameters. It is characterized by conditional switching points of the investors and a loan scheme of the IMF. Upon observing a loan of size  $L$ , investors stay if and only if their private signal is above the conditional switching point  $x(L)$ . The IMF only grants a loan when its private signal  $x_I$  is contained in the intervention interval  $[\underline{x}_I, \bar{x}_I]$ . In this case, the IMF grants a loan  $L(x_I)$  that is dependent on its private signal. Conditions on the precisions  $\alpha$ ,  $\beta$  and  $\gamma$  are needed to ensure that the private information of investors is sufficiently informative to avoid multiple equilibria. A condition on  $\hat{\theta}$  is needed to ensure that the IMF loan as defined in Equation (7) is positive (this condition is needed for the formal proofs, numerical analysis though

suggests that this condition is not very restrictive in practice). The following proposition states the conditions for a unique equilibrium and the related strategies of the IMF and the investors.<sup>9</sup>

PROPOSITION 1.

In the sequential model there exist a  $\gamma^* > 0$  and an  $\epsilon > 0$  such that if  $\frac{(\alpha+\gamma)^2}{\beta} \frac{\alpha+\beta+\gamma}{\alpha+2\beta+\gamma} \leq 2\pi$ ,  $\gamma \in (0, \gamma^*)$  and  $\hat{\theta} \in (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$  there is a unique equilibrium in which

i)  $\underline{x}_I = -\frac{\alpha}{\gamma}\hat{\theta}$ ,

ii)  $x(0)$  and  $\bar{x}_I$  are the unique solution of  $\begin{cases} \mathbb{E}[\theta - \ell(x(0))|x(0), 0] = 0, \\ \mathbb{E}[\theta - \ell(x(0))|\bar{x}_I] = 0, \end{cases}$

iii)  $\hat{x}_I = \frac{(\frac{1}{2}-\bar{L})(\alpha+\gamma)-\alpha\hat{\theta}}{\gamma}$ ,

iv)  $L(x_I) = \begin{cases} \bar{L} & \text{if } x_I \in [\underline{x}_I, \bar{x}_I] \text{ and } \hat{x}_I \geq \bar{x}_I, \\ & \text{or if } x_I \in [\underline{x}_I, \hat{x}_I] \text{ and } \hat{x}_I \in (\underline{x}_I, \bar{x}_I), \\ \frac{1}{2} - \frac{\alpha\hat{\theta} + \gamma x_I}{\alpha + \gamma} & \text{if } x_I \in (\hat{x}_I, \bar{x}_I] \text{ and } \hat{x}_I \in (\underline{x}_I, \bar{x}_I), \\ & \text{or if } x_I \in [\underline{x}_I, \bar{x}_I] \text{ and } \hat{x}_I \leq \underline{x}_I, \end{cases}$

v)  $x(L) = \frac{1}{2} - L$  if  $L \in (0, \bar{L})$  and  $\hat{x}_I \in (\underline{x}_I, \bar{x}_I)$ ,  
or if  $L \in (0, \frac{1}{2}]$  and  $\hat{x}_I \leq \underline{x}_I$ ,

vi)  $x(\bar{L})$  is the unique solution of  $\mathbb{E}[\theta - \ell(x(\bar{L}))|x(\bar{L}), \bar{L}] = 0$  if  $\hat{x}_I > \underline{x}_I$ .

The proposition gives an explicit expression for the equilibrium loan scheme, except for the upper bound  $\bar{x}_I$  of the intervention interval. It also gives explicit expressions for the investors' conditional switching points when an interior loan is observed. In i) the lower bound of the intervention interval is established. Given this lower bound, in ii) it is stated that the upper bound of the intervention interval and the investors' conditional switching point when no loan is observed, are best-replies to each other. In iii) the highest IMF signal  $\hat{x}_I$  is stated for which the maximum loan  $\bar{L}$  can be granted. The loan scheme on the intervention interval is summarized in iv). When  $\hat{x}_I \geq \bar{x}_I$ , the IMF resources are always constraining and the maximum loan is granted for all signals in the intervention interval. When  $\hat{x}_I \in (\underline{x}_I, \bar{x}_I)$ , the maximum loan is only granted on the lower part of the intervention interval. Not surprisingly, on the upper part the IMF loan is decreasing in  $x_I$ : when the IMF

<sup>9</sup>The equilibrium is unique up to the value of  $x(L)$  for loan sizes  $L$  that are never chosen by the IMF. Any value of  $x(L)$  above  $\frac{1}{2}$  suffices.

receives a higher signal, it expects the fundamental to be higher, so a smaller loan size suffices to set the expected return equal to zero. When  $\hat{x}_I \leq \underline{x}_I$ , the IMF resources are so large that they are never fully used. In this case, the IMF loan is decreasing on the entire intervention interval. Note that the IMF loan is at most  $\frac{1}{2}$ . In v) the conditional switching points are summarized for interior loans. When  $\hat{x}_I > \underline{x}_I$ , the IMF resources are so small that they are constraining on the lower part of the intervention interval and loans up to  $\bar{L}$  can occur. If  $\hat{x}_I \leq \underline{x}_I$  the IMF is not constrained and loans larger than  $\frac{1}{2}$  cannot occur. In vi) it is stated that when  $\hat{x}_I > \underline{x}_I$ , so that the IMF is constrained by its resources, the conditional switching point upon observing the maximum loan is simply the best-reply to the IMF granting the maximum loan for  $x_I \in [\underline{x}_I, \hat{x}_I]$ .

When the IMF grants an interior loan, it provides a loan that fills the gap between the expected fundamental and  $\frac{1}{2}$ . It thus provides an *ex ante* insurance that “neutralizes” the bad fundamental in such a way that  $\mathbb{E}[\theta|x_I] + L = \frac{1}{2}$ . Since the IMF only intervenes when  $\mathbb{E}[\theta|x_I] \geq 0$ , this shows that the IMF resources are never constraining when  $\bar{L} \geq \frac{1}{2}$ . In this case the IMF loan always fully reveals the IMF’s private information. When  $\bar{L}$  is below  $\frac{1}{2}$ , the loans for signals on the lower part of the interval are truncated at  $\bar{L}$ .

In the case of an interior loan, the fundamental is not only neutralized from the IMF’s point of view, but also from the investors’ point of view. The expected fraction of running investors turns out to be equal for all interior loan sizes. Both the switching point and the expected value of the fundamental conditional on the switching point equal  $(\alpha\hat{\theta} + \gamma x_I)/(\alpha + \gamma)$ . A closer look at Equation (4) then shows that whenever an interior loan is granted, the expected fraction of running investors equals  $\frac{1}{2}$ . The IMF thus strategically uses the signalling effect to neutralize the investors’ behavior. The liquidity effect of the loan then makes the IMF effective in setting the expected return to zero.

The IMF’s strategy is in fact aimed at neutralizing the investors in such a way that there will be a “half run”. This is a direct result of the IMF’s aim to make the expected return of the staying investors equal to zero. The IMF expects the fundamental to be  $(\alpha\hat{\theta} + \gamma x_I)/(\alpha + \gamma)$ . Since the investors’ signals are unbiased, the IMF expects the median investor to receive the signal  $(\alpha\hat{\theta} + \gamma x_I)/(\alpha + \gamma)$ . In the case of a fully revealing loan, this median investor uses the IMF signal to form expectations.

From Equation (3) it then follows that her expectation of the fundamental equals the IMF's expectation. When the IMF aims for an expected return of zero, the median investor thus also expects a zero return. Half of the investors have lower signals than the median investor and will run. The IMF thus implicitly aims for a "half run". This discussion shows that there is an appealing alternative modelling of the IMF behavior that nevertheless leads to identical equilibrium strategies. If the IMF expects the country to be solvent but illiquid, instead of caring about the return of all staying investors it can also exclusively focus on the expected return of the median investor.

The IMF has no incentive to deviate from the equilibrium strategy. For signals such that the IMF grants no loan this is clear: there is no reason why the IMF should grant a loan. Either the country is expected to be insolvent and does not qualify for it, or the country is expected to be illiquid so that it does not need a loan. When the equilibrium scheme prescribes a positive loan, there is also no incentive to deviate. On the one hand, for an interior loan a deviation to a larger loan than prescribed is not attractive. The IMF is concerned about its monetary balances and already reaches its aim of a zero expected return with the prescribed smaller loan. On the other hand, it is also not optimal for the IMF to grant a smaller loan. This would increase the switching point of the investors and hence would result in a larger fraction of running investors. Together with a smaller loan size this would imply a negative expected return.

When a loan is granted, the investors' reactions and the extent of the signalling effect depend on whether it is smaller than the maximum loan size or not. Knowledge of the maximum loan size  $\bar{L}$  is thus crucial. When it would not have been public knowledge, investors would need a prior distribution for  $\bar{L}$ . Since the IMF in turn would need to base its strategy on this prior distribution as well, the analysis would be considerably more complicated. The important exception is that if the IMF is known to have large monetary balances of at least  $\frac{1}{2}$ , the exact size does not matter.

At this point it is interesting to analyze the model in which the IMF is obliged to truthfully reveal its signal, or equivalently the model in which the investors are so well informed that they know the IMF signal. This model is formally identical to the model where the IMF grants loans based on publicly available information provided by a rating agency as in Carlson and Hale (2006). When the IMF does not

grant a loan, the switching point of the investors will decrease in the now known IMF signal. This shows that the upper bound of the intervention interval will be different. However, on the intervention interval the loan size as function of the IMF signal is still as specified in Proposition 1.

**1.4.2. Effectiveness of IMF intervention.** The IMF aims to prevent the country from being solvent but illiquid. In these cases, although the fundamental is positive, remaining investors would make a loss due to the coordination problem. The *ex ante* probability of the country being solvent but illiquid (SBI) is

$$\begin{aligned} \mathbb{P}[\text{SBI}] &= \int_0^1 \mathbb{P}\left[\theta - \Phi\left(\sqrt{\beta}(x(L(x_I)) - \theta)\right) + L(x_I) < 0 \mid \theta\right] \\ &\quad \times \frac{\sqrt{\alpha}}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha(\theta - \hat{\theta})^2} d\theta. \end{aligned}$$

The probability term inside the integral is a probability statement about the IMF's behavior. Since its signal is a random variable, the loan size  $L(x_I)$  and hence the investors' conditional switching points  $x(L(x_I))$  are random variables as well. In the Morris-Shin model where the IMF is not present this probability is either zero or one; in our model it can also take intermediate values. The following proposition relates the effectiveness of the IMF intervention to the IMF resources.

PROPOSITION 2.

*Suppose the conditions of Proposition 1 hold and that  $\hat{\theta} < \frac{1}{2}$ .*

*For large IMF resources the probability of the country being solvent but illiquid is decreased; however, for small resources it can be increased.*

In the proposition the Morris-Shin model without the IMF is compared with the model of this paper where the IMF is present. The proposition is about countries where problems are likely,  $\hat{\theta} < \frac{1}{2}$ , which are exactly the countries where IMF involvement may be expected. When the IMF has large resources it succeeds in reducing the probability that the country will be solvent but illiquid. In contrast, when the IMF has small resources, its presence can make this outcome more likely.

Key to understand these opposite effects are the principles defining the IMF's behavior. IMF intervention only occurs when the country is expected to be solvent but illiquid. This makes the intervention interval independent of the maximum loan size, as can be seen from the equilibrium expressions in Proposition 1. Suppose

first that the IMF has large resources that are not constraining, so  $\bar{L} \geq \frac{1}{2}$ . When the IMF grants a loan, it fully reveals its private information. This allows the IMF to neutralize the investors' behavior. The IMF loan then provides the needed liquidity for a zero expected return. Since the loan size fully reveals the IMF's private information, the signalling is used to coordinate the investors and its effect is always positive. Interestingly, this positive effect is not exclusively coming from the liquidity effect. As shown in the previous subsection, the expected fraction of running investors is a half. In the model without the IMF the expected fraction of running investors is larger.<sup>10</sup> The liquidity provision of the IMF thus leads to catalytic finance.

Now suppose that the IMF has small resources that are constraining on a large part of the intervention interval. The main effect of the IMF intervention then stems from the cases where the IMF grants the maximum loan. When doing so, it reveals that its resources are too small to make the expected return of staying investors non-negative. Hence, the signalling effect of the IMF loan tends to scare investors away. The IMF loan is a mixed blessing. Since the maximum loan is rather small, the signalling effect can be more important than the liquidity effect. In this case it is bad news that the IMF grants the maximum loan and the probability of the country being solvent but illiquid is increased. Note that this happens although the IMF has good intentions and does the best it can given its information. The *ex ante* effect of the IMF trying to help in an unconvincing way thus magnifies the country's problems.

In Figure 2 the effect of the IMF resources on the probability of the country being solvent but illiquid is depicted. This probability is strictly decreasing in the resources when  $\bar{L} < \frac{1}{2}$ . For  $\bar{L} \geq \frac{1}{2}$  the effect of the IMF is independent of its resources, since the IMF resources are never constraining. The probability of the country being solvent but illiquid in the Morris-Shin model without the IMF is also depicted in the figure, as well as the expected loan size. Clearly, for small resources the IMF presence is distorting, while for large resources it is beneficial. Note that an increase in the resources only leads to a relatively small increase in the expected loan size. This reflects the fact that the increased resources allow for higher loans on

---

<sup>10</sup>In the model without the IMF,  $\hat{\theta} < \frac{1}{2}$  implies that the investors' switching point is above  $\frac{1}{2}$ , so that the expected fraction of running investors is larger than  $\frac{1}{2}$ .

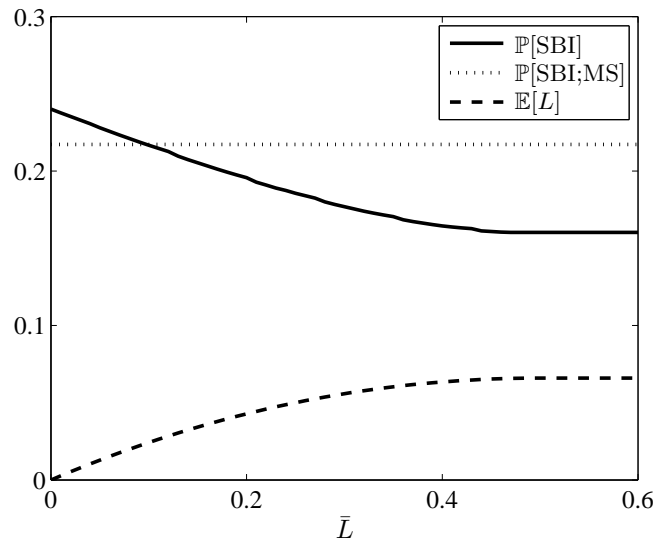


FIGURE 2. The probability of the country being solvent but illiquid is decreasing in the IMF resources  $\bar{L}$ . Compared to the benchmark model without the IMF, the IMF is effective for large resources; however, for small resources the IMF presence increases the probability of the country being solvent but illiquid. ( $\hat{\theta} = 0.4$ ,  $\alpha = \beta = \gamma = 1$ )

a decreasing part of the intervention interval which leads to a diminishing decrease in the probability of the country being solvent but illiquid.

Since the focus of the paper is the impact of the IMF presence, we will vary the precision of the IMF signal  $\gamma$  (see Metz (2002) for the effect of changes in the precision of the investors' private information  $\beta$ ). In Figure 3 the probability of the country being solvent but illiquid is depicted as a function of  $\gamma$  when the IMF resources are large enough to be never binding ( $\bar{L} \geq \frac{1}{2}$ ). The figure also shows the probability of the country being solvent but illiquid even though the IMF has granted a loan  $\mathbb{P}[\text{SBI}, L > 0]$ , the expected probability of IMF intervention  $\mathbb{P}[L > 0]$  and the expected loan size  $\mathbb{E}[L]$ . The probability of the country being solvent but illiquid in the Morris-Shin model without the IMF is depicted by the grey horizontal line.

For a very small precision  $\gamma$ , the probability of the country being solvent but illiquid is increasing in the precision of the IMF signal. This is caused by a sharp decline in the probability of IMF intervention. For larger  $\gamma$ , there is a small decrease in the probability when the precision of the IMF signal increases. The small decrease



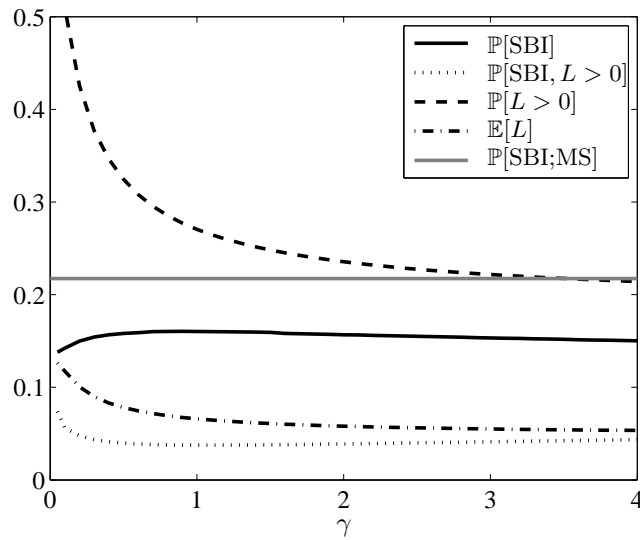


FIGURE 3. An increase in the precision of the IMF's information  $\gamma$  only slightly affects the probability of the country being solvent but illiquid. This is the result of two opposite effects: the IMF grants fewer loans, but upon intervening the loans are larger. The main effect of an increased precision is a lower expected loan. ( $\hat{\theta} = 0.4$ ,  $\alpha = \beta = \gamma = 1$ )

is the result of two opposite effects. Firstly, more precise information allows the IMF to make better assessments so that fewer loans are granted. Secondly, although the IMF intervenes less often, the expected loan size decreases less than proportionally. When intervening, the IMF thus grants larger loans. Although the intervention interval shrinks when the precision increases, better assessments make signals close to the lower bound of the intervention interval more likely and it is exactly these signals for which the largest loans are granted. When the precision increases, there will thus be fewer but larger loans. Although an increased IMF precision slightly lowers the probability of the country being solvent but illiquid, it thus mainly reduces the expected IMF loan.

From the figure it is clear that in general the IMF becomes slightly more effective when the precision of its signal increases. However, it is interesting to look at how the success of the IMF is perceived if one only considers the IMF interventions. Consider for example the probability of the country being solvent but illiquid when the IMF intervenes, so  $\mathbb{P}[\text{SBI}, L > 0]/\mathbb{P}[L > 0]$ . From the figure it can be seen that  $\mathbb{P}[\text{SBI}, L >$

0] is increasing for not too small precisions and  $\mathbb{P}[L > 0]$  is decreasing. When the IMF grants a loan, it thus becomes less successful and the perceived effectiveness decreases. The reason is that when the precision of the IMF signal increases, the IMF knows better when loans are really needed, but it is exactly for these cases that the IMF is less successful. Similarly, consider the probability that the IMF is involved if the country is solvent but illiquid, so  $\mathbb{P}[\text{SBI}, L > 0]/\mathbb{P}[\text{SBI}]$ . Apart from small IMF precisions, this measure increases. Again, the IMF becomes more associated with failed interventions when the precision of its information increases.

### 1.5. The Importance of Timing

The signalling effect cannot occur when the IMF and the investors make their decisions simultaneously. Moreover, when investors do not observe the loan size, they have to form expectations about it. In this section the sequential model discussed so far is compared to its simultaneous counterpart. The model in which the IMF can choose any loan in  $[0, \bar{L}]$  is unfortunately too complicated to analyze. We therefore analyze the simultaneous model in which the IMF has to decide whether or not to grant a loan of a pre-specified size  $\bar{L}$ .<sup>11</sup> To allow for comparison, we will confine the analysis to interval strategies for the IMF and switching point strategies for the investors. Given a weak assumption, this restriction is without loss of generality for the equilibrium analysis.

Investors make their decision without knowing the granted loan size  $L$ . They can only condition on their own information. The investors' strategy is thus an unconditional switching point  $x$  satisfying

$$\mathbb{E}[\theta - \ell(x) + L|x] = 0.$$

For the conditional expectation of  $\theta - \ell$ , expressions are derived in Subsection 1.3.2. When a loan is granted its size is fixed; the expected loan size thus equals the probability that a loan is granted times the size  $\bar{L}$ . The probability that a loan is granted equals the probability that the IMF signal is contained in the intervention interval  $[\underline{x}_I, \bar{x}_I]$ . The IMF signal conditional on investor  $i$ 's signal can be analyzed in a way close to the analysis of Subsection 1.3.2. It is normally distributed with

---

<sup>11</sup>Numerical results suggests that these two models lead to similar effects if the *ex ante* expected loan sizes are equal.

mean  $(\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta)$  and precision  $\gamma(\alpha + \beta)/(\alpha + \beta + \gamma)$ . The expected loan size conditional on the information of investor  $i$  is thus

$$\mathbb{E}[L|x_i] = \bar{L} \times \int_{\underline{x}_I}^{\bar{x}_I} \sqrt{\frac{\gamma(\alpha+\beta)}{\alpha+\beta+\gamma}} \frac{1}{2\pi} e^{-\frac{1}{2} \frac{\gamma(\alpha+\beta)}{\alpha+\beta+\gamma} \left(x_I - \frac{\alpha\hat{\theta} + \beta x_i}{\alpha+\beta}\right)^2} dx_I. \quad (8)$$

From this expression it follows that the expected loan size is positive, increasing in  $x_i$  for signals below the critical signal  $x^d$  that satisfies  $(\alpha\hat{\theta} + \beta x^d)/(\alpha + \beta) = \frac{1}{2}(\underline{x}_I + \bar{x}_I)$  and then decreasing. Investors with intermediate signals expect the country to be solvent but illiquid and thus qualifying for an IMF loan, while investors with high or low signals expect the country to be liquid or insolvent respectively and not qualifying for IMF support.

The best-reply of the IMF to the investors' switching point  $x$  is an intervention interval  $[\underline{x}_I, \bar{x}_I]$ . The lower bound of this interval is the signal for which the IMF expects the country to be just solvent. Hence,  $\underline{x}_I$  satisfies  $\mathbb{E}[\theta|\underline{x}_I] = 0$ . As in the sequential model, it follows that  $\underline{x}_I = -\alpha\hat{\theta}/\gamma$ . The upper bound of the IMF intervention interval is the signal for which the IMF expects the country to be just liquid. So,  $\bar{x}_I$  is defined by  $\mathbb{E}[\theta - \ell(x)|\bar{x}_I] = 0$ . For reasons already mentioned in Subsection 1.3.5 the upper bound  $\bar{x}_I$  is uniquely defined and  $\underline{x}_I < \bar{x}_I$ .

In the following proposition a sufficient condition for a unique equilibrium is stated and the effect of the IMF on the probability of the country being solvent but illiquid is characterized.

### PROPOSITION 3.

*In the simultaneous model there exists an  $\bar{L}^* > 0$  such that if  $c^{MS} < 2\pi$  and  $\bar{L} \in (0, \bar{L}^*]$  there is a unique equilibrium in interval/switching point strategies.*

*The presence of the IMF lowers the probability of the country being solvent but illiquid.*

The proposition states that when the Morris-Shin model without the IMF has a unique equilibrium, the model with the IMF also has a unique equilibrium in interval/switching point strategies when the loan size is not too large. The possibility of equilibria where the IMF and the investors do not have interval or switching point strategies cannot be excluded. However, it has already been proved that the best-reply of the investors to an interval strategy of the IMF is a switching point strategy and vice versa. When the IMF is restricted to interval strategies, or the

investors to switching point strategies, the equilibrium of the proposition is the unique equilibrium.<sup>12</sup>

The proposition also states that the IMF succeeds in lowering the probability of the country being solvent but illiquid. A comparison of the investors' switching points for the Morris-Shin model and this model shows why this is the case. Since  $\mathbb{E}[\theta - \ell(x) + L|x] > \mathbb{E}[\theta - \ell(x)|x]$  and as functions of  $x$  both are increasing,  $x < x^{MS}$ . By sometimes granting a loan the IMF thus lowers the switching point of the investors. Fewer investors run which implies a decrease in the probability of the country being solvent but illiquid.

The main difference between the sequential and the simultaneous model is that even for small resources  $\bar{L}$ , the presence of the IMF reduces the probability of the country being solvent but illiquid. This is especially striking since also in the sequential model with small resources the IMF grants the maximum loan for a large part of the intervention interval. Since there is no signalling in the simultaneous model, the only effect of the IMF is an expected positive liquidity effect which makes investors less likely to run. By shutting off the signalling channel in the simultaneous model, IMF loans are bound to have a positive effect.

## 1.6. Conclusion

This paper analyzes how the news of an IMF intervention affects the investors' behavior and their coordination problem. When the IMF has sufficiently large resources, the intervention is aimed at establishing a "half run" and off-setting the negative consequences. The IMF presence reduces the probability of the country being solvent but illiquid. On the other hand, the signalling effect of the loan can increase this probability when the IMF has small resources. Despite its good intentions, the IMF can thus aggravate the country's problems if its resources are small. Timing is key: when the IMF and investors act simultaneously, the liquidity effect

---

<sup>12</sup>The restriction on the allowed strategies is needed since the usual technique of iterated deletion of strictly interim dominated strategies cannot be applied. In contrast with the model of Corsetti et al. (2004) in which the large player is congruent to the small players, in our model it aims to counterbalance them. Any change in the behavior of the investors is thus partially offset by the IMF, which makes it impossible to establish equilibrium uniqueness without restricting the set of allowed strategies.

makes the IMF always successful since the negative signalling effect of the loan is necessarily absent.

An important direction for future research would be to combine the effect of the IMF on investors and the effect on the country. In a realistic setting the country chooses its effort in the first period, in the second period a fraction of the investors has the opportunity to withdraw their investments which could lead to the outset of a bank run, in the third period the IMF could decide to grant a loan, and in the fourth period all investors have to decide on rolling-over the investments. This model makes it possible to analyze conditional loans abstained from in this paper. The moral hazard problem of the country arises since the IMF loan is a strategic substitute to its own effort as in Morris and Shin (2006) and Corsetti et al. (2006). It would also allow for a more elaborated behavior of the investors and the IMF since the sequential setting allows them to react to each other in turn. Although analytical results might not be obtained, numerical analysis could provide additional intuition.

### Bibliography

- Angeletos, G.-M., Hellwig, C. and Pavan, A. (2006), ‘Signalling in a Global Game: Coordination and Policy Traps’, *Journal of Political Economy* **114**(3), 452–484.
- Angeletos, G.-M. and Werning, I. (2006), ‘Crises and Prices: Information Aggregation, Multiplicity and Volatility’, *American Economic Review* **96**(5), 1720–1736.
- Artis, M. J. (1996), ‘How Accurate Are The IMF’s Short-Term Forecasts? Another Examination of the World Economic Outlook’, *IMF Working Paper 96/89*.
- Atkeson, A. (2001), Discussion of Morris and Shin’s “Rethinking Multiple Equilibria in Macroeconomic Modelling”, in B. S. Bernanke and K. Rogoff, eds, ‘NBER Macroeconomics Annual 2000’, MIT Press, Cambridge, MA.
- Bannier, C. E. (2005), ‘Big Elephants in Small Ponds: Do Large Traders Make Financial Markets More Aggressive?’, *Journal of Monetary Economics* **52**(8), 1517–1531.
- Carlson, M. and Hale, G. (2006), ‘Courage to Capital? A Model of the Effects of Rating Agencies on Sovereign Debt Roll-over’, *Topics in Macroeconomics* **6**(2), article 8.
- Carlsson, H. and van Damme, E. (1993), ‘Global Games and Equilibrium Selection’, *Econometrica* **61**(5), 989–1018.
- Corsetti, G., Dasgupta, A., Morris, S. E. and Shin, S. H. (2004), ‘Does One Soros Make a Difference? A Theory of Currency Crises with Large and Small Traders’, *Review of Economic Studies* **71**(1), 87–113.

- Corsetti, G., Guimarães, B. and Roubini, N. (2006), 'International Lending of Last Resort and Moral Hazard: A Model of IMF's Catalytic Finance', *Journal of Monetary Economics* **53**(3), 441–471.
- Cottarelli, C. and Giannini, C. (2006), Bedfellows, Hostages, or Perfect Strangers? Global Capital Markets and the Catalytic Effect of IMF Crisis Lending, *in* A. Mody and A. Rebucci, eds, 'IMF-Supported Programs: Recent Staff Research', International Monetary Fund, pp. 202–227.
- Diamond, D. W. and Dybvig, P. H. (1983), 'Bank Runs, Deposit Insurance, and Liquidity', *Journal of Political Economy* **91**(3), 401–419.
- Eichengreen, B., Kletzer, K. and Mody, A. (2006), 'The IMF in a world of private capital markets', *Journal of Banking and Finance* **30**(5), 1335–1357.
- Hellwig, C., Mukherji, A. and Tsyvinski, A. (2006), 'Self-Fulfilling Currency Crises: the Role of Interest Rates', *American Economic Review* **96**(5), 1769–1787.
- IMF (1990), *Articles of Agreement of the International Monetary Fund*, available at <http://www.imf.org/external/pubs/ft/aa/index.htm>.
- Jeanne, O. and Zettelmeyer, J. (2001), 'International bailouts, moral hazard and conditionality', *Economic Policy* **16**(33), 408–431.
- Judd, K. L. (1985), 'The law of large numbers with a continuum of i.i.d. random variables', *Journal of Economic Theory* **35**(1), 19–25.
- Metz, C. E. (2002), 'Private and Public Information in Self-Fulfilling Currency Crises', *Journal of Economics* **76**(1), 65–85.
- Mody, A. and Saravia, D. (2006), 'Catalysing Private Capital Flows: Do IMF Programmes Work as Commitment Devices?', *The Economic Journal* **116**(513), 843–867.
- Morris, S. and Shin, H. S. (1998), 'Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks', *American Economic Review* **88**(3), 587–597.
- Morris, S. and Shin, H. S. (2001), Rethinking Multiple Equilibria in Macroeconomic Modelling, *in* B. S. Bernanke and K. Rogoff, eds, 'NBER Macroeconomics Annual 2000', MIT Press, Cambridge, MA.
- Morris, S. and Shin, H. S. (2003), Global Games - Theory and Applications, *in* M. Dewatripont, L. P. Hansen and S. Turnovsky, eds, 'Advances in Economics and Econometrics: Theory and Applications, 8th World Congress of the Econometric Society', Cambridge University Press, Cambridge.
- Morris, S. and Shin, H. S. (2006), 'Catalytic Finance: When Does It Work?', *Journal of International Economics* **60**(1), 161–177.
- Rochet, J.-C. and Vives, X. (2004), 'Coordination failures and the Lender of Last Resort: Was Bagehot Right after All?', *Journal of the European Economic Association* **2**(6), 1116–1147.
- Romer, C. D. and Romer, D. H. (2000), 'Federal Reserve Information and the Behavior of Interest Rates', *American Economic Review* **90**(3), 429–457.

Tarashev, N. A. (2007), ‘Speculative Attacks and the Informational Role of Interest Rates’, *Journal of the European Economic Association* **5**(1), 1–36.

United States General Accounting Office (2003), ‘International Financial Crises: Challenges Remain in IMF’s Ability to Anticipate, Prevent, and Resolve Financial Crises’, *Report No. GAO-03-734*, [www.gao.gov/new.items/d03734.pdf](http://www.gao.gov/new.items/d03734.pdf).

## Appendix 1.A. Proofs

### PROOF OF PROPOSITION 1.

i) This follows directly from  $\mathbb{E}[\theta|\underline{x}_I] = 0$ .

ii) Given a switching point  $x(0)$ , the upper bound of the intervention interval  $\bar{x}_I$  is defined by  $\mathbb{E}[\theta - \ell(x(0))|\bar{x}_I] = 0$ . In the main text it has already been proved that  $\bar{x}_I$  is unique and that  $\bar{x}_I > \underline{x}_I$ . For a given  $x(0)$  denote the upper bound as  $\bar{x}_I(x(0))$ . In equilibrium the investors’ switching point  $x(0)$  is a best-reply when the IMF does not grant a loan for signals outside  $[-\alpha\hat{\theta}/\gamma, \bar{x}_I(x(0))]$ . So,  $x(0)$  should satisfy  $\mathbb{E}[\theta - \ell(x(0))|x(0), 0] = 0$  given this intervention interval. Using Equation (5) gives  $\mathbb{E}[\theta|x(0), 0] = (\alpha\hat{\theta} + \beta x(0) + \gamma\mathbb{E}[x_I|x(0), 0]) / (\alpha + \beta + \gamma)$ . Since  $\mathbb{E}[x_I|x_i, 0]$  is increasing in  $x_i$ , standard arguments of global game theory show that in equilibrium the investors have a switching point strategy (see Morris and Shin (2003) for a proof based on iterated deletion of strictly dominated strategies). The conditional expectation of the IMF signal can be written as

$$\mathbb{E}[x_I|x(0), 0] \mathbb{P}[x_I \notin [\underline{x}_I, \bar{x}_I(x(0))]|x(0)] = \frac{\alpha\hat{\theta} + \beta x(0)}{\alpha + \beta} - \mathbb{E}[\mathbb{E}[x_I|x(0), L]|x_I \in [\underline{x}_I, \bar{x}_I(x(0))], x(0)] \mathbb{P}[x_I \in [\underline{x}_I, \bar{x}_I(x(0))]|x(0)].$$

Since the expected fraction of running investors is strictly below 1,  $(\alpha\hat{\theta} + \gamma\bar{x}_I(x(0))) / (\alpha + \gamma) < 1$  and  $\bar{x}_I(x(0))$  is bounded. This implies that  $\mathbb{P}[x_I \in [\underline{x}_I, \bar{x}_I(x(0))]|x(0)] \rightarrow 0$  when  $x(0) \rightarrow \pm\infty$ . Since the term  $\mathbb{E}[\mathbb{E}[\cdot]|\cdot]$  is contained in  $[\underline{x}_I, \bar{x}_I(x(0))]$ , it is bounded as well. It follows that  $\mathbb{E}[x_I|x(0), 0] \rightarrow (\alpha\hat{\theta} + \beta x(0)) / (\alpha + \beta)$  when  $x(0) \rightarrow \pm\infty$ . Since  $\mathbb{E}[\ell(x(0))|x(0), 0] \in [0, 1]$  this shows that  $\mathbb{E}[\theta - \ell(x(0))|x(0), 0] \rightarrow -\infty$  when  $x(0) \rightarrow -\infty$  and similarly that  $\mathbb{E}[\theta - \ell(x(0))|x(0), 0] \rightarrow \infty$  when  $x(0) \rightarrow \infty$ . Hence, continuity implies that there exists an  $x(0)$  that satisfies  $\mathbb{E}[\theta - \ell(x(0))|x(0), 0] = 0$ .

Uniqueness of  $x(0)$  follows when for the best-reply  $\bar{x}_I(x(0))$  the return  $\mathbb{E}[\theta - \ell(x(0))|x(0), 0]$  is strictly increasing in  $x(0)$ . The expression for  $\mathbb{E}[\theta|x(0), 0]$  can be

used to obtain

$$\frac{\partial}{\partial x(0)} \mathbb{E} \left[ \theta \middle| x(0), 0 \right] = \frac{\beta}{\alpha + \beta + \gamma} + \frac{\gamma \frac{\partial}{\partial x(0)} \mathbb{E} [x_I | x(0), 0]}{\alpha + \beta + \gamma}.$$

The derivative in the second term captures the effect in the expectation of the IMF signal when  $x(0)$  is changed and is bounded. For  $\gamma \rightarrow 0$  the second term becomes arbitrarily small. From  $\mathbb{E}[\ell | x(0), 0] = \mathbb{E}[\mathbb{E}[\ell | x(0), x_I] | x(0), 0]$ , the analogue of Equation (4) and using that  $(\partial/\partial y)\Phi(\sqrt{\alpha}(y - \mu)) = \sqrt{\alpha/2\pi} e^{-\frac{1}{2}\alpha(y-\mu)^2} \leq \sqrt{\alpha/2\pi}$ , it follows that

$$\frac{\partial}{\partial x(0)} \mathbb{E} \left[ \ell(x(0)) \middle| x(0), 0 \right] \leq \frac{1}{\sqrt{2\pi}} \frac{\alpha + \gamma}{\alpha + \beta + \gamma} \sqrt{\frac{\beta(\alpha + \beta + \gamma)}{\alpha + 2\beta + \gamma}}.$$

This shows that there exists a  $\gamma^*$  such that  $\gamma < \gamma^*$  and  $((\alpha + \gamma)^2/\beta)(\alpha + \beta + \gamma)/(\alpha + 2\beta + \gamma) \leq 2\pi$  guarantee uniqueness.

iii), iv) and v) In the text it was proved that the IMF grants a loan with an interior size  $L \in (0, \bar{L})$  for at most one signal  $x_I$ . Standard arguments show that when a positive loan is granted, the investors' best-reply is a switching point strategy. Since  $c^{MS} \leq 2\pi$  is implied by  $((\alpha + \gamma)^2/\beta)(\alpha + \beta + \gamma)/(\alpha + 2\beta + \gamma) \leq 2\pi$ , the investors' switching point is unique for every interior loan size.

The next step is to derive the loan size for signals in the intervention interval and the best-reply of investors to interior loans. Substitution of Equation (4) and  $\mathbb{E}[\theta | x(L), x_I]$  into Equation (1) and comparing this expression to Equation (7) shows that  $x(L) = (\alpha\hat{\theta} + \gamma x_I)/(\alpha + \gamma)$  is indeed a solution. It directly follows that  $L = \frac{1}{2} - (\alpha\hat{\theta} + \gamma x_I)/(\alpha + \gamma)$ . This loan scheme is strictly decreasing so an interior loan indeed reveals the private information of the IMF. It is straightforward to check that for  $\hat{x}_I$  defined in iii) the loan scheme prescribes the maximum loan. When  $\hat{x}_I > x_I$  the IMF resources are constraining and the maximum loan  $\bar{L}$  is granted on  $[x_I, \min\{\hat{x}_I, \bar{x}_I\}]$ .

In order to prove uniqueness, note that for every loan  $L \in (0, \bar{L})$  the investors' best-reply  $x(L)$  makes  $\mathbb{E}[\theta - \ell(x(L)) | x_I] - \mathbb{E}[\theta - \ell(x(L)) | x(L), L]$  equal to zero. Taking the derivative to  $x(L)$  gives

$$\begin{aligned} & - \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\beta(\alpha + \gamma)}{\alpha + \beta + \gamma}} e^{-\frac{1}{2} \frac{\beta(\alpha + \gamma)}{\alpha + \beta + \gamma} \left( x(L) - \frac{\alpha\hat{\theta} + \gamma x_I}{\alpha + \gamma} \right)^2} - \frac{\beta}{\alpha + \beta + \gamma} \dots \\ & + \frac{1}{\sqrt{2\pi}} \frac{\alpha + \gamma}{\alpha + \beta + \gamma} \sqrt{\frac{\beta(\alpha + \beta + \gamma)}{\alpha + 2\beta + \gamma}} e^{-\frac{1}{2} \frac{\beta(\alpha + \beta + \gamma)}{\alpha + 2\beta + \gamma} \left( x(L) - \frac{\alpha\hat{\theta} + \beta x(L) + \gamma x_I}{\alpha + \beta + \gamma} \right)^2}. \end{aligned}$$



When this is strictly negative for all  $x(L)$  there is a unique solution. The first term can be discarded, using the fact that for the third term the exponent part is smaller than one; it follows that  $((\alpha + \gamma)^2/\beta)(\alpha + \beta + \gamma)/(\alpha + 2\beta + \gamma) \leq 2\pi$  is a sufficient condition for uniqueness.

It remains to be proved that for  $\bar{x}_I$  the granted loan is positive. It is straightforward to show that for  $\hat{\theta} = \frac{1}{2}$  the midpoint of the interval is below  $\frac{1}{2}$ . This implies that an investor receiving  $\frac{1}{2}$  as a signal and observing the IMF not granting a loan, expects the fundamental to be above  $\frac{1}{2}$ . This shows that the investors' switching point is below  $\frac{1}{2}$ . When the IMF receives  $\frac{1}{2}$  as signal, it expects the fundamental to be  $\frac{1}{2}$  and the fraction of running investors to be less than  $\frac{1}{2}$ . This shows that  $\bar{x}_I < \frac{1}{2}$ , which implies that the loan for  $\bar{x}_I$  is strictly positive. Due to continuity this is the case for  $\hat{\theta}$  in a neighborhood of  $\frac{1}{2}$ .

vi) As before, the investors' best-reply is a switching point strategy. Existence of  $x(L)$  follows by noting that  $\mathbb{E}[x_I|x(L), L] \in [\underline{x}_I, \hat{x}_I]$ . Equation (5) then implies that  $\mathbb{E}[\theta - \ell + \bar{L}|x(\bar{L}), \bar{L}]$  increases continuously from  $-\infty$  to  $+\infty$  when  $x(\bar{L})$  increases from  $-\infty$  to  $+\infty$ . Straightforward computations show that  $(\partial/\partial x(L))\mathbb{E}[x_I|x(L), L] > 0$ , so the expectation of the IMF signal is increasing in the private information of an investor. Hence,  $(\partial/\partial x(L))\mathbb{E}[\theta|x(L), L] > \beta/(\alpha + \beta + \gamma)$ . In the same way as for  $x(0)$  it follows that  $((\alpha + \gamma)^2/\beta)(\alpha + \beta + \gamma)/(\alpha + 2\beta + \gamma) \leq 2\pi$  implies uniqueness of  $x(L)$ .  $\square$

## PROOF OF PROPOSITION 2.

Suppose that the IMF resources are sufficiently large so that they are never binding. When  $\hat{\theta} < \frac{1}{2}$ , the investors' switching point in the Morris-Shin model is above  $\frac{1}{2}$ . When the IMF is present, it reduces the probability that the country is solvent but illiquid when the conditional switching points are below  $\frac{1}{2}$ . If the IMF grants an interior loan, the conditional switching point  $\frac{1}{2} - L$  is indeed below  $\frac{1}{2}$ . In the proof of Proposition 1 iv) it was shown that for  $\hat{\theta} < \frac{1}{2}$  the midpoint of the intervention interval is below  $\frac{1}{2}$ . Suppose that the switching point  $x(0)$  equals  $\frac{1}{2}$ . An investor with signal  $\frac{1}{2}$  who observes that the IMF does not grant a loan, thus expects the fundamental to be above  $\frac{1}{2}$  and the fraction of running investors below  $\frac{1}{2}$ . This contradicts that  $\frac{1}{2}$  is a switching point. Since the equilibrium is unique,  $x(0)$  is below  $\frac{1}{2}$ .

When the IMF resources are small, the claim follows when parameters are found for which the probability of the country being solvent but illiquid is higher than in the Morris-Shin model without the IMF. Numerical analysis shows that this is the case for parameters  $\hat{\theta} = 0.4$  and  $\alpha = \beta = \gamma = 1$ , see Figure 2. Mathematical analysis fails since there are opposite effects. Manually weighting these effects according to their probability of occurrence is too complicated. To get a flavor, suppose that  $\bar{L}$  is so small that it is granted on the entire intervention interval. On the one hand, since the midpoint of this interval is below  $\hat{\theta}$ , the investors' switching point is higher than in the Morris-Shin model and there are more running investors. On the other hand, when the loan is not granted, the investors' switching point is lower than in the Morris-Shin model and there are fewer running investors. The combined effect follows from weighting these effects according to the simultaneous probability density of the fundamental and the IMF signal.  $\square$

### PROOF OF PROPOSITION 3.

Suppose that the investors act according to a candidate switching point  $x$ . In equilibrium  $x$  should satisfy  $\mathbb{E}[\theta - \ell(x) + L|x] = 0$ . The upper bound  $\bar{x}_I$  of the intervention interval is then defined by  $\mathbb{E}[\theta - \ell(x)|\bar{x}_I] = 0$ . In the main text it is already proved that  $\bar{x}_I$  is uniquely defined. Let  $\bar{x}_I(x)$  denote  $\bar{x}_I$  as function of  $x$ . Since  $\bar{x}_I$  is continuous in  $x$ , also  $\mathbb{E}[L|x]$  is continuous in  $x$ . But then it is clear that  $\mathbb{E}[\theta - \ell(x) + L|x]$  is continuous in  $x$ . Since  $\mathbb{E}[\theta|x]$  maps onto  $(-\infty, \infty)$  while  $\mathbb{E}[-\ell(x) + L|x]$  is contained in  $[-1, \bar{L}]$ , there exists a value  $x$  such that  $\mathbb{E}[\theta - \ell(x) + L|x] = 0$ .

This switching point  $x$  is unique if  $(\partial/\partial x)\mathbb{E}[\theta - \ell(x) + L|x] > 0$  for all  $x$ . The derivative of the expected fundamental minus the expected fraction of running investors is given by

$$\frac{\partial}{\partial x}\mathbb{E}[\theta - \ell(x)|x] > \frac{\beta}{\alpha + \beta} - \frac{1}{\sqrt{2\pi}}\frac{\alpha}{\alpha + \beta}\sqrt{\frac{\beta(\alpha + \beta)}{\alpha + 2\beta}}.$$

Equation (8) gives an expression for the expected loan size conditional on the signal of an investor. When  $x$  increases,  $\ell(x)$  increases and hence  $\bar{x}_I$  has to increase as well. This shows that  $(\partial/\partial x)\bar{x}_I(x) > 0$ . It now follows that

$$\frac{\partial}{\partial x}\mathbb{E}[L|x] = \frac{\bar{L}}{\sqrt{2\pi}}\sqrt{\frac{\gamma(\alpha + \beta)}{\alpha + \beta + \gamma}}\left(\left(\frac{\partial\bar{x}_I(x)}{\partial x} - \frac{\beta}{\alpha + \beta}\right)e^{-\frac{1}{2}\frac{\gamma(\alpha + \beta)}{\alpha + \beta + \gamma}\left(\bar{x}_I(x) - \frac{\alpha\hat{\theta} + \beta x}{\alpha + \beta}\right)^2}\right)$$

$$+ \frac{\beta}{\alpha + \beta} e^{-\frac{1}{2} \frac{\gamma(\alpha + \beta)}{\alpha + \beta + \gamma} \left( x_I - \frac{\alpha \hat{\theta} + \beta x}{\alpha + \beta} \right)^2} > -\bar{L} \frac{1}{\sqrt{2\pi}} \frac{\beta}{\alpha + \beta} \sqrt{\frac{\gamma(\alpha + \beta)}{\alpha + \beta + \gamma}}.$$

Combining the found expressions shows that there is a unique switching point  $x$  if

$$\sqrt{\frac{\alpha^2}{\beta} \frac{\alpha + \beta}{\alpha + 2\beta}} + \bar{L} \sqrt{\frac{\gamma(\alpha + \beta)}{\alpha + \beta + \gamma}} < \sqrt{2\pi}.$$

When  $c^{MS} < 2\pi$ , there exists an  $\bar{L}^*$  such that the inequality is satisfied for  $\bar{L} < \bar{L}^*$ .

The proof is finished if it is shown that when all investors act according to the switching point strategy  $x$  and when the IMF reacts optimally, it is indeed optimal for an investor to use the switching point strategy  $x$ . To prove this, it is sufficient to establish that  $(\partial/\partial x_i)\mathbb{E}[\theta - \ell(x) + L|x_i] > 0$ , since an investor who receives the switching point as signal is indifferent between running and staying. Since for a given switching point the expected fraction of running investors is decreasing in  $x_i$ , it is clear that  $(\partial/\partial x_i)\mathbb{E}[\theta - \ell(x)|x_i] > \beta/(\alpha + \beta)$ . In the same way as before it then follows that  $(\partial/\partial x)\mathbb{E}[L|x] > -\bar{L}\sqrt{1/2\pi}(\beta/(\alpha + \beta))\sqrt{\gamma(\alpha + \beta)/(\alpha + \beta + \gamma)}$ . The switching point strategy is thus the optimal strategy if  $\bar{L}\sqrt{\gamma(\alpha + \beta)/(\alpha + \beta + \gamma)} < \sqrt{2\pi}$ . But this is implied by the condition of the proposition.

In the main text it is proved that the presence of the IMF decreases the probability of the country being solvent but illiquid.  $\square$



## CHAPTER 2

# Liquidity Runs with Endogenous Information Acquisition

**ABSTRACT.** This paper analyzes a liquidity run model in which investors strategically acquire private information. Hence, equilibria can differ both in information structure and extent of the run. A two-dimensional equilibrium partitioning depending on expected fundamentals and the cost of information is presented. The dichotomy “no private information/private information” represents the equilibrium information structures for high and low costs respectively. Endogenous information acquisition thus embeds liquidity run models based on Diamond & Dybvig (1983) and Morris & Shin (2001) in a unified analytical framework. For intermediate costs these models slide into each other. As a result, intuitively less appealing equilibria are eliminated.

**KEYWORDS.** Bank runs, information acquisition, coordination games, option pricing.

**JEL CLASSIFICATION.** C73, D82, F34, G14.

## 2.1. Introduction

Investors want to be well-informed when they choose their portfolios. However, the amount and precision of the information they acquire in equilibrium will depend on the cost. Liquidity run models after Diamond and Dybvig (1983) and Morris and Shin (2001) take the information structure as given.<sup>1</sup> In the former there is only common information, while in the latter agents have noisy private information. This paper presents a liquidity run model where private information acquisition is endogenous: investors optimally decide whether to acquire a signal about the fundamentals taking its cost as given. While models with a fixed information structure focus on fundamentals and self-fulfilling prophecies as causes of liquidity runs, the model of this paper also analyzes how the availability, quality and cost of private information affect the occurrence and extent of runs.

The main results of this paper are as follows. First, the dichotomy “private information/no private information” only describes the information structure for extremely low and high costs of information. For intermediate costs, the equilibrium information structure crucially depends on the prior of the fundamentals. Second, the equilibrium multiplicity that occurs in Diamond-Dybvig models for intermediate priors is eliminated for a range of cost-prior pairs. Of the two symmetric equilibria in Diamond-Dybvig models the intuitively more appealing one survives: all investors run for relatively bad priors, while for relatively good priors all stay. Third, there always exists at least one equilibrium for all costs and priors due to complementarities in information acquisition. When investors base their decisions on private information, their less predictable behavior increases the uncertainty about the investment return, which makes private information more valuable. Fourth, an increase in the precision of private information or a decrease in its cost favor information acquisition. For bad priors of the fundamentals this increase in the quality/price ratio of information helps to deter a run, for good priors, however, more investors will run.

Endogenous information acquisition alters the equilibrium structure compared to models with a fixed information structure. It is the difference between the cost

---

<sup>1</sup>These models are here interpreted in a setting where investors have to decide whether or not to withdraw their money from a certain investment. Alternatively, this paper could have been phrased in terms of investors that have to decide on rolling-over the debt of a country. More general interpretations of the underlying coordination problem are also possible.

of information and its expected added value in terms of the investment return that determines whether or not an investor acquires information. The exogenous cost can be seen as monetary costs when the investors hire an investment agency or as a cost in terms of effort and time when they search for information themselves. Either way, it reflects that the investor has to incur a cost in order to obtain more precise information, so that the quality of the information is a strategic decision. The value of information depends on the uncertainty about the investment return, which itself depends on the investors' prior. Information acquisition is thus related to the fundamentals of the investment project - a point raised by Rey (2001) in her comments on Morris-Shin models. The value of information as function of the prior can be understood in the light of option pricing results. The more likely that the information will be used, in the sense that it will change an investor's behavior relative to not having information, the higher its value.

This paper presents a two dimensional equilibrium partitioning of priors and costs. Two equilibrium candidates are of the Diamond-Dybvig type: no private information and either a full run or no run. The other possible equilibrium is of the Morris-Shin type: investors have private information and run when it is bad. The prior of the fundamentals together with the cost of information determines whether in equilibrium investors acquire private information, which in turn determines the occurrence and extent of the liquidity run. In Diamond-Dybvig models equilibrium multiplicity occurs for an intermediate range of priors. The private information introduced in Morris-Shin models eliminates this multiplicity by allowing for a unique hybrid equilibrium in which some investors stay and others run. In the model of this paper, even for intermediate priors an equilibrium without private information can be unique when the cost of information is intermediate. Interestingly, the multiplicity is not eliminated by endowing investors with private information, but by giving them the opportunity to acquire it.

When the cost of private information is very high or very low, the one dimensional equilibrium partitions of the original models arise. For high costs the prior partition consists of three regions as in Diamond-Dybvig type models, see Sbracia and Zaghini (2001) and Metz (2002). When the cost is low, investors acquire noisy private information, which lead to the information structure underlying global games, see Carlsson and van Damme (1993) and Morris and Shin (2003). For a relatively

high precision of the private information, this heterogeneity replaces the multiplicity that can occur in the absence of private information by a hybrid equilibrium. The prediction of a unique equilibrium for sufficiently precise private information is confirmed in an experimental study discussed by Heinemann, Nagel and Ockenfels (2004). In response to criticism on the original model, e.g. by Atkeson (2001) and Rey (2001), more realistic models with noisy private information have been developed. Several preserve this uniqueness result; others, however, lead to multiplicity, see e.g. Tarashev (2007), Hellwig et al. (2006) and Angeletos and Werning (2006). This suggests that the dichotomy “private information/no private information” in relation to “uniqueness/multiplicity” gives an overly simplified picture.

Several papers focus on information acquisition in settings with strategic complementarities. In the model of Nikitin (2004), investors can acquire complete information about the return of all investment opportunities. Although the three equilibria resemble the equilibria occurring in this paper, the model is rather complex and the equilibrium analysis is limited to showing that the three equilibria can occur instead of elaborating on how the interaction of fundamentals and prices affects the existence of these equilibria. Hellwig and Veldkamp (2006) analyze the effect of costly private information on beauty contest models. Although the structure of the model is similar to the structure of this paper’s model, only the squared distance to a realized random variable matters for the payoffs. This is a key difference since it makes the *ex ante* expected value of the random variable irrelevant for the equilibrium analysis, while it is precisely this that is at the heart of this paper’s analysis.

In Section 2.2 the liquidity model is analyzed when the information structure is fixed. Endogenous information acquisition is introduced in Section 2.3. After deriving detailed expressions for the value of information in Section 2.4, its relation with the underlying fundamentals is explored in Section 2.5. The main results of this paper are discussed in Section 2.6, where the implications of endogenous information acquisition for the equilibria are analyzed. In Section 2.7 the model is extended so that investors can choose the precision of their information. Section 2.8 concludes. The analysis of non-symmetrical equilibria is deferred to Appendix 2.A and proofs to Appendix 2.B.



## 2.2. The Liquidity Run Model with Fixed Information Structures

Consider the coordination problem at the heart of the Diamond-Dybvig model and the Morris-Shin model. There is a continuum of identical investors with a total measure equal to 1. The utility function of the investors is a linear function of their money holdings. In period 0, all investors put one unit in the same investment project. In period 1, investors receive new common information about the investment return that replaces all previous information. This new information causes investors to reconsider their investment. In this period they then simultaneously decide whether to stay and remain invested or to run and withdraw the money. Investors who run are fully refunded. They can then use the money for a risk-free investment alternative with a normalized (net) return of 0 in the next period. Investors who stay until period 2 receive a (net) return of  $\theta - \ell$ , where  $\theta$  is a random variable summarizing the fundamentals of the project and  $\ell \in [0, 1]$  is the fraction of investors who ran in period 1.<sup>2</sup> The idea is that if investors withdraw their money, the project has to be downsized, which negatively affects the return of the remaining investors. Hence, the return function combines the fundamentals and the cost of premature liquidation. The fundamental  $\theta$  is normally distributed with *ex ante* expectation  $\hat{\theta} \in \mathbb{R}$  and precision  $\alpha > 0$ . These parameters are provided to the public at the beginning of period 1.<sup>3</sup>

In the Diamond-Dybvig world, investors have only common information about the project. Based on this common information, an investor decides whether to stay or to run. Denote the two symmetric (pure) strategy profiles “all-stay” and “all-run” by  $S$  and  $R$  respectively. The expected return in the  $S$  profile where all investors stay equals the *ex ante* expectation of the fundamental  $\hat{\theta}$ . The  $S$  profile is then an equilibrium if staying gives a (weakly) higher return than running, so if and only if  $\hat{\theta} \geq 0$ . Likewise, the  $R$  profile where all investors run is an equilibrium only for  $\hat{\theta} \leq 1$  since the expected return equals  $\hat{\theta} - 1$ . Note especially that when the *ex ante* expectation of the fundamental is contained in the interval  $[0, 1]$  both

<sup>2</sup>For the sake of clarity the return in case  $\theta - \ell < -1$  is not truncated. This is not essential for the results. Note that this can also be interpreted as representing a logarithmic utility function when the return of the investment equals  $e^{\theta - \ell}$ .

<sup>3</sup>The fundamental can alternatively be thought of as having an improper uniform distribution on the real line. For a realized fundamental  $\theta$  and normally distributed noise  $\varepsilon$  with zero mean and precision  $\alpha$ ,  $\hat{\theta} = \theta + \varepsilon$  then represents imperfect *public* information about the fundamental.

profiles are equilibria. Since there is only common information, investors have no means of coordination which invites multiple equilibria. The model thus captures sudden big jumps in investments observed in the real world. However, for  $\hat{\theta}$  close to the borders of the interval, one of the equilibria is intuitively much more appealing than the other: for relatively bad *ex ante* expectations of the fundamentals investors are more likely to run, while for relatively good *ex ante* expectations they are more likely to stay. Hence, for  $\hat{\theta}$  close to 0 or 1 the equilibria of Diamond-Dybvig models are not closely in line with intuition.

In the Morris-Shin world, investors are endowed with noisy private information about the realized fundamental  $\theta$  of the project. Next to the common information  $\hat{\theta}$  and  $\alpha$ , investor  $i$  has a signal  $x_i = \theta + \varepsilon_i$ . The noise  $\varepsilon_i$  is drawn from a normal distribution with zero mean and precision  $\beta > 0$  and is independent across investors. The focus on noisy private information instead of complete information, reflects the idea that different investors combine information from various noisy sources which leads to different private information. Whether an investor stays or runs depends on her private information. Intuitively, she runs when her private information is bad. A symmetric equilibrium candidate is then characterized by a common switching point  $x^*$  such that investor  $i$  runs if and only if  $x_i < x^*$ . An investor who receives the switching point as private information is indifferent between running and staying. The expected return of running is 0, so the switching point is defined by  $\mathbb{E}[\theta - \ell|x^*] = 0$ . To measure the informativeness of the public information relative to the private information define  $\gamma = (\alpha^2/\beta)(\alpha + \beta)/(\alpha + 2\beta)$ . Morris and Shin (2001) then show that when  $\gamma \leq 2\pi$ , so when the public information has a relatively low informativeness, there is a unique equilibrium in which all investors have the same switching point strategy characterized by  $x^*$ . Hellwig (2002) shows that if this condition does not hold, there exists an *ex ante* expectation of the fundamental  $\hat{\theta}$  for which there are multiple equilibria. The condition can thus be seen as a necessary condition for a unique equilibrium when no assumptions about  $\hat{\theta}$  are made. It is assumed here that the private information is sufficiently precise to satisfy this condition with strict inequality. To stress the reliance on private information, the equilibrium “all-use-switching-point- $x^*$ ” is denoted by  $I$ . Since there is a unique equilibrium, the economic outcomes are fully determined by the parameters. The model has the intuitively appealing property that for better fundamentals fewer

investors run. However, the smooth response to changes in the *ex ante* expectation of the fundamental in Morris-Shin models bars sharp declines in investments, i.e. sharp rises in collective withdrawals, which is one of the appealing results of Diamond-Dybvig models.

### 2.3. The Liquidity Run Model with Endogenous Information Acquisition

The Diamond-Dybvig model and the Morris-Shin model show that the outcome of the coordination problem crucially depends on the information structure. Conversely, imposing a particular information structure determines the nature of the outcome. Endogenizing the information structure clearly avoids this exogenous selection of equilibria.

With endogenous information acquisition, it is not up to the modeler to decide whether or not investors are endowed with private information. In the model, investors decide themselves whether or not to acquire private information. More precisely, before making the investment decision in period 1, but after learning  $\hat{\theta}$  and  $\alpha$ , investors simultaneously decide upon acquiring private noisy information about the realization  $\theta$  of the fundamental. Hence, investors consider acquiring information for the same reason that they consider running. When investor  $i$  acquires information, she receives a signal  $x_i = \theta + \varepsilon_i$ , where, as before, the noise  $\varepsilon_i$  is drawn from a normal distribution with zero mean and precision  $\beta$ . In this section  $\beta$  is fixed, identical for all investors and public knowledge. In Section 2.7 investors can choose the precision of the information they acquire. An investor acquires information when the added value in terms of the investment return is higher than the cost of the information. The cost is assumed to be exogenously determined. It can be seen as a purely monetary cost of information, but it may also reflect the efforts needed to collect the information. The exogenous cost captures the fact that the investor should spend time or resources in order to obtain more precise information about the return, so that she strategically chooses the quality of her information. An investor does not know the decisions of other investors about the information acquisition when she makes her investment decision.

Throughout the paper the focus is on symmetric equilibria; non-symmetric equilibria are discussed in Appendix 2.A. For explanatory convenience the following rule

is used when there is a mass zero of indifferent investors: an investor stays if she is indifferent between staying and running and she acquires information if she is indifferent between acquiring and not acquiring. This assumption is not essential for the results.

When investors do not acquire information, the Diamond-Dybvig world arises. Let the profiles  $S$  and  $R$  now refer to “no-information/all-stay” and “no-information/all-run” respectively. Similarly, let the profile  $I$  now refer to “information/all-use-switching-point- $x^*$ ”, which represents the Morris-Shin world that arises when investors acquire information. In the previous section it was shown for a fixed information structure when these profiles are equilibria. When the information structure is endogenous, there is an additional decision node at which profitable deviations may occur. Hence, equilibria for a fixed information structure are not necessarily equilibria when information is endogenously acquired. In order to determine whether a profile is an equilibrium, the value of information needs to be contrasted with its cost.

#### 2.4. The Value of Information

Loosely speaking, the value of information is the difference of the expected return with and without information, or, in other words, the maximum amount an investor is willing to pay for that information. In order to get a more precise definition, let for given private information and candidate equilibrium the function  $r : \emptyset \cup \mathbb{R} \times \{I, R, S\} \rightarrow \mathbb{R}$  denote the expected return for an investor who reacts optimally to the strategies of the other investors. The profiles  $I$ ,  $R$  and  $S$  are thus loosely interpreted as prescribing the strategies of all investors but one. For  $Q \in \{I, R, S\}$  the expected return is then defined as

$$r(\emptyset, Q) = \max\{0, \mathbb{E}[\theta - \ell|Q]\},$$

$$r(x_i, Q) = \max\{0, \mathbb{E}[\theta - \ell|x_i, Q]\}.$$

Information that is not yet known is denoted with a capital letter, so the random variable  $X_i$  denotes the unrevealed information of investor  $i$ . Since an investor does not know the realization of the fundamental, she perceives the information  $X_i$  as having a normal distribution with mean  $\hat{\theta}$  and precision  $(1/\alpha + 1/\beta)^{-1} = \alpha\beta/(\alpha + \beta)$ . The value of private information  $v^Q$  is now the difference of the expected return with

and without private information, so

$$v^Q = \mathbb{E}[r(X_i, Q)] - r(\emptyset, Q).$$

In the remainder of this section detailed expressions for the value of information are derived for the three candidate equilibria.

First consider the  $S$  profile where all investors stay. The analysis can be restricted to non-negative *ex ante* expectations of the fundamental,  $\hat{\theta} \geq 0$ , since the discussion of the model without private information already showed that this condition is necessary for the profile to be an equilibrium. Because it is necessarily (weakly) optimal for an investor without information to stay, this directly implies  $r(\emptyset, S) = \hat{\theta}$ . Suppose investor  $i$  deviates by acquiring private information and that she receives information  $x_i$ . Bayesian updating shows that  $\theta|x_i$ , the expected fundamental conditional on information  $x_i$ , has a normal distribution with mean  $(\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta)$  and precision  $\alpha + \beta$ . Hence, the expected return equals  $r(x_i, S) = \max\{0, (\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta)\}$ . This shows that the investor will invest if and only if it leads to a positive expected return, so if  $x_i \geq -\alpha\hat{\theta}/\beta$ . The value of information in the  $S$  profile as function of the *ex ante* expectation of the fundamental  $\hat{\theta} \geq 0$  is thus given by

$$\begin{aligned} v^S(\theta) &= \int_{-\frac{\alpha}{\beta}\hat{\theta}}^{\infty} \frac{\alpha\hat{\theta} + \beta x_i}{\alpha + \beta} \frac{\sqrt{\frac{\alpha\beta}{\alpha+\beta}}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\alpha\beta}{\alpha+\beta}(x_i-\hat{\theta})^2} dx_i - \hat{\theta} \\ &= -\hat{\theta}\Phi\left(-\sqrt{\frac{\alpha+\beta}{\beta/\alpha}}\hat{\theta}\right) + \frac{\sqrt{\frac{\beta/\alpha}{\alpha+\beta}}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\alpha+\beta}{\beta/\alpha}\hat{\theta}^2}, \end{aligned} \quad (1)$$

where  $\Phi$  denotes the normal cumulative density function.

The value of information in the  $R$  profile where all investors run follows in a similar way. The analysis can be restricted to  $\hat{\theta} \leq 1$ . Since without private information an investor should withdraw her money, this implies  $r(\emptyset, R) = 0$ . Suppose that investor  $i$  has acquired information  $x_i$  so that the expected return equals  $r(x_i, R) = \max\{0, (\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta) - 1\}$ . The negative externality of the withdrawing investors makes her invest if and only if  $x_i \geq (\alpha + \beta)/\beta - \alpha\hat{\theta}/\beta$ . The value

of information in the  $R$  profile is then

$$\begin{aligned} v^R(\hat{\theta}) &= \int_{\frac{\alpha+\beta}{\alpha}-\frac{\alpha}{\beta}\hat{\theta}}^{\infty} \left( \frac{\alpha\hat{\theta} + \beta x_i}{\alpha + \beta} - 1 \right) \frac{\sqrt{\frac{\alpha\beta}{\alpha+\beta}}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\alpha\beta}{\alpha+\beta}(x_i-\hat{\theta})^2} dx_i \\ &= (\hat{\theta} - 1)\Phi \left( \sqrt{\frac{\alpha + \beta}{\beta/\alpha}} (\hat{\theta} - 1) \right) + \frac{\sqrt{\frac{\beta/\alpha}{\alpha+\beta}}}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{\alpha+\beta}{\beta/\alpha}(\hat{\theta}-1)^2}. \end{aligned} \quad (2)$$

Additional intuition about the value of information can be obtained by letting  $Y_i = (\alpha\hat{\theta} + \beta X_i)/(\alpha + \beta) - 1$  denote the conditional expected return of staying in the  $R$  profile. The value of information is then simply  $v^R(\hat{\theta}) = \mathbb{E}[\max\{0, Y_i\}]$ . This shows that the value of information reflects the value of a (European) call option with strike price 0. Intuitively, without information an investor will withdraw her money, so when she acquires information she has acquired an option to remain invested, which she will only exercise if her information is good. For the  $S$  profile where everyone stays, the conditional expected return equals  $Z_i = (\alpha\hat{\theta} + \beta X_i)/(\alpha + \beta)$ . The value of information is given by  $v^S(\hat{\theta}) = \mathbb{E}[\max\{0, Z_i\}] - \hat{\theta}$ . By writing  $\hat{\theta} = \mathbb{E}[Z_i]$  it follows that  $v^S(\hat{\theta}) = \mathbb{E}[\max\{-Z_i, 0\}]$ . Hence in the  $S$  profile the value of information reflects the value of a put option with strike price 0. Intuitively, although it seems that upon acquiring information an investor gets a call option, which reflects that she will only stay in case of good information, she will also lose a share, which reflects that without information she stays. The information makes a difference only for bad information, hence the put option structure. In fact, this resembles the put-call parity, which states that the value of a call option (plus the strike price which is here 0) equals the value of a put option plus a share.

Now consider the Morris-Shin type  $I$  profile where all investors acquire information and take the investment decision according to a switching point strategy. In contrast with the  $S$  and  $R$  profiles, the value of information in the  $I$  profile does not only come from the ability to better discriminate between good and bad realizations of the fundamental. In the  $I$  profile investors base their decision whether to run or not on their information. Since private information makes predictions about the private information of other investors more precise, it is thus also useful for predicting the fraction of withdrawing investors.

When investor  $i$  has no information, her expectation of the fundamental is  $\hat{\theta}$ . The law of large numbers (see Judd (1985)) can be applied to show that the fraction of

withdrawing investors is equal to the probability that investor  $j$  receives information that is worse than the switching point. The expected return is thus

$$r(\emptyset, I) = \max \left\{ 0, \hat{\theta} - \Phi \left( \sqrt{\frac{\alpha\beta}{\alpha+\beta}} (x^* - \hat{\theta}) \right) \right\}. \quad (3)$$

Now suppose that investor  $i$  decides to acquire information and receives  $x_i$ . Above it was shown that  $\theta|x_i$  has a normal distribution with mean  $(\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta)$  and precision  $\alpha + \beta$ . Since  $X_j|x_i = (\theta + \varepsilon_j)|x_i = \theta|x_i + \varepsilon_j$ , it is clear that  $X_j|x_i$  has a normal distribution with the same mean but with precision  $(1/(\alpha + \beta) + 1/\beta)^{-1} = \beta(\alpha + \beta)/(\alpha + 2\beta)$ . The expected return of staying conditional on  $x_i$  is then given by

$$\mathbb{E}[\theta - \ell|x_i, I] = \frac{\alpha\hat{\theta} + \beta x_i}{\alpha + \beta} - \Phi \left( \sqrt{\frac{\beta(\alpha + \beta)}{\alpha + 2\beta}} \left( x^* - \frac{\alpha\hat{\theta} + \beta x_i}{\alpha + \beta} \right) \right).$$

Note that the expected return of staying is strictly increasing in  $x_i$ . Intuitively, when investor  $i$  receives better information she expects a better realization of the fundamental and a smaller fraction of withdrawing investors. The definition of the switching point gives that  $\mathbb{E}[\theta - \ell|x^*, I] = 0$ . Hence, investors only stay if their information is weakly better than  $x^*$ . The expected return when an investor has yet unknown information thus equals

$$\mathbb{E}[r(X_i, I)] = \int_{x^*}^{\infty} \mathbb{E}[\theta - \ell|x_i, I] \frac{\sqrt{\frac{\alpha\beta}{\alpha+\beta}}}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{\alpha\beta}{\alpha+\beta} (x_i - \hat{\theta})^2} dx_i. \quad (4)$$

Equations (3)-(4) give a detailed expression for the value of information in the  $I$  profile.

## 2.5. The Value of Information and the Fundamentals

The value of information in each profile depends on the *ex ante* expectation of the fundamental. Understanding this relation is key in understanding how the *ex ante* expectation of the fundamental  $\hat{\theta}$  affects the behavior of the investors. Intuitively, whether a higher *ex ante* expectation of the fundamental implies a higher or a lower value of information depends on whether the uncertainty about the sign of the return is increased or decreased. The following proposition quantifies this claim. Apart from relating the values of information in the three profiles to  $\hat{\theta}$ , their mutual relations are clarified. To guarantee that the  $I$  profile is well defined, it is assumed that  $\alpha$  and  $\beta$  are such that  $\gamma < 2\pi$ .

## PROPOSITION 1.

Assume that  $\gamma < 2\pi$ .

- i) In profile  $Q \in \{I, R, S\}$  the value of information  $v^Q$  is positive.
- ii) The value of information as function of the ex ante expectation of the fundamental  $\hat{\theta}$  in the  $I$  profile is strictly increasing for  $\hat{\theta} < \frac{1}{2}$  and strictly decreasing for  $\hat{\theta} > \frac{1}{2}$ , in the  $R$  profile it is strictly increasing and in the  $S$  profile it is strictly decreasing.
- iii) There exists a threshold  $\bar{\theta} \in (0, \frac{1}{2})$  such that if  $\hat{\theta} \leq \bar{\theta}$  the value of information is highest in the  $S$  profile, if  $\hat{\theta} \in [\bar{\theta}, 1 - \bar{\theta}]$  it is highest in the  $I$  profile and if  $\hat{\theta} \geq 1 - \bar{\theta}$  it is highest in the  $R$  profile.
- iv) For bad ex ante expectations of the fundamental,  $\hat{\theta} \leq \frac{1}{2}$ , the value of information is lowest in the  $R$  profile, while for good ex ante expectations of the fundamental,  $\hat{\theta} \geq \frac{1}{2}$ , it is lowest in the  $S$  profile.
- v) The values of information in the  $R$  and  $S$  profiles are symmetrical images around  $\hat{\theta} = \frac{1}{2}$  in the sense that  $v^R(\frac{1}{2} - \delta) = v^S(\frac{1}{2} + \delta)$ ,  $\delta \geq -\frac{1}{2}$ . The value of information in the  $I$  profile is symmetrical around  $\hat{\theta} = \frac{1}{2}$  in the sense that  $v^I(\frac{1}{2} - \delta) = v^I(\frac{1}{2} + \delta)$ ,  $\delta \geq 0$ .

From the proposition it follows that the main properties of the values of information  $v^I(\hat{\theta})$ ,  $v^R(\hat{\theta})$  and  $v^S(\hat{\theta})$  are as depicted in Figure 1.

Statement i) claims that information always has a positive value. This implies that there always exist strictly positive costs for which the value of information is higher than its cost.

When the *ex ante* expectation of the fundamental increases, according to statement ii) the value of information in the  $S$  profile decreases. Intuitively this follows from the fact that when  $\hat{\theta}$  increases, the probability of making a loss in the  $S$  profile decreases; hence the maximum cost an investor is willing to pay for receiving information decreases. A symmetric argument shows that the foregone positive return in the  $R$  profile increases when  $\hat{\theta}$  increases and that the value of information thus also increases. Statement ii) also shows that  $v^I$  behaves similarly to  $v^R$  for  $\hat{\theta} < \frac{1}{2}$  and similarly to  $v^S$  for  $\hat{\theta} > \frac{1}{2}$ . When  $\hat{\theta} < \frac{1}{2}$  an investor in the  $I$  profile would run when she has no information (this is proved in Lemma 6 in Appendix 2.B). The foregone positive return increases when  $\hat{\theta}$  becomes larger, so the maximum cost that an investor is willing to pay for information increases. When  $\hat{\theta} > \frac{1}{2}$  an investor



would stay when she has no information. The expected return is increasing in  $\hat{\theta}$  and does so faster than the expected return if she had had information. Hence, the maximum cost an investor is willing to pay for information decreases.

Statement ii) shows again the close link with option pricing. In the  $R$  profile the value of information reflects a call option with strike price 0. Clearly, when  $\hat{\theta}$  increases, the option will be exercised more often and the value of the information increases. Similarly, an increase in  $\hat{\theta}$  makes a put option with strike price 0 less valuable, which shows why the value of information decreases in the  $S$  profile. Since without private information an investor should run in the  $I$  profile for  $\hat{\theta} < \frac{1}{2}$  and stay for  $\hat{\theta} > \frac{1}{2}$ , the value of information reflects a call option in the first case and a put option in the latter.

When  $\hat{\theta}$  is relatively low, according to statement iii) information in the  $S$  profile is most valuable. Intuitively, although the expected return is positive, it is not unlikely that the realized return will be negative. Investors want to be able to run in these cases, which makes the information valuable. Similarly, when the fundamental is likely to be good, the expected return of the investment in the  $R$  profile is negative but there are many realizations of the fundamental for which it is positive. Investors want to be able to distinguish these cases, which gives information a high value. Finally, when  $\hat{\theta}$  is intermediate in the  $I$  profile, the expected fraction of withdrawing investors is also intermediate. Information has a high value since it enables investors to predict which of the two is largest.

The value of information in the  $I$  profile is never the lowest of the three, as claimed in statement iv). The intuition of why for  $\hat{\theta} \leq \frac{1}{2}$  the value of information in the  $R$  profile is lower than in the  $I$  profile is straightforward. Since in the latter there is always a positive fraction of investors who stay, the expected return of staying will always be higher than in the  $R$  profile.

Finally, statement v) shows that the  $R$  and  $S$  profiles are not only symmetrical in the sense that in the one all investors stay and in the other all investors run, but that this symmetry goes further:  $v^R(\frac{1}{2} - \delta) = v^S(\frac{1}{2} + \delta)$ . The intuition is as follows. In the  $S$  profile an investor without information stays, while with information she can run for very bad information. The value of information is thus minus the expected return in case of bad information. In the  $R$  profile the value of information is the expected return in the case of good information. For fundamentals that are  $\frac{1}{2} + \delta$

and  $\frac{1}{2} - \delta$  respectively, these values are the same. Statement v) also shows that  $v^I(\frac{1}{2} - \delta) = v^I(\frac{1}{2} + \delta)$ . Although somewhat more involved, the intuition is as before: for  $\hat{\theta} = \frac{1}{2} - \delta$  information allows investors to stay when positive returns are expected, while for  $\frac{1}{2} + \delta$  it allows investors to run when negative returns are expected.

## 2.6. Equilibrium Implications of Endogenous Information Acquisition

Compared to a fixed information structure, endogenous information acquisition imposes an additional equilibrium condition. The  $R$  profile where all investors run without acquiring information can only be an equilibrium if the value of information is *lower* than its price. This should also be the case for the  $S$  profile where all investors stay without acquiring information to be an equilibrium. However, the  $I$  profile, where all investors acquire information and use a switching point strategy, is an equilibrium if the value of information is *higher* than its price.

The existence of an equilibrium for all expected values of the fundamental now follows from the fact that  $v^I(\hat{\theta}) > \min\{v^R(\hat{\theta}), v^S(\hat{\theta})\}$  (statement iv) of Proposition 1). The intuition for  $\hat{\theta} \leq \frac{1}{2}$  is as follows. In this case an investor without information in the  $I$  profile should withdraw her money. The difference between  $v^I(\hat{\theta})$  and  $v^R(\hat{\theta})$  is thus only caused by different returns when information is acquired. In the  $I$  profile there are always some investors who stay. Conditional on the same private information, the expected return in the  $I$  profile is higher than in the  $R$  profile. An investor with private information is more willing to stay in the  $I$  profile than in the  $R$  profile (her switching point is lower). The conclusion is that strategic complementarities in information acquisition ensure the existence of an equilibrium.

Let  $c$  denote the cost of information. The following corollary follows directly from Proposition 1 and reconciles the Diamond-Dybvig and the Morris-Shin world in a formal way. Figure 1 provides a generic graphical example.

### COROLLARY 2.

*Assume that  $\gamma < 2\pi$ .*

*i) The  $I$  profile is the unique equilibrium if and only if  $c < \min\{v^R(\hat{\theta}), v^S(\hat{\theta})\}$ . The  $R$  and  $S$  profiles are the only equilibria candidates if and only if  $c > v^I(\hat{\theta})$ .*

*ii) The  $R$  profile is the unique equilibrium if and only if both  $c > v^I(\hat{\theta})$  and in addition  $\hat{\theta} \geq 0$  implies  $c < v^S(\hat{\theta})$ .*

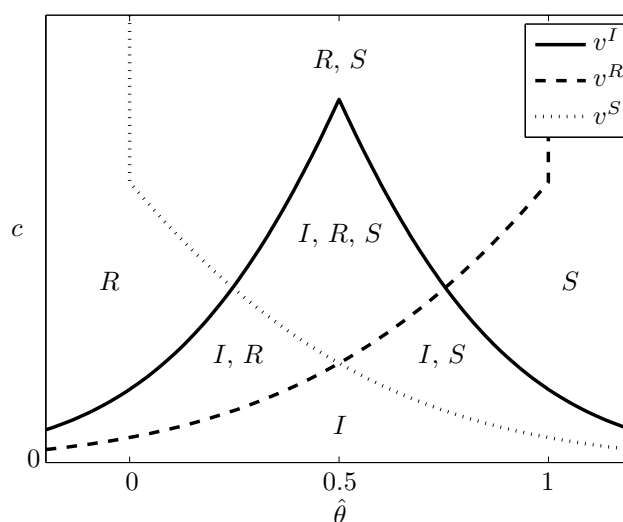


FIGURE 1. The value of information as function of the *ex ante* expectation of the fundamental and the implied equilibrium regions.<sup>4</sup>  
 $(\alpha = 1, \beta = 1)$

The *S* profile is the unique equilibrium if and only if both  $c > v^I(\hat{\theta})$  and in addition  $\hat{\theta} \leq 1$  implies  $c < v^R(\hat{\theta})$ .

iii) The sets  $\{(\hat{\theta}, c) \in [0, 1] \times (0, \infty) | Q \text{ is the unique equilibrium}\}$ ,  $Q \in \{I, R, S\}$ , are non-empty.

In Appendix 2.A it is proved that if the *I*, *R* or *S* profile is the unique equilibrium among symmetric profiles, it is also the unique equilibrium if non-symmetric profiles are allowed for.

Statement i) relates the Diamond-Dybvig and the Morris-Shin world. If the cost of information is sufficiently high, investors do not acquire private information and the Diamond-Dybvig world arises. Only the *R* equilibrium and the *S* equilibrium can exist. If the cost is sufficiently low, the Morris-Shin world arises where investors have private information and the *I* profile is the unique equilibrium. From statement i) of Proposition 1 it follows that  $v^R(\hat{\theta})$  and  $v^S(\hat{\theta})$  are strictly positive for every  $\hat{\theta}$ . This implies that for every  $\hat{\theta}$  there are positive costs for which the *I* equilibrium is unique. However, although there are finite costs which lead to the Diamond-Dybvig world regardless of the *ex ante* expectation of the fundamental - in fact any cost

<sup>4</sup>The vertical lines starting at  $(0, v^S(0))$  and  $(1, v^R(1))$  are included to indicate that for costs higher than  $v^S(0) = v^R(1)$  the *S* and *R* equilibrium cannot exist for  $\hat{\theta} < 0$  and  $\hat{\theta} > 1$  respectively.

higher than  $v^I(\frac{1}{2})$  achieves this - only freely available private information leads to the Morris-Shin world for sure.

The model with endogenous information acquisition is accomplishing more than embedding the two models. While in the original Diamond-Dybvig the  $R$  equilibrium exists for  $\hat{\theta} \leq 1$  and the  $S$  equilibrium for  $\hat{\theta} \geq 0$ , statement ii) shows that the possibility of acquiring information reduces the equilibrium regions.

Statement iii) emphasizes the implications of the first two statements for intermediate *ex ante* expectations of the fundamentals. For  $\hat{\theta} \in [0, 1]$  the  $R$  and  $S$  profiles are both equilibria if private information is not available. If information can be acquired, however, the  $I$  profile is the unique equilibrium for sufficiently low (but still positive) costs. Also, statements iii) and iv) of Proposition 1 show that there exists a threshold  $\bar{\theta} \in (0, \frac{1}{2})$  such that  $\hat{\theta} \in (1 - \bar{\theta}, 1]$  implies  $v^R(\hat{\theta}) > v^I(\hat{\theta}) > v^S(\hat{\theta})$ . Hence, for  $\hat{\theta} \in (1 - \bar{\theta}, 1]$  and  $c \in (v^I(\hat{\theta}), v^R(\hat{\theta}))$  the  $S$  profile is the unique equilibrium. Similarly, the  $R$  profile is the unique equilibrium for  $\hat{\theta} \in [0, \bar{\theta})$  and  $c \in (v^I(\hat{\theta}), v^S(\hat{\theta}))$ . Compared to the model without private information the  $R$  equilibrium is eliminated for  $\hat{\theta}$  close to 1, while the  $S$  equilibrium is eliminated when  $\hat{\theta}$  is close to 0. It is the intuitively more appealing equilibrium that survives:  $R$  when the *ex ante* expectation of the fundamental is bad,  $S$  when it is good.

It is interesting to look more carefully at the two ways the multiplicity of the original Diamond-Dybvig model with  $\hat{\theta} \in [0, 1]$  disappears for some combinations of  $\hat{\theta}$  and  $c$ . First, for low costs of information investors *have* private information, which replaces the multiple equilibria with a unique hybrid switching point equilibrium. Ruling out multiplicity was in fact the very reason that private information was introduced in Morris and Shin (2001). Second, for low but not very low costs and  $\hat{\theta}$  close to 0 or 1, the original multiplicity disappears since investors *can have* private information. It is the sheer possibility of being able to acquire information that eliminates one of the equilibria.

For all  $\hat{\theta}$  there are intermediate costs such that the Diamond-Dybvig and Morris-Shin worlds are blended. For example, when  $\hat{\theta} \leq \frac{1}{2}$  both the  $I$  and the  $R$  equilibrium occur when the cost is between  $v^R(\hat{\theta})$  and  $v^I(\hat{\theta})$ . For  $\hat{\theta}$  close to  $\frac{1}{2}$  even the  $S$  equilibrium joins and the multiplicity increases. Hence, sunspots are not ruled out when the cost is not convincingly low or high. But the jump is not necessarily

extreme in the sense that all investors suddenly change behavior. The hybrid  $I$  equilibrium where some investors stay and others run can smooth a jump.

The model with endogenous information acquisition presents a partitioning of the *ex ante* expectation of the fundamental in two dimensions. Liquidity run models without private information typically have a one dimensional partitioning where a run occurs when  $\hat{\theta}$  is bad, no run occurs when  $\hat{\theta}$  is good, and both a run or no run can occur for intermediate  $\hat{\theta}$ . Liquidity run models with private information typically have the trivial partitioning of a unique hybrid equilibrium where a fraction of the investors runs for all  $\hat{\theta}$ . The extent of the run is then decreasing in  $\hat{\theta}$ . In the model of this paper these one-dimensional partitionings arise for high costs ( $c > v^I(\frac{1}{2})$ ) or for free private information ( $c = 0$ ). In general, the partitioning of  $\hat{\theta}$  depends on the cost of information and concerns three different equilibria. The *ex ante* expectation of the fundamental and the cost together determine whether or not a run occurs and its extent.

At this point, it should be noted that even if the investors initially have imprecise *private* information about the realization of the fundamental, the same intuition applies. When the precision of the initial private information is sufficiently low and no additional information can be acquired, the Diamond-Dybvig profiles are replaced by two Morris-Shin profiles. These profiles are the unique stable equilibria candidates in switching point strategies. One profile is close to the  $R$  profile and has a very large fraction of investors running; the other profile resembles the  $S$  profile and has a very large fraction of investors staying. When the precision of the initial information is very low, only a small fraction of investors will acquire additional private information, which shows that these equilibrium candidates are very close to the equilibrium candidates of the model without initial information. The original Morris-Shin  $I$  profile is replaced by a profile where some investors with very bad or very good initial private information do not acquire additional information. The same reasoning as in the model without initial private information applies to determine which of these profiles are equilibria.

In the original Diamond-Dybvig model with  $\hat{\theta} \in [0, 1]$ , neither the *ex ante* expectation of the fundamental nor its precision plays a role in the equilibrium selection. However, in the model of this paper, besides  $\beta$  also  $\alpha$  has an effect on the maximum costs due to the availability of private information. Since for identical costs a higher

precision makes acquiring private information more attractive, an increase in  $\beta$  can be expected to have the same effect as a decrease in the cost, which suggests that the value of information increases. When  $\alpha$  decreases, the relative importance of the private information increases, so intuitively the effects are similar to an increase in  $\beta$ . The following proposition makes this precise.

**PROPOSITION 3.**

*Assume that  $\gamma < 2\pi$ . The value of information in the  $I$ ,  $R$  and  $S$  profiles is decreasing in  $\alpha$  and increasing in  $\beta$ .*

When the precision of the fundamental decreases or the precision of private information increases, an investor with information is better able to distinguish the cases where the return will be positive from the cases where it will be negative. This ensures that in the  $R$  and the  $S$  profiles she will expect a higher return which increases the value of information. Also in the  $I$  profile an increase in the precision of private information or a decrease in the precision of the fundamental has the same effect. For *ex ante* bad fundamentals ( $\hat{\theta} < \frac{1}{2}$ ) the intuition is straightforward: better information is positive since it allows investors to better distinguish good *ex post* fundamentals, which makes investors more willing to stay. The expected return when acquiring information thus increases while without information the investor will still withdraw her money. This shows that the value of information increases. For *ex ante* good fundamentals ( $\hat{\theta} > \frac{1}{2}$ ) the intuition is not so clear: investors can also better distinguish the cases of good and bad realizations of the fundamentals. Since without private information it is optimal to stay, investors are now more inclined to run. This reduces the expected return when acquiring information, but it also reduces the expected return when not acquiring information. The symmetry of  $v^I$  around  $\hat{\theta} = \frac{1}{2}$  shows that the second effect is stronger.

Combining Corollary 2 and Proposition 3 gives insight into how the equilibria depend on the precisions of the private information and the fundamental. When the project is likely to have a bad fundamental,  $\hat{\theta} < \frac{1}{2}$ , relatively cheap information with a high precision will help to attract investors. The reason is straightforward. Information with a high precision has a high value. When its cost is low, investors are inclined to acquire information. This is beneficial for the existence of the  $I$  equilibrium but not for the existence of the  $R$  equilibrium. A low volatility of the

fundamental has an opposite effect. When a bad fundamental is likely, a low volatility of the fundamental makes a bad realization very likely. Information is not very useful in the  $R$  profile so the  $R$  equilibrium will probably exist. A very volatile fundamental makes private information very useful so that the  $I$  profile can be expected to be an equilibrium. A decrease in  $\alpha$  and an increase in  $\beta$  thus favor information acquisition. When the cost is not too high, the information acquisition favors the  $I$  equilibrium at the expense of the  $R$  equilibrium. Since in the  $I$  equilibrium some investors stay, the run is less severe than in the  $R$  equilibrium. This reflects the effect of a decrease in  $\alpha$  or an increase in  $\beta$  in the  $I$  equilibrium itself, where investors are more willing to stay when private information becomes relatively more important (see the proof of the proposition).

When the *ex ante* expectation of the fundamental is relatively good,  $\hat{\theta} > \frac{1}{2}$ , a decrease in  $\alpha$  and an increase in  $\beta$  favor information acquisition for the same reason. However, the effect on the run is opposite. When the cost is not too high, the information acquisition favors the  $I$  equilibrium at the expense of the  $S$  equilibrium. More investors withdraw their money and the run is more severe. Again this reflects the effects inside the  $I$  equilibrium. That an increase in the relative precision of private information has opposite effects for good and bad *ex ante* expectations of the fundamentals is a common feature in the global games literature; see, for example, Metz (2002) and Sbracia and Zaghini (2001). Prati and Sbracia (2002) provide empirical evidence for this prediction.

For global game models where agents have private information, it is common practice to discuss the limiting case where the private information becomes arbitrarily precise. Combining Corollary 2 and Proposition 3 shows that when  $\beta$  increases towards infinity, the cost range for which the  $I$  profile is unique expands. For this limiting case the constraint on the cost of information becomes less severe, and for not too high costs the agents will indeed acquire private information.<sup>5</sup>

## 2.7. Information with Endogenous Precision

In this section the restriction that investors can only acquire information with an exogenously given precision  $\beta$  is loosened. The precision is endogenized by letting

---

<sup>5</sup>The limiting case where  $\alpha \rightarrow 0$  does not provide additional insights. When the fundamental has almost the improper uniform distribution on the real line, the potential profit goes to infinity, and so does the value of information.

investors choose their preferred precision. The cost of information is a linear function of its precision, so information with precision  $\beta$  costs  $\beta\hat{c}$ , where  $\hat{c} > 0$  denotes the cost of information per unit precision. Modelling the information acquisition in this way reflects the possibility of investors buying  $\beta \in [0, \infty)$  units of information with unit precision and cost  $\hat{c}$  each. Note that the cost of information is a convex function of the variance: information with half the variance costs double. Qualitatively similar results will be obtained for other convex pricing schemes that increase in the precision.

Given the behavior of the other investors, investor  $i$  chooses the precision  $\beta$  that maximizes the expected return minus the cost of the acquired information. The expected return conditional on information  $x_i$  with precision  $\beta$  is denoted by  $r(x_i, Q; \beta)$ , where the profile  $Q \in \{I, R, S\}$  captures the behavior of the other investors. Since not having private information is identical to having unrelated information, which is information with zero precision, notation is slightly abused by letting  $\beta = 0$  refer to this case (i.e.  $\mathbb{E}[r(X_i, Q; 0)] = \mathbb{E}[r(\emptyset, Q)]$ ). The problem investor  $i$  faces is then<sup>6</sup>

$$\max_{\beta \geq 0} \mathbb{E}[r(X_i, Q; \beta)] - \beta\hat{c}.$$

First, consider for  $\hat{\theta} \geq 0$  the  $S$  profile where all investors stay. The expected return if no information is acquired is simply  $\hat{\theta}$ , while for  $\beta > 0$  an expression follows from Equation (1) by adding  $\hat{\theta}$ . The  $S$  profile is an equilibrium if investor  $i$  prefers not to acquire information. Instead of analyzing the new investors' problem to find the maximum cost per unit precision  $\hat{c}$  for which investor  $i$  acquires information, the relation with the baseline model is exploited. In Section 2.4 the maximum cost an investor is willing to pay for information with precision  $\beta$  is derived. To explicitly indicate the dependence on  $\beta$  this maximum cost is denoted here by  $v^S(\hat{\theta}; \beta)$ . Hence, when the cost per unit precision equals  $v^S(\hat{\theta}; \beta)/\beta$  investor  $i$  acquires information. The standardized value of information for which an investor wants to acquire information is then given by  $\hat{v}^S(\hat{\theta}) = \max_{\beta > 0} v^S(\hat{\theta}; \beta)/\beta$ . When the cost of information per unit precision is higher than the standardized value of information, an investor will not acquire information and hence the  $S$  profile is an equilibrium.

---

<sup>6</sup>Technical details are deferred to the proof of Proposition 4 in the appendix.



Since Proposition 1 states that all  $v^S(\hat{\theta}; \beta)/\beta$  decrease in  $\hat{\theta}$ , their upper envelope  $\hat{v}^S$  is decreasing in  $\hat{\theta}$  as well.

Now consider for  $\hat{\theta} \leq 1$  the  $R$  profile where all investors run. In the same way as for the  $S$  profile it follows that the standardized value of information is given by  $\hat{v}^R(\hat{\theta}) = \max_{\beta > 0} v^R(\hat{\theta}; \beta)/\beta$ . The  $R$  profile is only an equilibrium when investor  $i$  does not acquire information, i.e. when the cost of information per unit precision is higher than the standardized value of information. From Proposition 1 it follows that  $\hat{v}^R$  is increasing in  $\hat{\theta}$  and that  $\hat{v}^S$  and  $\hat{v}^R$  are symmetric images around  $\hat{\theta} = \frac{1}{2}$ .

In the  $I$  profile all investors acquire information. Assume that all investors different from  $i$  choose precision  $\beta_j > 0$  and act according to a switching point strategy where the switching point  $x_j^*$  satisfies  $\mathbb{E}[\theta - \ell|x_j^*] = 0$ . The uniqueness condition is not satisfied for very small values of  $\beta_j$ . For these precisions the set of allowed strategies is restricted to switching point strategies.<sup>7</sup> In a symmetric equilibrium, the  $\beta$  that maximizes investor  $i$ 's problem should be equal to  $\beta_j$ . An expression for the expected return can be obtained in a similar way as for the baseline model. Unfortunately, the maximization problem cannot be analytically solved. As a consequence, also the maximum cost per unit precision for which the  $I$  profile is an equilibrium can only be obtained numerically. This maximum cost is denoted by  $\hat{v}^I$  and referred to as the standardized value of information.

In Figure 2 the standardized value of information for all three profiles is shown as function of the *ex ante* expectation of the fundamental. The following proposition states that for all  $\alpha$  and  $\beta$  the standardized value of information as function of  $\hat{\theta}$  in the  $R$  and the  $S$  profiles behave as depicted in the figure, and that the standardized value of information in the  $I$  profile is symmetrical.

**PROPOSITION 4.**

*i) The standardized values of information as function of the ex ante expectation of the fundamental  $\hat{\theta}$  in the  $R$  and the  $S$  profile,  $\hat{v}^R$  and  $\hat{v}^S$ , are strictly increasing respectively decreasing.*

*ii) The standardized value of information is lower in the  $R$  profile than in the  $S$  profile if  $\hat{\theta} < \frac{1}{2}$  and higher if  $\hat{\theta} > \frac{1}{2}$ .*

<sup>7</sup>Alternatively, for a given precision of the fundamental  $\alpha$  the set of allowed private precisions could have been restricted to  $\{0\} \cup (\beta(\alpha), \infty)$ , where  $\beta(\alpha) = (\alpha/8\pi)(\alpha - 2\pi + \sqrt{(\alpha + 2\pi)^2 + 8\pi\alpha}) > 0$  is the unique precision of private information such that  $\gamma = \sqrt{2\pi}$ .

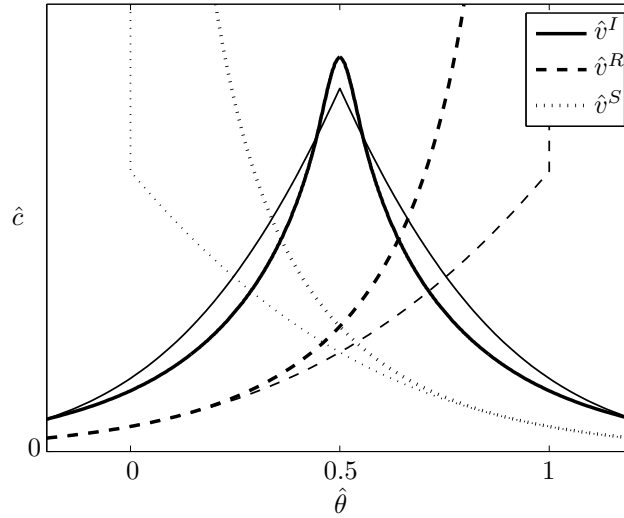


FIGURE 2. The standardized value of information as function of the *ex ante* expectation of the fundamental. ( $\alpha = 1$ , the thin lines show the value of information for  $\beta = 1$ )

*iii) The standardized values of information in the R and S profiles are symmetrical images around  $\hat{\theta} = \frac{1}{2}$  in the sense that  $\hat{v}^R(\frac{1}{2} - \delta) = \hat{v}^S(\frac{1}{2} + \delta)$ ,  $\delta \geq -\frac{1}{2}$ . The standardized value of information in the I profile is symmetrical around  $\hat{\theta} = \frac{1}{2}$  in the sense that  $\hat{v}^I(\frac{1}{2} - \delta) = \hat{v}^I(\frac{1}{2} + \delta)$ ,  $\delta \geq 0$ .*

Comparing this proposition to Proposition 1 shows that the standardized values of information in the *R* and *S* profiles behave similarly to their non-standardized counterparts. The strategic complementarities in information acquisition again imply that the standardized value of information in the *I* profile is higher than in one of the other profiles. Hence, equilibrium existence is guaranteed. Since for all  $\beta$  it is clear that  $\hat{v}^R(\hat{\theta}) \geq v^R(\hat{\theta}; \beta)/\beta$  and  $\hat{v}^S(\hat{\theta}) \geq v^S(\hat{\theta}; \beta)/\beta$ , the regions where the *R* and the *S* equilibria are eliminated are expanded. Loosely speaking, the statements made in Corollary 2 are strengthened if the precision is free to choose. Figure 2 suggests that the statements for the *I* profile are qualitatively the same. Numerical analysis shows that it depends on  $\alpha$  and  $\beta$  whether or not  $\hat{v}^I$  is higher than  $v^I/\beta$  at the tails and/or the peak. However, the form of  $\hat{v}^I$  remains roughly similar. The reason that  $\hat{v}^I$  increases for  $\hat{\theta} < \frac{1}{2}$  and decreases for  $\hat{\theta} > \frac{1}{2}$  is the same as before: the closer the *ex ante* expectation of the fundamental to  $\frac{1}{2}$ , i.e. the more uncertainty

there is about the sign of the return, the more valuable the information is. The conclusion is that the statements of Corollary 2 are robust, if not strengthened, when the investors can choose the precision of their information.

## 2.8. Conclusion

The occurrence and the extent of liquidity runs is affected by endogenous information acquisition. Only when the cost of private information is high or very low, can the information structure be taken as fixed without affecting the results. For intermediate costs the artificial dichotomy “private information/no private information” is too simplistic. This is most clear when priors are intermediate. In this case, regardless of whether the prior is relatively good or bad, both a full run or no run can occur when private information is not available, while a partial run occurs if the fixed information structure contains private information. However, in the model with endogenous information acquisition, no run occurs when the prior is relatively good, whereas a full run occurs when the prior is relatively bad.

Interestingly, the strategic complementarities in the run/stay decision lead to strategic complementarities in the acquire-information/not-acquire-information decision. When investors acquire private information, their behavior becomes less predictable, which increases the uncertainty about the return. This raises the value of information, which in turn justifies the information acquisition. This mechanism ensures the existence of an equilibrium for all parameters. It also implies that the equilibrium areas overlap, so that for some parameters multiple equilibria exist.

A promising direction for future research would be to embed the model in a dynamic context. The unique equilibria for some cost-prior combinations together with the multiple equilibria that occur for other combinations suggest a role for hysteresis. Specifically, for countries with improving fundamentals this implies a lock-in effect since the fundamental has to improve considerably before investors become sufficiently interested to acquire information and consider investing.

## Bibliography

Angeletos, G.-M. and Werning, I. (2006), ‘Crises and Prices: Information Aggregation, Multiplicity and Volatility’, *American Economic Review* **96**(5), 1720–1736.

- Atkeson, A. (2001), Discussion of Morris and Shin's "Rethinking Multiple Equilibria in Macroeconomic Modelling", in B. S. Bernanke and K. Rogoff, eds, 'NBER Macroeconomics Annual 2000', MIT Press, Cambridge, MA.
- Carlsson, H. and van Damme, E. (1993), 'Global Games and Equilibrium Selection', *Econometrica* **61**(5), 989–1018.
- Diamond, D. W. and Dybvig, P. H. (1983), 'Bank Runs, Deposit Insurance, and Liquidity', *Journal of Political Economy* **91**(3), 401–419.
- Heinemann, F., Nagel, R. and Ockenfels, P. (2004), 'The Theory of Global Games on Test: Experimental Analysis of Coordination Games with Public and Private Information', *Econometrica* **72**(5), 1583–1599.
- Hellwig, C. (2002), 'Public Information, Private Information, and the Multiplicity of Equilibria in Coordination Games', *Journal of Economic Theory* **107**, 191–222.
- Hellwig, C., Mukherji, A. and Tsyvinski, A. (2006), 'Self-Fulfilling Currency Crises: the Role of Interest Rates', *American Economic Review* **96**(5), 1769–1787.
- Hellwig, C. and Veldkamp, L. (2006), Knowing What Others Know: Coordination Motives in Information Acquisition, mimeo.
- Judd, K. L. (1985), 'The law of large numbers with a continuum of i.i.d. random variables', *Journal of Economic Theory* **35**(1), 19–25.
- Metz, C. E. (2002), 'Private and Public Information in Self-Fulfilling Currency Crises', *Journal of Economics* **76**(1), 65–85.
- Morris, S. and Shin, H. S. (2001), Rethinking Multiple Equilibria in Macroeconomic Modelling, in B. S. Bernanke and K. Rogoff, eds, 'NBER Macroeconomics Annual 2000', MIT Press, Cambridge, MA.
- Morris, S. and Shin, H. S. (2003), Global Games - Theory and Applications, in M. Dewatripont, L. P. Hansen and S. Turnovsky, eds, 'Advances in Economics and Econometrics: Theory and Applications, 8th World Congress of the Econometric Society', Cambridge University Press, Cambridge.
- Nikitin, M. (2004), Information Acquisition, Coordination, and Fundamentals in a Financial Crisis, mimeo.
- Prati, A. and Sbracia, M. (2002), 'Currency Crises and Uncertainty About Fundamentals', *IMF Working Paper 02/3*.
- Rey, H. (2001), Discussion of Morris and Shin's "Rethinking Multiple Equilibria in Macroeconomic Modelling", in B. S. Bernanke and K. Rogoff, eds, 'NBER Macroeconomics Annual 2000', MIT Press, Cambridge, MA.
- Sbracia, M. and Zaghini, A. (2001), 'Expectations and information in second generation currency crises models', *Economic Modelling* **18**, 203–222.
- Tarashev, N. A. (2007), 'Speculative Attacks and the Informational Role of Interest Rates', *Journal of the European Economic Association* **5**(1), 1–36.

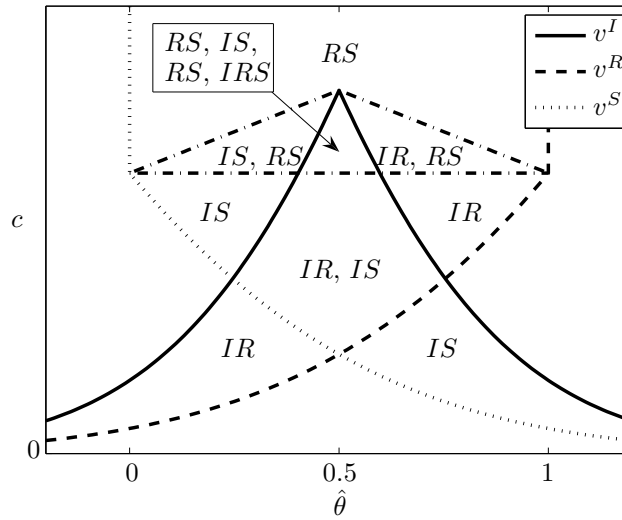


FIGURE 3. The non-symmetrical equilibria regions. ( $\alpha = 1, \beta = 1$ )

**Appendix 2.A. Non-Symmetrical Equilibria**

In the model of Diamond and Dybvig (1983) there exists a non-symmetrical equilibrium (a mixed strategy equilibrium). In the original paper this is not analyzed since “it is not economically meaningful”. Below it is shown that all non-symmetrical equilibria of the model with endogenous information acquisition have similar characteristics. It is still assumed that the private information is sufficiently precise to guarantee  $\gamma < 2\pi$ . If an investor acquires information, the existence of dominant regions then implies that in equilibrium investors with information act according to a switching point strategy.

The discussion of non-symmetrical equilibria below is reflected in Figure 3. Comparing this figure with Figure 1 gives the following proposition.

**PROPOSITION 5.**

*Assume that  $\gamma < 2\pi$ . If there is a unique equilibrium among the symmetrical profiles, then a non-symmetrical equilibrium does not exist.*

The intuition is straightforward: a non-symmetrical equilibrium can only arise if there are at least two equilibria that can be mixed.

In the non-symmetrical equilibrium of the original model with  $\hat{\theta} \in [0, 1]$  some investors run and others stay. In the model with endogenous information acquisition, a fraction  $\hat{\theta}$  of the investors runs and a fraction  $1 - \hat{\theta}$  stays. Denote this

non-symmetrical equilibrium by  $RS$ . Since  $\mathbb{E}[\theta - \ell | RS] = 0$  investors are indeed indifferent between running and staying. This equilibrium is not economically meaningful because the extent of the run is increasing in the *ex ante* expectation of the fundamental. The  $RS$  profile is not an equilibrium for all costs. Since  $\mathbb{E}[\theta - \ell | RS]$  has a normal distribution with zero mean and precision  $\alpha$ , it follows that for all  $\hat{\theta}$  the value of information is given by  $v^S(0) = v^R(1)$ . For costs of at least this value the  $RS$  equilibrium exists.

Denote by  $IR$  a profile where a fraction  $\lambda \in (0, 1)$  of the investors acquires information and has a switching point strategy while the remaining fraction  $1 - \lambda$  runs without acquiring information. Denote by  $r(x_i, IR)$  the expected return given information  $x_i$ . In equilibrium an investor without information should weakly prefer to withdraw her money, so  $\mathbb{E}[\theta - \lambda \ell | IR] - (1 - \lambda) \leq 0$ , and the value of information should equal the cost  $c$ , so  $\mathbb{E}[r(X_i, IR)] = c$ . The switching point  $x^*$  is determined by  $\mathbb{E}[\theta - \lambda \ell | x^*, IR] - (1 - \lambda) = 0$ . An investor who acquires information stays with positive probability while without information she runs for sure. Hence, the fraction of remaining investors is increasing in  $\lambda$ , which implies that the switching point  $x^*$  is decreasing in  $\lambda$ . The expected return given information  $x_i$  in this equilibrium is given by  $r(x_i, IR) = \max\{0, \mathbb{E}[\theta - \lambda \ell | x_i, IR] - (1 - \lambda)\}$ . When  $x^*$  decreases,  $r(x_i, IR)$  is positive for a larger range of private information and for every  $x_i$  in this range the expected fraction of withdrawing investors is smaller. Hence  $\mathbb{E}[r(X_i, IR)]$  is decreasing in  $x^*$  and thus increasing in  $\lambda$ . In the proof of Lemma 6 in Appendix 2.B it is shown that  $x^*$  is decreasing in  $\hat{\theta}$  which implies that  $\mathbb{E}[r(X_i, IR)]$  is increasing in  $\hat{\theta}$ . This shows that for a fixed cost,  $\lambda$  should decrease in  $\hat{\theta}$ . So, when the *ex ante* expectation of the fundamental improves, more investors run. Similarly, when  $c$  increases,  $\lambda$  also increases. Hence, for higher costs more investors want information.

For  $\hat{\theta} \leq \frac{1}{2}$ , the  $IR$  equilibrium only exists in the interior of the region where both the  $I$  and the  $R$  equilibrium exist. Consider  $\hat{\theta} < \frac{1}{2}$ . In the last paragraph it was argued that  $\lambda$  is increasing in  $c$ . There is only one cost for which  $\lambda = 0$ , and this cost is given by  $v^R(\hat{\theta})$ . Similarly,  $v^I(\hat{\theta})$  is the unique cost for which  $\lambda = 1$  and the expected return equals its cost. It follows that only for  $c \in (v^R(\hat{\theta}), v^I(\hat{\theta}))$  does the  $IR$  profile exist with  $\lambda \in (0, 1)$ . When  $\lambda$  increases, the expected fraction of staying investors increases as well, hence the expected return of an investor who does not acquire information and stays is increasing in  $\lambda$ . But Lemma 6 of Appendix 2.B

shows that when  $c = v^I(\hat{\theta})$  an investor without information should run. Thus, also for  $c \in (v^R(\hat{\theta}), v^I(\hat{\theta}))$  an investor without information should run. The conclusion is that for these costs the  $IR$  profile is a non-symmetrical equilibrium.

For  $\hat{\theta} > \frac{1}{2}$  the region where the  $IR$  equilibrium exists is more complex. In the  $I$  equilibrium the expected return given unknown information is  $\mathbb{E}[r(X_i, I)]$ . For  $\hat{\theta} \leq \frac{1}{2}$  this value equals  $v^I(\hat{\theta})$ . Similar reasoning to above now gives that the  $IR$  profile only exists with  $\lambda \in (0, 1)$  if  $c \in (v^R(\hat{\theta}), \mathbb{E}[r(X_i, I)])$ . The expected return of an investor who stays without acquiring information is strictly increasing in  $\lambda$ . For costs close to  $v^R(\hat{\theta})$  it is negative. Similar to the proof that  $v^I(\hat{\theta})$  is increasing in  $\hat{\theta}$  for  $\hat{\theta} < \frac{1}{2}$ , it can be proved that  $\mathbb{E}[r(X_i, I)]$  is increasing in  $\hat{\theta}$ . Since this implies that  $\mathbb{E}[r(X_i, I)] > v^I(\hat{\theta})$  an investor who stays without acquiring information expects a positive return when the cost is close to  $\mathbb{E}[r(X_i, I)]$ . There thus exist a unique  $\bar{c}^{IR}(\hat{\theta}) \in (v^R(\hat{\theta}), \mathbb{E}[r(X_i, I)])$  such that for this cost the expected return of staying without information equals zero. It is clear that  $\bar{c}^{IR}(\frac{1}{2}) = v^I(\frac{1}{2})$  and that  $\bar{c}^{IR}(1) = v^R(1)$  (note that  $v^I(\frac{1}{2}) > v^R(1) = v^S(0)$  since the uncertainty about the behavior of the other investors increases the value of information in the  $I$  equilibrium). The dash-dotted line in Figure 3 for  $\hat{\theta} > \frac{1}{2}$  representing  $\bar{c}$  is found by numerical methods. For  $c \in (v^R(\hat{\theta}), \bar{c}^{IR}(\hat{\theta}))$  the  $IR$  equilibrium exists with  $\lambda \in (0, 1)$ .

Denote by  $IS$  a profile where a fraction  $\lambda \in (0, 1)$  of the investors acquires information and has a switching point strategy while the remaining fraction  $1 - \lambda$  stays without acquiring information. The switching point is then determined by  $\mathbb{E}[\theta - \lambda \ell | x^*] = 0$ . Arguments similar to above show that the switching point  $x^*$  is now increasing in  $\lambda$ . This profile can only be an equilibrium if the cost of information satisfies  $\mathbb{E}[r(X_i, IS)] - r(\emptyset, IS) = c$ . Due to symmetry with the  $IR$  case, the left-hand side is increasing in  $\lambda$ . Also due to symmetry, the left-hand side is decreasing in  $\hat{\theta}$ . An increase in  $\hat{\theta}$  or an increase in  $c$  thus leads to an increase in  $\lambda$ . So, when the *ex ante* expectation of the fundamental improves less investors stay and when the cost of information increases more investors acquire information.

From the symmetry with respect to the  $IR$  profile, it is clear that for  $\hat{\theta} \geq \frac{1}{2}$  the  $IS$  equilibrium only exists in the interior of the region where both the  $I$  and  $S$  equilibrium exist. For  $\hat{\theta} < \frac{1}{2}$ , there exists a  $\bar{c}^{IS}(\hat{\theta})$  such that for this cost the expected return of an investor who stays without acquiring information is zero. It

follows that  $\bar{c}^{IS}(0) = v^R(0)$ ,  $\bar{c}^{IS}(\frac{1}{2}) = v^I(\frac{1}{2})$ . The *IS* equilibrium with  $\lambda \in (0, 1)$  only exists for  $c \in (v^S(\hat{\theta}), \bar{c}^{IS}(\hat{\theta}))$ .

Denote by *IRS* a profile where a fraction of the investors stays without acquiring information, a fraction runs without acquiring information and the remaining investors acquire information. This equilibrium only exists in the interior of the region where the *IR*, the *IS* and the *RS* equilibria exist. So, for  $\hat{\theta} \leq \frac{1}{2}$  only if  $c \in (v^S(0), \bar{c}^{IS}(\hat{\theta}))$  and for  $\hat{\theta} \geq 0$  only if  $c \in (v^R(1), \bar{c}^{IR}(\hat{\theta}))$ . For lower costs more investors want to acquire information, which increases the uncertainty and thus makes information more valuable. For higher costs investors do not want to acquire information. For a worse *ex ante* expectation of the fundamental investors prefer to run, for a better one they prefer to stay. For reasons explained above, the expected fraction of running investors is increasing in the *ex ante* expectation of the fundamental, while the fraction of investors who acquire information is increasing in the cost.

## Appendix 2.B. Proofs

First the following lemma is proved where  $x^*(\hat{\theta})$  denotes the switching point as a function of the *ex ante* expectation of the fundamental.

LEMMA 6.

- i)  $r(\emptyset, I) = 0$  if  $\hat{\theta} \leq \frac{1}{2}$  and  $r(\emptyset, I) = \mathbb{E}[\theta - \ell|I]$  if  $\hat{\theta} \geq \frac{1}{2}$
- ii)  $x^*(\frac{1}{2} + \delta) = 1 - x^*(\frac{1}{2} - \delta)$ ,  $\delta \geq 0$

Note that i) states that an investor without information in the *I* equilibrium should run if  $\hat{\theta} < \frac{1}{2}$  and stay if  $\hat{\theta} > \frac{1}{2}$ .

PROOF OF LEMMA 6.

i) Define  $A(\hat{\theta}, \alpha, x_i, \beta) = (\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta) - \Phi(\sqrt{\gamma}((\alpha\hat{\theta} + \beta x_i)/(\alpha + \beta) - \hat{\theta}))$ . In equilibrium  $A(\hat{\theta}, \alpha, x^*, \beta) = 0$ . Taking the derivatives to  $\hat{\theta}$  and  $x^*$  gives

$$\begin{aligned} \frac{\partial}{\partial \hat{\theta}} A &= \frac{\alpha}{\alpha + \beta} \left( 1 + \frac{\beta}{\alpha} \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma\left(\frac{\alpha\hat{\theta} + \beta x^*}{\alpha + \beta} - \hat{\theta}\right)^2} \right) > 0, \\ \frac{\partial}{\partial x^*} A &= \frac{\beta}{\alpha + \beta} \left( 1 - \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma\left(\frac{\alpha\hat{\theta} + \beta x^*}{\alpha + \beta} - \hat{\theta}\right)^2} \right) > 0, \end{aligned}$$

where the inequality in the last line is implied by the condition  $\gamma < 2\pi$ . Now the implicit function theorem gives  $(\partial/\partial \hat{\theta})x^*(\hat{\theta}) = -(\partial A/\partial \hat{\theta})/(\partial A/\partial x^*) < 0$ . Using the



expression for  $\mathbb{E}[\ell|I]$  of Equation (3) it follows that

$$\frac{\partial}{\partial \hat{\theta}} \mathbb{E}[\ell|I] = \frac{\sqrt{\frac{\alpha\beta}{\alpha+\beta}}}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{\alpha\beta}{\alpha+\beta} (x^*(\hat{\theta})-\hat{\theta})^2} \left( \frac{\partial}{\partial \hat{\theta}} x^*(\hat{\theta}) - 1 \right) < 0.$$

But then  $\mathbb{E}[\theta - \ell|I]$  is increasing in  $\hat{\theta}$ . Note that  $x^*(\frac{1}{2}) = \frac{1}{2}$  and thus  $\mathbb{E}[\theta - \ell|I] = 0$  if  $\hat{\theta} = \frac{1}{2}$ . This implies that  $\mathbb{E}[\theta - \ell|I] \leq 0$  if and only if  $\hat{\theta} \leq \frac{1}{2}$ , with equality if and only if  $\hat{\theta} = \frac{1}{2}$ .

ii) The definition of  $x^*(\hat{\theta})$  gives for  $\frac{1}{2} - \delta$

$$\begin{aligned} & \frac{\alpha(\frac{1}{2} - \delta) + \beta x^*(\frac{1}{2} - \delta)}{\alpha + \beta} \\ &= \Phi \left( \frac{\beta(\alpha + \beta)}{\alpha + 2\beta} \left( x^*(\frac{1}{2} - \delta) - \frac{\alpha(\frac{1}{2} - \delta) + \beta x^*(\frac{1}{2} - \delta)}{\alpha + \beta} \right) \right). \end{aligned}$$

Now subtract both sides from 1 to arrive at

$$\begin{aligned} & \frac{\alpha(\frac{1}{2} + \delta) + \beta(1 - x^*(\frac{1}{2} - \delta))}{\alpha + \beta} \\ &= \Phi \left( \frac{\beta(\alpha + \beta)}{\alpha + 2\beta} \left( (1 - x^*(\frac{1}{2} - \delta)) - \frac{\alpha(\frac{1}{2} + \delta) + \beta(1 - x^*(\frac{1}{2} - \delta))}{\alpha + \beta} \right) \right). \end{aligned}$$

This last line is exactly the definition of  $x^*(\frac{1}{2} + \delta)$ . □

PROOF OF PROPOSITION 1.

v) The statement about  $v^R$  and  $v^S$  follows directly from Equations (1) and (2). The proof of the statement about  $v^I$  is more involved. The variables for  $\hat{\theta} = \frac{1}{2} - \delta$  need to be related to the variables for  $\hat{\theta} = \frac{1}{2} + \delta$ . The value of  $\hat{\theta}$  is added as a subscript to  $\mathbb{E}$  and  $\mathbb{P}$  in order to explicitly denote the dependence of expectations and probabilities on  $\hat{\theta}$ .

In the same way as Equation (1) it follows that

$$\begin{aligned} \mathbb{E}_{\hat{\theta}}[\theta|X_i \geq x^*, I] \mathbb{P}_{\hat{\theta}}[X_i \geq x^*] &= \int_{x^*}^{\infty} \frac{\alpha \hat{\theta} + \beta x_i}{\alpha + \beta} \frac{\sqrt{\frac{\alpha\beta}{\alpha+\beta}}}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{\alpha\beta}{\alpha+\beta} (x_i-\hat{\theta})^2} dx_i \quad (5) \\ &= \hat{\theta} \Phi \left( \sqrt{\frac{\alpha\beta}{\alpha + \beta}} (\hat{\theta} - x^*) \right) + \frac{\sqrt{\frac{\beta/\alpha}{\alpha+\beta}}}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{\alpha\beta}{\alpha+\beta} (\hat{\theta}-x^*)^2}. \end{aligned}$$

Lemma 6 implies that  $x^*(\frac{1}{2} + \delta) - (\frac{1}{2} + \delta) = -(x^*(\frac{1}{2} - \delta) - (\frac{1}{2} - \delta))$ . Now use Equation (5) to obtain that

$$\begin{aligned} & \mathbb{E}_{\frac{1}{2}-\delta}[\theta|X_i \geq x^*(\frac{1}{2} - \delta), I] \mathbb{P}_{\frac{1}{2}-\delta}[X_i \geq x^*(\frac{1}{2} - \delta)] \\ & \quad \dots - \mathbb{E}_{\frac{1}{2}+\delta}[\theta|X_i \geq x^*(\frac{1}{2} + \delta), I] \mathbb{P}_{\frac{1}{2}+\delta}[X_i \geq x^*(\frac{1}{2} + \delta)] \\ & = (\frac{1}{2} - \delta) \mathbb{P}_{\frac{1}{2}+\delta}[X_i < x^*(\frac{1}{2} + \delta)] - (\frac{1}{2} + \delta) \mathbb{P}_{\frac{1}{2}+\delta}[X_i \geq x^*(\frac{1}{2} + \delta)] \\ & = -(\frac{1}{2} + \delta) + \mathbb{P}_{\frac{1}{2}+\delta}[X_i < x^*(\frac{1}{2} + \delta)]. \end{aligned} \quad (6)$$

Now note that  $\mathbb{E}_{\frac{1}{2}-\delta}[\ell|X_i \geq x^*(\frac{1}{2} - \delta), I] \mathbb{P}_{\frac{1}{2}-\delta}[X_i \geq x^*(\frac{1}{2} - \delta)] = \mathbb{P}_{\frac{1}{2}-\delta}[X_j < x^*(\frac{1}{2} - \delta) \wedge X_i > x^*(\frac{1}{2} - \delta)]$ . This in turn equals

$$\begin{aligned} & \mathbb{E}_{\frac{1}{2}-\delta} \left[ \mathbb{P}[X_j < x^*(\frac{1}{2} - \delta) | \theta = \tilde{\theta}] \mathbb{P}[X_i \geq x^*(\frac{1}{2} - \delta) | \theta = \tilde{\theta}] \right] \\ & = \mathbb{E}_{\frac{1}{2}+\delta} \left[ \mathbb{P}[X_j < 1 - x^*(\frac{1}{2} + \delta) | \theta = 1 - \tilde{\theta}] \mathbb{P}[X_i < 1 - x^*(\frac{1}{2} + \delta) | \theta = 1 - \tilde{\theta}] \right] \\ & = \mathbb{E}_{\frac{1}{2}+\delta} \left[ \mathbb{P}[X_j \geq x^*(\frac{1}{2} + \delta) | \theta = \tilde{\theta}] \mathbb{P}[X_i < x^*(\frac{1}{2} + \delta) | \theta = \tilde{\theta}] \right]. \end{aligned} \quad (7)$$

where it is used that  $X_j|\theta$  and  $X_i|\theta$  are independent, that the involved precisions do not change, that the normal distribution is symmetrical and that  $\theta - (\frac{1}{2} - \delta) = -((1 - \theta) - (\frac{1}{2} + \delta))$ . When  $i$  and  $j$  are interchanged, the expression in the last line of Equation (7) is equal to  $\mathbb{E}_{\frac{1}{2}+\delta}[\ell|X_i \geq x^*(\frac{1}{2} + \delta), I] \mathbb{P}_{\frac{1}{2}+\delta}[X_i \geq x^*(\frac{1}{2} + \delta)]$ . This shows that the expected contribution of  $\ell$  to the return of an investor is the same for  $\hat{\theta} = \frac{1}{2} - \delta$  and  $\hat{\theta} = \frac{1}{2} + \delta$ .

Combine this finding with Equation (6) to obtain

$$\mathbb{E}_{\frac{1}{2}-\delta}[r(X_i, I)] = \mathbb{E}_{\frac{1}{2}+\delta}[r(X_i, I)] - \left( (\frac{1}{2} + \delta) - \mathbb{P}_{\frac{1}{2}+\delta}[X_i < x^*(\frac{1}{2} + \delta)] \right). \quad (8)$$

Lemma 6 showed that  $r(\emptyset, I) = 0$  for  $\hat{\theta} \leq \frac{1}{2}$  and  $r(\emptyset, I) = \mathbb{E}[\theta - \ell|I]$  for  $\hat{\theta} \geq \frac{1}{2}$ . Recognizing that the last term of Equation (8) is exactly the expected return of an investor without information who stays in the  $I$  equilibrium, gives  $v^I(\frac{1}{2} - \delta) = v^I(\frac{1}{2} + \delta)$ .

ii) Due to the symmetry of  $v^R$  and  $v^S$ , which was proved in v), it suffices to prove the statement for  $v^S$ . Using Equation (1) gives

$$\frac{\partial}{\partial \hat{\theta}} v^S(\hat{\theta}) = \Phi \left( \sqrt{\frac{\alpha + \beta}{\beta/\alpha}} \hat{\theta} \right) - 1 < 0.$$

Due to symmetry it suffices to prove the statement about  $v^I$  for  $\hat{\theta} \leq \frac{1}{2}$ . From the lemma it follows that  $v^I(\hat{\theta}) = \mathbb{E}[r(X_i, I)]$  for  $\hat{\theta} \leq \frac{1}{2}$ . The definition of  $v^I(\hat{\theta})$  shows that a change in  $\hat{\theta}$  has an effect via the expected return if the information is better than the switching point, and via the expected return due to a different investment decision if the information equals the switching point. For the first effect it follows that

$$\begin{aligned} \frac{\partial}{\partial \hat{\theta}} \mathbb{E}[\theta - \ell | x_i, I] &= \frac{\alpha}{\alpha + \beta} - \frac{\sqrt{\frac{\beta(\alpha + \beta)}{\alpha + 2\beta}}}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{\beta(\alpha + \beta)}{\alpha + 2\beta} \left(x^*(\hat{\theta}) - \frac{\alpha\hat{\theta} + \beta x_i}{\alpha + \beta}\right)^2} \\ &\dots \times \left( \frac{\partial}{\partial \hat{\theta}} x^*(\hat{\theta}) - \frac{\alpha}{\alpha + \beta} \right) > 0, \end{aligned}$$

where it was used that  $(\partial/\partial \hat{\theta})x^*(\hat{\theta}) < 0$  as was found in the proof of Lemma 6. Hence,  $(\partial/\partial \hat{\theta})r(x_i, I) > 0$  for  $\hat{\theta} < \frac{1}{2}$ , so for the values of the information for which the investor would already have stayed, her expected return will be higher. The proof is finished when noting that  $(\partial/\partial \hat{\theta})x^*(\hat{\theta}) < 0$  implies that the range of information for which an investor expects a positive return increases.

iv) Due to symmetry it suffices to prove the statement only for  $\hat{\theta} < \frac{1}{2}$ . The symmetry of  $v^S$  and  $v^R$  and ii) imply that  $v^S(\hat{\theta}) > v^R(\hat{\theta})$ . It remains to be proved that  $v^I(\hat{\theta}) > v^R(\hat{\theta})$ . From the lemma it is clear that  $v^I(\hat{\theta}) = \mathbb{E}[r(X_i, I)]$  for  $\hat{\theta} < \frac{1}{2}$ . Since there is always a strictly positive fraction of investors who stay in the  $I$  equilibrium, clearly  $\mathbb{E}[\theta - \ell | x_i, I] > \mathbb{E}[\theta - \ell | x_i, R]$  and thus  $x^* < (\alpha + \beta)/\beta - \alpha\hat{\theta}/\beta$ . Hence,  $r(x_i, I) \geq r(x_i, R)$  for all  $x_i$  while a strict inequality holds for  $x_i > x^*$ . Since these values have a positive probability mass, it follows that  $\mathbb{E}[r(X_i, I)] > \mathbb{E}[r(X_i, R)]$ , which implies that  $v^I(\hat{\theta}) > v^R(\hat{\theta})$  for  $\hat{\theta} < \frac{1}{2}$ .

i) Due to iv) and v) it suffices to prove that  $v^R(\hat{\theta}) > 0$  for  $\hat{\theta} \leq \frac{1}{2}$ . Since  $r(x_i, R) \geq 0$  for all  $x_i$  while a strict inequality holds for  $x_i > (\alpha + \beta)/\beta - \alpha\hat{\theta}/\beta$ , which happens with positive probability, it follows that  $v^R(\hat{\theta}) = \mathbb{E}[r(X_i, R)] > 0$ .

iii) Due to symmetry it suffices to prove the statement only for  $\hat{\theta} \leq \frac{1}{2}$ . The symmetry of  $v^S$  and  $v^R$  and ii) imply that  $v^S(\hat{\theta}) > v^R(\hat{\theta})$  for  $\hat{\theta} < \frac{1}{2}$ . Given ii) and iv) it needs to be proved that  $v^S(0) > v^I(0)$ . But this holds since from arguments similar to the ones used in iv) it follows that  $\mathbb{E}[r(X_i, S)] > \mathbb{E}[r(X_i, I)]$ .  $\square$

### PROOF OF PROPOSITION 3.

Due to symmetry of  $v^R$  and  $v^S$  it suffices to prove the statement for  $v^S$ . Equation

(1) can be used to obtain

$$\begin{aligned}\frac{\partial}{\partial \alpha} v^S(\hat{\theta}) &= -\frac{1}{2} \frac{\beta(2\alpha + \beta)}{\alpha^2(\alpha + \beta)^2} \frac{\sqrt{\frac{\alpha+\beta}{\beta/\alpha}}}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{\alpha+\beta}{\beta/\alpha} \hat{\theta}^2} < 0, \\ \frac{\partial}{\partial \beta} v^S(\hat{\theta}) &= \frac{1}{2} \frac{1}{(\alpha + \beta)^2} \frac{\sqrt{\frac{\alpha+\beta}{\beta/\alpha}}}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{\alpha+\beta}{\beta/\alpha} \hat{\theta}^2} > 0.\end{aligned}$$

Due to the symmetry of  $v^I$  in  $\hat{\theta} = \frac{1}{2}$  it suffices to give the proof for  $\hat{\theta} \leq \frac{1}{2}$ , and due to continuity it even suffices to only consider  $\hat{\theta} < \frac{1}{2}$ . From the lemma it follows that an investor without information withdraws her money. The value of information is then given by  $\mathbb{E}[r(X_i, I)]$ . For reasons explained in the proof of Proposition 1 i) this value is decreasing in the switching point  $x^*$ . To compute the derivative of  $x^*$  to  $\alpha$  and  $\beta$  the function  $A$  as defined in the proof of Lemma 6 is used. There it was found that  $(\partial/\partial x^*)A > 0$ . Now compute

$$\begin{aligned}\frac{\partial}{\partial \alpha} A &= \frac{\beta(\hat{\theta} - x^*)}{(\alpha + \beta)^2} \left( 1 + \frac{1}{2} \frac{3\alpha\beta + 4\beta^2}{\alpha^2 + 2\beta\alpha} \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma\left(\frac{\alpha\hat{\theta} + \beta x^*}{\alpha + \beta} - \hat{\theta}\right)^2} \right), \\ \frac{\partial}{\partial \beta} A &= \frac{\alpha(x^* - \hat{\theta})}{(\alpha + \beta)^2} \left( 1 - \frac{1}{2} \frac{\alpha^2 - 2\beta^2}{\alpha^2 + 2\alpha\beta} \frac{\sqrt{\gamma}}{\sqrt{2\pi}} e^{-\frac{1}{2}\gamma\left(\frac{\alpha\hat{\theta} + \beta x^*}{\alpha + \beta} - \hat{\theta}\right)^2} \right).\end{aligned}$$

Since  $\gamma < 2\pi$  and  $x^* > \hat{\theta}$  when  $\hat{\theta} < \frac{1}{2}$  (see the proof of the lemma), it follows that  $(\partial/\partial \alpha)A < 0$  and  $(\partial/\partial \beta)A > 0$ . The implicit function theorem gives that  $(\partial/\partial \alpha)x^* > 0$  and  $(\partial/\partial \beta)x^* < 0$ , which finishes the proof.  $\square$

#### PROOF OF PROPOSITION 4.

i) When it is proved that  $\hat{v}^S$  exists the statements follow directly from the main text and the symmetry between  $\hat{v}^S$  and  $\hat{v}^R$ . First it is proved that the investor's problem is well-defined. Since an investor will stay when her information is better than  $-\alpha\hat{\theta}/\beta$ , she will always stay if the precision goes to zero when  $\hat{\theta} > 0$ . For  $\hat{\theta} = 0$ , Equation (1) shows that  $\mathbb{E}[r(X_i, S; \beta)]$  goes to 0 when  $\beta$  becomes arbitrarily small. Hence,  $\lim_{\beta \rightarrow 0} \mathbb{E}[r(X_i, S; \beta)] - \beta\hat{c} = \mathbb{E}[r(X_i, S; 0)]$  and the objective function is continuous in  $\beta$ . Note that  $r$  is bounded, since  $\mathbb{E}[|Y|]$  is bounded when  $Y$  has a normal distribution. Hence, the total return for very high  $\beta$  becomes negative. This shows that  $\max_{\beta \geq 0} \mathbb{E}[r(X_i, S; \beta)] - \beta\hat{c}$  is well-defined.

To prove that  $\max v^S(\hat{\theta}; \beta)/\beta$  exists for  $\hat{\theta} > 0$ , Equation (1) can be used to obtain

$$\lim_{\beta \rightarrow 0} \frac{v^S(\hat{\theta}; \beta)}{\beta} = \lim_{\beta \rightarrow 0} \frac{\sqrt{\frac{1}{\alpha\beta(\alpha+\beta)}}}{\sqrt{2\pi}} e^{-\frac{1}{2} \frac{\alpha+\beta}{\beta/\alpha} \hat{\theta}^2} = 0.$$

Since the expected return is always finite the maximum cost per unit precision goes to zero for  $\beta \rightarrow \infty$ . The non-negativity of the expected return now implies that the maximum is well-defined for  $\hat{\theta} > 0$ . Note that for  $\hat{\theta} = 0$  this maximum cost is infinity.

ii) This follows from the symmetry of  $\hat{v}^R$  and  $\hat{v}^S$  and the fact that  $\hat{v}^S$  is decreasing in  $\hat{\theta}$ .

iii) The first statement is proved in the text. The statement about  $\hat{v}^I$  is more complicated. A proof of the equilibrium existence for low unit costs is sketched for  $\hat{\theta} < \frac{1}{2}$  (below it is proved that  $\hat{v}^I$  is symmetric around  $\hat{\theta} = \frac{1}{2}$ ). Similar to the  $S$  profile it can be proved that the maximization problem is well defined. This maximization problem can then be seen as a mapping from  $\beta_j$  to  $\beta$ . For  $\beta_j$  close to 0, there is an equilibrium where almost all the investors run. Comparison with the  $R$  profile shows that for low enough unit costs investor  $i$  wants precision  $\beta > \beta_j$ . Since the expected return is bounded, for very large  $\beta_j$  investor  $i$  wants precision  $\beta < \beta_j$ . When the mapping is continuous a fixed point is guaranteed, which shows the existence of an equilibrium for low unit costs.

To prove the symmetry of  $\hat{v}^I$  in  $\frac{1}{2}$ , take  $\beta_j$  as given. It now suffices to show that the maximization problems for  $\hat{\theta} = \frac{1}{2} + \delta$  and for  $\hat{\theta} = \frac{1}{2} - \delta$  are identical up to a constant. From Lemma 6 it follows that  $x_j^*(\frac{1}{2} + \delta; \beta_j) = 1 - x_j^*(\frac{1}{2} - \delta; \beta_j)$ , where the dependence on  $\hat{\theta}$  and either  $\beta_j$  is explicitly denoted. Investor  $i$  stays if her private information is higher than  $x^*(\hat{\theta}; \beta)$ , which is determined by  $\mathbb{E}[\theta - \ell | x^*(\hat{\theta}; \beta)] = 0$ . In the same way as the proof of Lemma 6 it now follows that  $x^*(\frac{1}{2} + \delta; \beta) = 1 - x^*(\frac{1}{2} - \delta; \beta)$ .

The derivation in Equation (6) still holds when the dependence of  $x^*$  on  $\beta$  is explicitly denoted. The derivation in Equation (7) shows that  $\mathbb{E}_{\frac{1}{2}-\delta}[\ell | X_i \geq x^*(\frac{1}{2} - \delta; \beta), I] \mathbb{P}_{\frac{1}{2}-\delta}[X_i \geq x^*(\frac{1}{2} - \delta; \beta)] = \mathbb{P}_{\frac{1}{2}+\delta}[X_j \geq x_j^*(\frac{1}{2} + \delta; \beta_j) \wedge X_i < x^*(\frac{1}{2} + \delta; \beta)]$ . Using the fact that  $\mathbb{P}[A^c \cap B^c] = 1 - \mathbb{P}[A] - \mathbb{P}[B] + \mathbb{P}[A \cap B]$  then gives

$$\begin{aligned} & \mathbb{E}_{\frac{1}{2}-\delta}[\ell | X_i \geq x^*(\frac{1}{2} - \delta; \beta), I] \mathbb{P}_{\frac{1}{2}-\delta}[X_i \geq x^*(\frac{1}{2} - \delta; \beta)] \\ &= 1 - \mathbb{P}_{\frac{1}{2}+\delta}[X_j < x_j^*(\frac{1}{2} + \delta; \beta_j)] - \mathbb{P}_{\frac{1}{2}+\delta}[X_i \geq x^*(\frac{1}{2} + \delta; \beta)] \\ & \quad \dots + \mathbb{E}_{\frac{1}{2}+\delta}[\ell | X_i \geq x^*(\frac{1}{2} + \delta; \beta), I] \mathbb{P}_{\frac{1}{2}+\delta}[X_i \geq x^*(\frac{1}{2} + \delta; \beta)]. \end{aligned}$$

This shows that

$$\begin{aligned}
& \mathbb{E}_{\frac{1}{2}-\delta}[r(X_i, I, \beta)] - \mathbb{E}_{\frac{1}{2}+\delta}[r(X_i, I, \beta)] \\
&= -\left(\frac{1}{2} + \delta\right) + \mathbb{P}_{\frac{1}{2}+\delta}[X_i < x^*\left(\frac{1}{2} + \delta; \beta\right)] \\
&\quad \dots - 1 + \mathbb{P}_{\frac{1}{2}+\delta}[X_j < x_j^*\left(\frac{1}{2} + \delta; \beta_j\right)] + \mathbb{P}_{\frac{1}{2}+\delta}[X_i \geq x^*\left(\frac{1}{2} + \delta; \beta\right)] \\
&= -\left(\frac{1}{2} + \delta\right) + \mathbb{P}_{\frac{1}{2}+\delta}[X_j < x_j^*\left(\frac{1}{2} + \delta; \beta_j\right)].
\end{aligned}$$

This expression is independent of  $\beta$ , which finishes the proof. □

## CHAPTER 3

# Fixing the Quorum: Representation versus Abstention

**ABSTRACT.** The majority of the participating voters in referenda does not necessarily reflect the majority of the whole population since voters can abstain. This paper shows that a quorum exists for which the outcome of the referendum coincides with the population preference. However, a second equilibrium can exist in which the proposal is always rejected. When insufficient information makes the optimal quorum unknown, it is in general more harmful to set the quorum too high than too low. Robustness of the results is analyzed by allowing pressure groups to encourage or discourage participation after the quorum is set.

**KEYWORDS.** Electoral engineering, quorum, referendum, voting/not-voting decision, voting rules.

**JEL CLASSIFICATION.** D72.

### 3.1. Introduction

In June 2005 a referendum was held in Italy to block a fertility law. The law banned research using stem cells from embryos and imposed stringent requirements on test-tube pregnancies. Adversaries of the law initiated the referendum to aim for its abrogation. To succeed, a quorum of 50% was to be met and a majority of the participating voters had to support the abrogation. This gave advocates of the law two different possibilities to avoid abrogation: i) encouraging no-voters to take the effort to vote so that they would form the majority; ii) discouraging no-voters from voting so that the quorum would not be met and the referendum would be invalid. In Italy, the advocates of the law chose for the second option, for example the speakers of the Senate and the Chamber of Deputies as well as the Roman Church discouraged people from voting. The New York Times (2005) writes that “Italian prelates have told parishioners to head to the beach instead of the polling places on Sunday and Monday, so that the quorum will not be met.” The strategy succeeded, the turnout was too low and the referendum was invalid. Since 26% of the population voted, while almost 90% of the participating voters were in favor, 23.4% of the population was in favor and voted. When a handful of people in favor was discouraged by the forecasts of an invalid referendum, encouraging no-voters to cast their ballot could indeed have led to a valid referendum in which the abrogation would have been approved.

Referenda are becoming increasingly widespread in democratic countries (Waters (2003) and Matsusaka (2005) discuss recent trends, see also the web sites of The Initiative & Referendum Institute). One of the main reasons is the wish to give voters a direct say in the issues at stake. An additional reason might be that direct democracy would contribute to voters’ involvement with and trust in the political system. However, referenda are known to be imperfect decision making tools in the sense that a counter-intuitive relationship between the voters’ preferences and the outcome can occur. Nurmi (1998) lists various voting paradoxes, including problems stemming from multiple proposals or multiple alternatives and the possibility of conflicting opinions between the majorities of the voters and their representatives. As the referendum in Italy shows, a quorum gives rise to an additional potential problem by giving opponents of change an additional tool to reach their aim. Fishburn and Brahm (1983) call this the “no-show” paradox.



The objective of this paper is twofold. In the first part we address the question whether there is any theoretical support for imposing a quorum in a referendum. The focus of the second part is on the robustness of the results. More specifically, we first look at the magnitude of the distortion when the quorum is set either too low or too high and then at the impact of pressure groups which can affect the voter turnout after the quorum is set.

The role of the quorum is analyzed in a stylized referendum model with heterogeneous voters. The existence of a quorum makes the turnout a decisive variable for determining the outcome. But even for referenda without a quorum, the voting/not-voting decision is an important aspect of explaining the outcome. Analyzing this decision usually leads to the conclusion that people who vote do not form a representative subset of the population. For example, Fort and Bunn (1998) find for referenda concerning nuclear power that actual participation has more explanatory power for the yes/no decision than both economic and preference variables. Successfully navigating the hurdles of registering, going to the booth etc. made a no-vote more likely. In the model, this asymmetry between opponents and proponents is reflected by their possibly different probabilities of voting.

In the first part of the paper we show that with the appropriate choice of the quorum and the default outcome that occurs if the quorum is not met, the population majority outcome can be attained. To see how the referendum should be designed, suppose that proponents are more likely to cast their ballots than opponents. In order to offset the bias towards accepting, the default outcome needs to be rejection. A higher quorum needs more participating voters. To be precise, it needs a higher fraction of yes-voters in the population since they are more likely to vote. A higher quorum thus reduces the cases where the majority of participating voters is in favor while the majority of the population is not. The population majority outcome is attained for the quorum for which they equal.

Interestingly, when voters care more about the outcome when they are participating, the optimal quorum does not necessarily lead to the population majority outcome. A second equilibrium can exist in which the default outcome always occurs. In this case, the referendum clearly is an imperfect tool for decision making.

The second part of the paper analyzes the robustness of the results in two ways. When the social planner has insufficient knowledge about the population parameters

or insufficient political power to set the quorum at its optimal level, a non-optimal quorum can arise. When the default outcome is set correctly, we show that setting the quorum too low is less harmful than setting it too high. The reason is that the default outcome will always occur when the quorum is too high, while when the quorum is too low both outcomes might still occur. Since in most real-life applications there is not much flexibility in setting the quorum, this finding implicates that only topics for which *both* sides have a high expected turnout should be subjected to referenda. A non-optimal quorum can also arise when pressure groups have the possibility to affect the turnout after the quorum is set, like in the Italian referendum discussed above. When the default outcome is rejecting the proposal, yes-pressure groups should always encourage people to vote. For no-pressure groups it is optimal to encourage voters to participate only if it is likely that there are relatively many no-voters, otherwise they should be discouraged from voting.

Since the basis of democracy is that all people are equally important, we consider the preference of the population majority as the benchmark outcome. We thus abstain from social welfare considerations that balance an “optimal outcome” with the cost of representation. The model can easily be adapted to address different intensities of voters’ preferences. In case a quorum is exogenously imposed to guarantee a certain level of representativeness, the second part of the paper can be read as analyzing the difference between the referendum outcome and the population majority outcome. We assume that participation is voluntary, as compulsory voting would trivially result in the population majority outcome (however, Franklin (1999) and Jakee and Sun (2006) raise arguments against compulsory voting).

Theoretical support for the importance of the population majority outcome follows from the axiomatization of May (1952) as the only voting rule that is decisive, anonymous, not-favoring any of the outcomes and positively responding (i.e. when one voter changes opinion then the group decision becomes more favorable towards that opinion). However, when voters can abstain from participating, Crte-Real and Pereira (2004) find that in general no voting rule that is independent of the abstainers’ preferences can achieve the population majority outcome. They show that this outcome can be achieved if in the case of a turnout below the quorum, the underlying reasons determine the outcome. In the equilibrium setting of this

paper's model, this interpretation of an insufficient turnout is done *ex ante* when the referendum is designed.

The model is based on the decision-theoretic approach initiated by Downs (1957). Voters participate in the referendum when they receive a positive net utility from voting. Following Riker and Ordeshook (1968) and in line with empirical evidence discussed extensively by Blais (2000), the net utility of voting depends on the outcome of the referendum, the cost of casting the ballot and a "consumption benefit" that represents the fulfillment of a voter's "civic duty". The main difference between their and our model is how a voter derives utility from the outcome of the referendum and from participating. In their model, they consider the benchmark of a utility function that is linear in the outcome of the referendum. However, there might be nonlinearities involved with respect to the outcome and participation. More specifically, the utility of the referendum outcome might depend on whether a voter has participated or not. On top of this, when there are many potential voters, the probability that a particular voter's action is decisive is almost zero. Myerson (2000) derives estimates of the order  $10^{-9}$ . Hence, unless the utility difference between the outcomes is extremely large relative to the cost of voting, the nonlinear effect might be far more important. It is not clear what the direction of this nonlinear effect should be: there are convincing arguments for all possibilities. When it is zero the outcome of the referendum does not affect a voter's participation decision. When it is negative, a voter exhibits an underdog-mentality: the less likely her preferred outcome, the more likely she will vote. When the nonlinear effect is positive, a voter likes to be part of the winning side. In this paper we consider all types. Moreover, we show that if all types can occur simultaneously, the average type drives the results.

Although the literature on voting is vast, there are few papers on referenda. Herrera and Mattozzi (2007) discuss a group-based referendum model. As in this paper, the turnout of each group is endogenous. However, instead of having the referendum outcome directly affecting the voters' utility, their groups weigh the cost of increasing the turnout with its effect on the referendum outcome. They find a "quorum paradox": the equilibrium turnout might only exceed the quorum if the quorum is not imposed. Myatt (2007) discusses a model in which a finite number of privately informed voters have to choose between two alternatives that are preferred to the status quo. In contrast with the model of this paper, strategic

voting can occur when a voter fears that her most preferred alternative will not receive sufficient support. Marquette and Hinckley (1988) and Kanazawa (1998) suggest that a voter's recall of previous elections is also relevant for current turnout. Closely related to the model of this paper, Kanazawa (1998) proposes to substitute the Riker-Ordeshook probability regarding the current election with the probability that the voter's preferred outcome occurred when she participated in past elections. Hence, instead of computing the probability that her preferred candidate wins as in this paper's model, a voter uses an estimation based on past experience.

The outline of the paper is as follows: Section 2 presents the model, Section 3 shows that there is a quorum for which the population majority outcome can occur and analyzes its properties, Section 4 addresses the robustness of the results by considering a not-optimally set quorum and allowing for pressures groups. Appendices 3.A and 3.B contain precise formulations of claims made in the main text. Proofs are deferred to Appendix 3.C.

### 3.2. The Referendum Model

**3.2.1. The Referendum.** A referendum is held in order to decide whether a proposal should be accepted or rejected. Each voter has three options: i) to vote in favor of the proposal; ii) to vote against it; iii) not to vote. Voters who do indeed vote are called *participating voters*. The referendum is only valid if a quorum is met, that is if more than a certain fraction of the voters is indeed voting. The proposal is accepted if the referendum is valid and if the majority of the participating voters is in favor.<sup>1</sup> When the quorum is met but a majority of the participating voters is against, the proposal is rejected. In case the referendum is invalid, a preset default outcome determines whether the proposal is accepted or not. Although in some real-life referenda the default outcome is not explicitly set, in most cases it is rather clear what will happen when the referendum is not valid. For example, in the referendum about the European Constitution in the Netherlands there was no formal default outcome. Though, all major political parties were in favor and it was clear that the European Constitution would be accepted in case the quorum would not be met. In this paper, designing a referendum is thus choosing the quorum and default option.

---

<sup>1</sup>When the intensities of the voters' preferences differ, a qualified majority can be used to protect a minority from the majority, see Appendix 3.A for details.

There is a continuum of voters with measure one. Each voter knows whether she is in favor of the proposal or against it, but there is uncertainty about the overall fraction of voters in favor of the topic.<sup>2</sup> The assumption that the preferences of voters are endogenously determined is rather standard. However, Rosema (2004) discusses the psychology of voting and finds that possible election outcomes are used in the decision what to vote. Making voters' preferences endogenous though, justifies research on its own and is outside the scope of this paper. Hence, denote by  $y$  the proportion of voters in favor of the proposal. The very reason that a referendum is needed, is that the value of  $y$  is unknown. Hence,  $y$  is a random variable which takes its values in an interval  $[\underline{y}, \bar{y}] \subset [0, 1]$ . The distribution of  $y$  is common knowledge. This can be the case if for example forecasting agencies provide correct projections when not everyone has made up her mind yet. The proportion of voters in favor has full support on  $[\underline{y}, \bar{y}]$ . The model is not relevant when the majority is either always in favor or always against, so it is assumed that  $\underline{y} < \frac{1}{2} < \bar{y}$ .

When the proportion of yes-voters  $y$  were observable, no referendum is needed to have the proposal accepted or rejected according to the majority of the voters. This benchmark case is referred to as the *population majority outcome*. To be precise, denote by  $A$  the event that the proposal is accepted and by  $R = A^c$  the event that it is rejected. The population majority outcome is then defined as  $A$  when  $y > \frac{1}{2}$  and  $R$  when  $y < \frac{1}{2}$ . When  $y = \frac{1}{2}$ , the population majority outcome prescribes both  $A$  and  $R$  with probability  $\frac{1}{2}$ . However, for notational convenience  $A$  is prescribed but we assume that this case does not occur, i.e.  $\mathbb{P}[y = \frac{1}{2}] = 0$ .

Since voters have the possibility to abstain from voting, the proportion of yes-voters  $y$  is not directly observable. This paper analyzes whether a referendum can be designed in such a way that the population majority outcome always occurs.

**3.2.2. The Voters.** A voter who is in favor of the proposal is referred to as a yes-voter, a voter who is against the proposal as a no-voter. The typical yes-voter will be indicated by index  $i$  and the typical no-voter by index  $j$ . Whether a voter will indeed participate depends on her net benefit of doing so. A voter participates in the referendum if her net utility of doing so is positive. In our model, this net

---

<sup>2</sup>It is possible to allow for voters who are indifferent with respect to the proposal by assuming that this group has a fixed size and that due to a lack of motivation these voters do never participate in the referendum.

utility of voting has the form proposed by Riker and Ordeshook (1968). As in their model, the net utility consists of three terms: i) a cost of voting; ii) a “consumption benefit from the act of voting” and iii) a utility from the outcome of the referendum depending on its probability of occurrence. The main difference between their model and mine is how the utility depends on the outcome of the referendum.

A voter who decides to indeed cast her vote, incurs a cost  $c > 0$  representing the effort to go to the ballot box. Since there is a continuum of voters, the impact of a single voter on the outcome is nil. If voters were only concerned about the strategic benefit of voting and its cost, this would lead to the well-known paradox that none of the voters would take the effort to cast a ballot. Cultural theories of voting argue that the incorporation of “civic engagement” eliminates the paradox. In an empirical study, Blais, Young and Lapp (2000) find support for this hypothesis. In explaining voter turnout, the cost of voting and a return depending on the outcome of the referendum matter, but only among the voters with a relatively weak civic engagement.

In the model this civic engagement is a moral pressure to vote that differs across voters. Let  $m_i$  be the moral pressure of yes-voter  $i$ . The moral pressure of a yes-voter has a uniform distribution on the interval  $[\bar{m}^y - \frac{\alpha}{2}, \bar{m}^y + \frac{\alpha}{2}]$  so that the average moral pressure of yes-voters is given by  $\bar{m}^y$ . Similarly, assume that the moral pressure of no-voters has a uniform distribution on the interval  $[\bar{m}^n - \frac{\alpha}{2}, \bar{m}^n + \frac{\alpha}{2}]$ . The moral pressure is felt as a disutility when a voter is not voting. Since there are no strong arguments why yes- and no-voters should have differently shaped moral pressure distributions, they are taken as identical. Hence, the scaling parameter  $\alpha$  that determines the within-group heterogeneity is the same for both sides. The average moral pressures though can be different. This allows for the proposal to unequally affect the yes- and no-voters, so that one side might be more inclined to vote. Different average moral pressures can thus cause a bias towards accepting or rejecting the proposal.

The dependence on the outcome is modeled in the following way. A yes-voter wants the proposal to be accepted and derives utility in this case. The utility a yes-voter derives from acceptance of the proposal can depend on whether the voter indeed participates in the referendum or not. Let the utility of an accepted proposal

for a participating yes-voter be  $\gamma^v$ , while it is  $\gamma^{nv}$  for a non-participating yes-voter.<sup>3</sup> Similarly, when the proposal is rejected, a participating no-voter derives utility  $\gamma^v$  while a non-participating no-voter derives utility  $\gamma^{nv}$ . For  $\gamma^v > \gamma^{nv}$ , voters derive more utility from their preferred outcome when they have participated. When the reversed inequality holds, a voter likes her preferred outcome best when it occurs without costing her any effort. If  $\gamma^v \neq \gamma^{nv}$ , the additional bias towards accepting or rejecting the proposal might either offset or strengthen the bias stemming from different average moral pressures.

The outcome of the referendum is unknown when the voters have to make their decisions. The *ex ante* expected utility thus depends on the probability of acceptance or rejection. Theoretically these probabilities can depend on whether a voter participates or not, so denote the probability of acceptance by  $\mathbb{P}^v[A]$  when a voter participates and by  $\mathbb{P}^{nv}[A]$  when she does not. For a yes-voter, the expected utility derived from the outcome of the referendum is thus  $\gamma^v \mathbb{P}^v[A]$  or  $\gamma^{nv} \mathbb{P}^{nv}[A]$  depending on whether she is participating or not.

The utilities of a yes-voter are summarized in Table 1, for a no-voter identical expressions hold when the probability of acceptance is replaced by the probability of rejection.<sup>4</sup> The net utility of voting is shown in the third line. The first term is a utility difference caused by voter  $i$ 's impact on the outcome, the second term is a utility difference due to different valuations of the outcome when a voter participates or not. Econometricians would call the latter an interaction effect. It captures nonlinearities that arise from the participation and the outcome. Riker and Ordeshook (1968) assume that the utility of the outcome does not depend on the voter's decision, so  $\gamma^v = \gamma^{nv}$ . The outcome thus only affects voters' decisions

---

<sup>3</sup>This is equivalent to the more elaborate modelling where disutility is derived from rejection of the proposal. For example, when participating yes-voters derive utility  $\beta^{vA} \mathbb{P}[A]$  in case of acceptance and  $\beta^{vR} \mathbb{P}[R]$  in case of rejection, the total utility is  $(\beta^{vA} - \beta^{vR}) \mathbb{P}[A] - \beta^{vR}$ . Defining  $\gamma^v$  as  $\beta^{vA} - \beta^{vR}$  and noting that the constant can be absorbed by rescaling of  $\bar{m}^y$ , as will be made clear below, gives the result.

<sup>4</sup>We implicitly assume that whenever a voter cast her ballot, she votes according to whether she is in favor or against. In other words, all voters are sincere. It is necessary to assume this since each voter is atomistic and her decision is not affecting the outcome. However, sincere voting is guaranteed when the voter's morality leads to a large negative utility when she votes for the non-preferred outcome.

	utility of yes-voter $i$	
voting	$\gamma^v \mathbb{P}^v[A] - c$	
not voting	$\gamma^{nv} \mathbb{P}^{nv}[A] - m_i$	
net utility of voting	$\gamma^v (\mathbb{P}^v[A] - \mathbb{P}^{nv}[A]) + (\gamma^v - \gamma^{nv}) \mathbb{P}^{nv}[A] - c + m_i$	
Riker-Ordeshook	$\gamma^v (\mathbb{P}^v[A] - \mathbb{P}^{nv}[A])$	$-c + m_i$
this model	$(\gamma^v - \gamma^{nv}) \mathbb{P}[A]$	$-c + m_i$

TABLE 1. Riker and Ordeshook (1968) assume that the utility of the outcome does not depend on participation, so  $\gamma^v = \gamma^{nv}$ . However, when the impact of a single voter is nihil, the probability of acceptance  $\mathbb{P}[A]$  is independent of voter  $i$ 's participation and the outcome only affects the participation decision through differences between  $\gamma^v$  and  $\gamma^{nv}$ .

through different probabilities of acceptance. However, the probability that a particular voter is pivotal is extremely small when the population is large. For example, consider a population of 5 million voters of which 50.1% is expected to be in favor. Feddersen (2004) uses a formula derived by Myerson (2000) to find estimates for the probability of a pivotal vote of the order  $10^{-9}$ . This shows that even when  $\gamma^v$  and  $\gamma^{nv}$  are close, different valuations of the outcome may be far more important than the utility difference caused by the voter's impact. Although voter's tend to overestimate their impact, as for example found by Blais et al. (2000), their biases should be of a very high order to outweigh the effects of different valuations.

To focus on how different valuations affect the referendum outcome, we abstain from the small impact of a single voter by assuming a continuum of voters. Hence, no strategic concerns are incorporated in the decision making process at the individual level.<sup>5</sup> The probability of acceptance does not depend on the voter's action and is denoted by  $\mathbb{P}[A]$ ; the probability of rejection is then  $\mathbb{P}[R] = 1 - \mathbb{P}[A]$ . The expression of the net utility shows that the levels of the utilities derived from acceptance or rejection are not relevant for the behavior of the voters, only their difference matters.

<sup>5</sup>In Section 3.4 we will give interest groups the possibility to coordinate the individuals. This allows individuals to indirectly strategically affect the outcome.



Define  $\gamma = \gamma^v - \gamma^{nv}$  as the excess utility of the preferred outcome of voting relative to not-voting.

It is not clear what the sign of  $\gamma$  should be, or even whether it should be non-zero. We hence do not make any assumptions and discuss the model for all possible values of  $\gamma$ . When  $\gamma = 0$ , the outcome of the referendum is not relevant for the decision of a voter whether to vote or not. For this reason we refer to these voters as *simple-hearted voters*.

When  $\gamma < 0$ , the outcome of the referendum will give a higher utility when the voter does not cast her vote. This captures the feeling of a voter who likes her preferred outcome best if she does not have to do anything for it to occur. A higher probability of her preferred outcome makes a voter less willing to vote. This resembles the “underdog effect” reported by Levine and Palfrey (2007) in a laboratorial experiment: voters supporting the less popular alternative have higher participation rates. Another way of interpreting this behavior is suggested by Haan and Kooreman (2003). For a finite number of voters they show that the side with the highest number of supporters can still be the most likely to lose due to free-riding behavior. When  $\gamma < 0$  voters balance their moral pressures with the outcome of the referendum, and we therefore refer to them as *calculating voters*.

When  $\gamma > 0$ , the more likely it is that the preferred outcome will occur, the more likely a voter will participate. This represents a voter who wants to be part of the winning team: the higher the probability of winning, the more likely she wants to take action to support it. This is in line with the expressive voting model of Schuessler (2000) in which benefits from attachment to a collective lead to a preference for the winning party. For example, Ashworth, Geys and Heyndels (2006) find evidence that although in Belgian municipal elections turnout is highest when the largest party obtains a small majority, turnout is again stimulated when there is a clear winner with at least two thirds of the votes. Further support that some voters want to be a winner is given by Bartels (1988) who shows that the public opinion before US presidential elections tends towards the winner of the most recent primary election. Remarkably, Clausen (1968) finds that in post-election recall surveys the winning candidate’s support is overestimated and concludes that apparently too many people “remember” to have contributed to the victory. Since voters cluster together when  $\gamma > 0$ , we refer to them as *affectionate voters*.

The above expressions show that the cost  $c$  of casting the ballot can be absorbed in the mean moral pressures  $\bar{m}^y$  and  $\bar{m}^n$ . Without loss of generality, the exposition of the model can thus focus on the case  $c = 0$ . It also shows that there is an alternative interpretation of the model in which all voters have the same moral pressures, but differ in their cost of voting.

**3.2.3. Equilibrium.** Since all voters have the same information, they make the same inference about  $\mathbb{P}[A]$  and  $\mathbb{P}[R]$ . For notational convenience we assume that when a voter is indifferent between voting or abstaining will vote. An equilibrium can then be characterized by two switching points  $-\gamma p$  and  $-\gamma r$  such that yes-voter  $i$  only votes if  $m_i \geq -\gamma p$ , no-voter  $j$  only votes if  $m_j \geq -\gamma r$ ,  $\mathbb{P}[A] = p$  and  $\mathbb{P}[R] = r$ . Since  $p + r = \mathbb{P}[A] + \mathbb{P}[R] = 1$ , an equilibrium is fully characterized by  $p$ . To find the equilibria, it thus suffices to analyze for all  $p \in [0, 1]$ , whether  $p - \mathbb{P}[A] = 0$  when the yes- and no-voter switching points are  $-\gamma p$  and  $-\gamma(1 - p)$  respectively.

Let  $Y = \mathbb{P}[m_i \geq -\gamma p]$  denote the probability that yes-voter  $i$  will vote. Invoking the law of large numbers, see Judd (1985),  $Y$  also denotes the proportion of yes-voters who are voting. Hence,  $Y$  will be referred to as the propensity to vote of yes-voters. Similarly, define the propensity to vote of no-voters  $N = \mathbb{P}[m_j \geq -\gamma(1 - p)]$ . Then

$$Y = \frac{1}{2} + \min \left\{ \max \left\{ \frac{\bar{m}^y + \gamma p}{\alpha}, -\frac{1}{2} \right\}, \frac{1}{2} \right\}. \quad (1)$$

A similar expression holds for  $N$ . Note that  $Y$  and  $N$  are both functions of  $p$ .

When the proportion of yes-voters equals  $y$ , the measure of participating yes-voters is given by  $yY$  and the measure of participating no-voters by  $(1 - y)N$ . The *participation rate* is thus given by  $yY + (1 - y)N$ . When  $q \in [0, 1]$  denotes the quorum, the referendum is valid if  $yY + (1 - y)N \geq q$ . This is the *quorum condition*. When the referendum is valid, the proposal is accepted if the majority of the participating voters is in favor, so if  $yY \geq (1 - y)N$  (for notational convenience the proposal is accepted when exactly half of the voters is in favor). This is the *majority condition*. In case the referendum is not valid, the preset default outcome  $D \in \{A, R\}$  determines the outcome.

Table 2 relates the probabilities of accepting the proposal with the propensities to vote and the quorum. Suppose that the default outcome is rejecting the proposal,  $D = R$  (the case  $D = A$  follows from symmetric arguments). First suppose that yes-voters are more likely to participate than no-voters, so  $Y > N$ . A higher proportion

condition	constraint	$\mathbb{P}[A]$
$Y > N$ and $q \leq \frac{2NY}{Y+N}$	majority	$\mathbb{P}[y \geq \frac{N}{Y+N}]$
$Y > N$ and $q \geq \frac{2NY}{Y+N}$	quorum	$\mathbb{P}[y \geq \frac{q-N}{Y-N}]$
$Y = N$	both	$\mathbb{P}[y \geq \frac{1}{2}] \mathbf{1}_{\{Y \geq q\}}$
$Y < N$	both	$\mathbb{P}[\frac{q-N}{Y-N} \geq y \geq \frac{N}{Y+N}]$

TABLE 2. Binding constraints and the probability of accepting the proposal when the default outcome is rejection.

$y$  of yes-voters makes a valid referendum more likely since more voters will actually vote (a yes-voter is more likely to vote than a no-voter), and it makes it more likely that the proposal is accepted (there are more participating yes-voters). When the quorum is below  $2NY/(Y+N)$ , the quorum is relatively easily met and the majority condition determines the probability of acceptance (note that for  $q = 2NY/(Y+N)$  the majority and quorum constraint coincide). For a higher quorum instead it is determined by the quorum constraint. Now suppose that  $Y < N$ . A higher fraction of yes-voters  $y$  makes a valid referendum less likely since less voters will actually vote (a yes-voter is less likely to vote than a no-voter), but if the referendum is valid it is more likely that the proposal is accepted (there are more participating yes-voters). Both constraints are binding, the quorum constraint from above, the majority constraint from below. Note that when  $Y = N$ , the quorum can only be met if  $q \leq Y = N$ . In this case the probability of accepting is determined by the majority condition.

An equilibrium in case  $D = R$  is thus a solution of  $p - \mathbb{P}[A] = 0$ , where  $\mathbb{P}[A]$ ,  $Y$  and  $N$  are as discussed above. This equilibrium characterization is at the core of the analysis.

### 3.3. The Quorum and the Population Majority Outcome

**3.3.1. Simple-Hearted Voters.** Suppose that the voters are simple-hearted, so  $\gamma = 0$ . The expectations about the outcome of the referendum do not affect the voter's decision whether to vote or not. This implies that the choice of the quorum does not affect the propensities to vote. Any bias that stems from different average moral pressures can thus be directly addressed by a quorum. The following

proposition states that with the right choice of the quorum and the default option, the population majority outcome occurs.

**PROPOSITION 1.** (Simple-Hearted Voters and the Population Majority Outcome)

*Assume that  $\gamma = 0$  and  $\bar{m}^y, \bar{m}^n \in (-\frac{\alpha}{2}, \frac{\alpha}{2})$ .*

*i) When  $\bar{m}^y = \bar{m}^n$ , the population majority outcome is only achieved in the unique equilibrium of the referendum with a quorum of at most  $q^* = \frac{1}{2} + \frac{\bar{m}^y + \bar{m}^n}{2\alpha}$  and default outcome  $D \in \{A, R\}$ .*

*ii) When  $\bar{m}^y \neq \bar{m}^n$ , the population majority outcome is only achieved in the unique equilibrium of the referendum with quorum  $q^* = \frac{1}{2} + \frac{\bar{m}^y + \bar{m}^n}{2\alpha}$  and default outcome  $D = R$  if  $\bar{m}^y > \bar{m}^n$  and  $D = A$  otherwise.*

In order to discuss the implications of the proposition, it is insightful to look first at the propensities to vote. The condition that  $\bar{m}^y$  and  $\bar{m}^n$  are contained in  $(-\frac{\alpha}{2}, \frac{\alpha}{2})$  implies that they are given by  $Y^* = \frac{1}{2} + \bar{m}^y/\alpha$  and  $N^* = \frac{1}{2} + \bar{m}^n/\alpha$  and that they are contained in  $(0, 1)$ , see Equation (1). This assures that on each side some voters do abstain from voting while others cast their votes. It hence excludes the less relevant cases where all voters of a side vote or all of them do not vote. The first statement of the proposition assumes that the propensities to vote are equal for yes- and no-voters. Obviously, a majority of yes-voters in the whole population,  $y \geq \frac{1}{2}$ , will then lead to a majority of yes-voters among the participating voters. The participation rate is constant and equal to  $yY^* + (1-y)N^* = Y^* = N^*$ . In this case, any quorum below or equal to the propensity  $Y^*$  or  $N^*$  is automatically met and the default outcome is free to choose (in the proposition the average propensity  $\frac{1}{2}(Y^* + N^*)$  is used to stress the similarity with the optimal quorum in the second statement). Since the majority of the participating voters perfectly reflects the majority among the population, the population majority outcome is achieved. Note especially that the quorum  $q = 0$  is allowed, which is identical to the case of not having a quorum. Intuitively, when the propensities to vote are equal, there is no bias towards accepting or rejecting the proposal and no quorum is needed. However, since the participation rate is constant, any sufficiently low quorum does no harm.

The second statement assumes that the propensities to vote are different. With the found expressions for  $Y^*$  and  $N^*$ , the optimal quorum can be expressed as the average propensity to vote  $\frac{1}{2}(Y^* + N^*)$ . To see why this is the case, assume that  $\bar{m}^y > \bar{m}^n$  (symmetric arguments hold for the opposite case). This assumption

implies that  $Y^* > N^*$ . Yes-voters are more likely to vote and without a quorum there is a bias towards accepting the proposal. When a quorum is introduced, it can only offset this bias if the default outcome is rejecting the proposal,  $D = R$ . The participation rate  $yY^* + (1 - y)N^*$  is strictly increasing in  $y$ . This shows that a majority of the population is in favor of the proposal,  $y \geq \frac{1}{2}$ , if and only if the participation rate is higher than  $\frac{1}{2}(Y^* + N^*)$ . The population majority outcome can thus be achieved by the quorum  $q^* = \frac{1}{2}(Y^* + N^*)$ . Note that the majority constraint is redundant: whenever the referendum is valid, a majority of the participating voters is in favor of the proposal. Instead of the fraction of participating voters in favor, the participation rate is the decisive variable. The model thus has a strong prediction: for a correctly set quorum the default outcome will never occur as the outcome of a valid referendum.

At first sight it might seem counterintuitive that the optimal quorum is increasing in the propensity to vote of both yes- and no-voters: the bias towards accepting is increased when yes-voters become more likely to vote, but it is decreased when no-voters become more likely to vote. An increased bias might need a higher quorum and a decreased bias a lower quorum. This reasoning correctly assesses the effect on the bias *in the absence of a quorum*. However, when the optimal quorum is imposed, the previous paragraph showed that the majority constraint is redundant. An increase in the propensity to vote of yes-voters has an identical effect on the quorum constraint as an increase in the propensity to vote of no-voters. More voters will indeed vote, so the quorum is more likely to be met and the probability of accepting the proposal is increased. To achieve the population majority outcome, an increase in the quorum is needed.

**3.3.2. Calculating Voters.** Now suppose that the voters are calculating, so  $\gamma < 0$ . The potential disutility of an unnecessary vote makes that less voters indeed take the effort to cast their ballots compared to the simple-hearted voters. *Ceteris paribus*, this leads to a lower optimal quorum. To construct a referendum that achieves the population majority outcome, the probability of a majority of yes-voters among the whole population is needed. Let  $\xi$  denote this probability, so  $\xi = \mathbb{P}[y \geq \frac{1}{2}]$ . From the assumptions on the distribution of  $y$  it follows that  $\xi \in (0, 1)$ . The following proposition states that with the right design of the referendum, the population majority outcome occurs.

PROPOSITION 2. (Calculating Voters and the Population Majority Outcome)

Assume that  $\gamma < 0$  and  $\bar{m}^y, \bar{m}^n \in (-\frac{\alpha}{2} - \gamma, \frac{\alpha}{2})$ .

i) When  $\bar{m}^y = \bar{m}^n + \gamma(1 - 2\xi)$ , the population majority outcome is only achieved in the unique equilibrium of the referendum with a quorum of at most  $q^* = \frac{1}{2} + \frac{\bar{m}^y + \bar{m}^n + \gamma}{2\alpha}$  and default outcome  $D \in \{A, R\}$ .

ii) When  $\bar{m}^y \neq \bar{m}^n + \gamma(1 - 2\xi)$ , the population majority outcome is only achieved in the unique equilibrium of the referendum with quorum  $q^* = \frac{1}{2} + \frac{\bar{m}^y + \bar{m}^n + \gamma}{2\alpha}$  and the default outcome  $D = R$  if  $\bar{m}^y > \bar{m}^n + \gamma(1 - 2\xi)$  and  $D = A$  otherwise.

The intuition for the proposition follows again from first looking to the propensities to vote. In the population majority outcome the probability that the proposal is accepted is given by  $\xi$ . The probability that the proposal is rejected is then given by  $1 - \xi$ . This means that the propensities to vote of yes-voters and no-voters are given by  $Y^* = \frac{1}{2} + (\bar{m}^y + \gamma\xi)/\alpha$  and  $N^* = \frac{1}{2} + (\bar{m}^n + \gamma - \gamma\xi)/\alpha$  respectively. The condition that  $\bar{m}^y$  and  $\bar{m}^n$  are contained in  $(-\frac{\alpha}{2} - \gamma, \frac{\alpha}{2})$  implies that for all  $\xi \in (0, 1)$  the propensities to vote  $Y^*$  and  $N^*$  are between 0 and 1. In other words, the condition ensures that for a fraction  $\gamma/\alpha$  of the voters indeed their voting decisions depend on their expectations (that  $\gamma < \alpha$  follows from the same condition). The first statement of the proposition now claims that when the propensities to vote are equal for yes- and no-voters, the referendum with a quota below or equal to  $\frac{1}{2}(Y^* + N^*)$  achieves the population majority outcome. The reason is the same as for the simple-hearted voters: with equal propensities to vote the fractions of yes- and no-voters among the participating voters are identical to the population fractions. No quorum is needed, but a sufficiently small quorum does not affect the outcome of the referendum since the participation rate is constant.

When the propensities are not equal, according to the second statement a quorum is needed to achieve the population majority outcome. In fact, the optimal quorum is again the average of the propensities to vote, but now evaluated at the equilibrium,  $q^* = \frac{1}{2}(Y^* + N^*)$ . To get more intuition, assume that  $\bar{m}^y > \bar{m}^n + \gamma(1 - 2\xi)$  (symmetric arguments hold for the opposite case). This implies that  $Y^* > N^*$ . Similar to the model with simple-hearted voters, a quorum with rejecting as default outcome,  $D = R$ , is needed to offset the bias towards accepting. The participation rate  $yY^* + (1 - y)N^*$  is strictly increasing in  $y$ . The majority of the population is in favor if and only if the participation rate is higher than  $\frac{1}{2}(Y^* + N^*)$ . Since in this

case the yes-voters constitute a majority, the quorum  $q^* = \frac{1}{2}(Y^* + N^*)$  achieves the population majority outcome.

Compared to the model with simple-hearted voters, there are two important differences. Firstly, *ceteris paribus* the optimal quorum is lower in case of calculating voters. Comparing the expressions for  $q^*$  in the second statements of Propositions 1 and 2 shows that in the model with calculating voters the quorum is  $-\gamma/\alpha$  lower. Some of the voters who would have cast their ballot when they would have been simple-hearted, prefer not to do so when they are calculating. A lower quorum is needed to offset a lower participation rate. This shows that when the referendum is designed for a population of simple-hearted voters while instead the voters are calculating, the quorum is set too high. In case  $\bar{m}^y > \bar{m}^n + \gamma(1 - 2\xi)$ , the quorum will only be met when the true proportion of yes-voters is at least  $y^*$  for  $y^* > \frac{1}{2}$ . The proposal is thus rejected for  $y \in [\frac{1}{2}, y^*)$ . When  $\mathbb{P}[y \in [\frac{1}{2}, y^*)] > 0$ , the referendum with the incorrectly set quorum will not achieve the population majority outcome and there is a tendency towards the default outcome  $R$ .

A second difference compared to the model with simple-hearted voters is that the design of the optimal referendum requires knowledge of  $\xi = \mathbb{P}[y \geq \frac{1}{2}]$ . Somewhat surprisingly, this knowledge is not needed for setting the optimal quorum. Instead, the knowledge of  $\xi$  is needed for setting the default outcome optimally. Intuitively, for the optimal quorum only the sum of the reductions in voters matters, while for the optimal default outcome the difference matters. When  $\gamma = 0$  the propensity to vote is independent of the expectations. However, when  $\gamma < 0$  the propensities to vote will in general depend on  $\gamma$ . Only when a population majority of yes- and no-voters is equally likely, so  $\xi = \frac{1}{2}$ , the default outcomes coincide with those in case of simple-hearted voters. When  $\xi \neq \frac{1}{2}$ , there will be fewer participating yes- and no-voters in equilibrium than in case of simple-hearted voters. When  $\xi > \frac{1}{2}$ , the decrease in yes-voters is larger than the decrease in no-voters. The choice of the default outcome needs to take account of this effect. The term  $\gamma(1 - 2\xi)$  in the conditions accomplishes this. This effect is increasing in the extent to which voters calculate,  $\gamma$ . Note that the model with simple-hearted voters can be seen as the limiting case of the model with calculating voters and  $\gamma \rightarrow 0$ .

**3.3.3. Affectionate Voters.** Now consider the model with affectionate voters, so  $\gamma > 0$ . The expectations about the outcome of the referendum again matter. But

now the higher the probability that the preferred outcome occurs, the more likely that a voter indeed casts her ballot. *Ceteris paribus*, this leads to more participating voters and hence to a higher optimal quorum than in case of simple-hearted voters. Compared to those voters, the affectionate voters have a tendency to behave in a coordinated way. This gives rise to the possibility of multiple equilibria. The following proposition states that although the referendum can be designed such that the population majority outcome occurs, under a certain condition there is indeed another equilibrium.

**PROPOSITION 3.** (Affectionate Voters and the Population Majority Outcome)

Assume that  $\gamma > 0$  and  $\bar{m}^y, \bar{m}^n \in (-\frac{\alpha}{2}, \frac{\alpha}{2} - \gamma)$ .

*i) The population majority outcome is achieved in an equilibrium of the referendum designed as specified in Proposition 2.*

*ii) For the quorum  $q^*$ , the equilibrium mentioned in i) is the unique equilibrium when  $|\bar{m}^y - \bar{m}^n| \geq \gamma$ , otherwise there is a single alternative equilibrium which is characterized by  $\mathbb{P}[D] = 1$ .*

The first statement shows that the expressions for the optimal quorum in case of calculating voters also hold for affectionate voters. Compared to the model with simple-hearted voters, the optimal quorum is higher with affectionate voters since voters are more likely to participate. Comparing the expressions for the optimal quorum of the three models shows that the quorum is increasing in the extent of affection  $\gamma$  (or decreasing in the extent voters calculate  $-\gamma$ ).

The proposition states that multiple equilibria can indeed arise. The second statement claims that when  $\bar{m}^y$  and  $\bar{m}^n$  are sufficiently close to each other, the optimal quorum does not necessarily lead to the population majority outcome.<sup>6</sup> In fact, this quorum can discourage the opponents of the default outcome from voting, an effect that is aggravated by the tendency to coordinate. This might give rise to an equilibrium where none of the voters expects the quorum to be met and because the voters adapt their behavior to this expectation, the quorum will indeed never be met. When  $|\bar{m}^y - \bar{m}^n| < \gamma$  the fact that voters base their decisions to vote on expectations together with their tendency to coordinate gives rise to self-fulfilling equilibria. When instead the difference between  $\bar{m}^y$  and  $\bar{m}^n$  is sufficiently big, the

<sup>6</sup>In case  $\bar{m}^y = \bar{m}^n + \gamma(1 - 2\xi)$  and  $q < q^*$  the equilibrium can be unique, but there can also be two other equilibria, see Appendix 3.B for details.



equilibrium with  $\mathbb{P}[D] = 1$  is not feasible anymore. To see why, suppose  $\bar{m}^y \geq \bar{m}^n + \gamma$ . Even when  $\mathbb{P}[R] = 1$  the propensity to vote of yes-voters is (weakly) higher as that of no-voters. There will be a positive probability of accepting the proposal, which is a contradiction.

A graphical representation provides additional insight in why the equilibrium is necessarily unique for the calculating voters but not for the affectionate voter. In Figures 1 and 2,  $p - \mathbb{P}[A]$  is shown as function of  $p$  for calculating and affectionate voters respectively. Recall that in equilibrium  $p - \mathbb{P}[A] = 0$ . In case of calculating voters,  $\gamma < 0$ , the propensity to vote  $Y = \frac{1}{2} + (\bar{m}^y + \gamma p)/\alpha$  is decreasing in  $p$ . The propensity to vote  $N = \frac{1}{2} + (\bar{m}^n + \gamma - \gamma p)/\alpha$  is increasing in  $p$  at the same rate. The participation rate for  $y = \frac{1}{2}$  is thus independent of  $p$ . But as discussed above, for the optimal quorum only the quorum constraint is binding. This implies that for all  $p$  the quorum constraint is also satisfied if and only if  $y \geq \frac{1}{2}$ . For small  $p$  the probability of accepting the proposal is then  $\xi$  until the no-voters are more likely to participate than yes-voters. In this case the quorum constraint and the majority constraint cannot be simultaneously met and  $\mathbb{P}[A] = 0$ . The function  $p - \mathbb{P}[A]$  is thus strictly increasing and has a un upwards jump. Since it is increasing, it crosses the x-axis at most once. The choice of the default outcome implies that the jump is after  $\xi$ , so that indeed an equilibrium exists.

In case of affectionate voters,  $\gamma > 0$ ,  $Y$  is increasing in  $p$  and  $N$  decreasing. Arguments opposite to the ones above show that  $\mathbb{P}[A]$  is zero for small  $p$ , while it jumps to  $\xi$  for larger  $p$ . This implies that  $p - \mathbb{P}[A]$  is not strictly increasing in  $p$ . There can be two equilibria: one with  $\mathbb{P}[A] = 0$  and one with  $\mathbb{P}[A] = \xi$ . The choice of the default outcome guarantees that the latter equilibrium exists. When  $|\bar{m}^y - \bar{m}^n| < \gamma$ , yes-voters have a lower propensity to vote than no-voters for  $p = 0$ . This implies that the quorum constraint and the majority constraint cannot be simultaneously. Since then  $\mathbb{P}[A] = 0$ , there is a second equilibrium in which the default outcome always occurs.

**3.3.4. Heterogenous Voter Types.** We now allow for heterogenous voters. To be more specific, the population can consist of simple-hearted, calculating and affectionate voters. Moreover, the parameters  $\alpha$  and  $\gamma$  can differ across voters. This means that a voter  $k$  is defined by her preference, i.e. in favor or against the proposal, and the parameters  $(\bar{m}_k, \alpha_k, \gamma_k)$ . Define the parameter set  $\mathcal{P} = \mathbb{R} \times (0, \infty) \times \mathbb{R}$ . Now

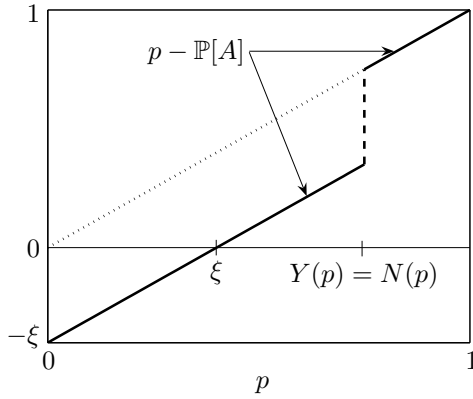


FIGURE 1. For calculating voters the optimal quorum leads to a unique equilibrium with the population majority outcome ( $\mathbb{P}[A] = \xi$ ).

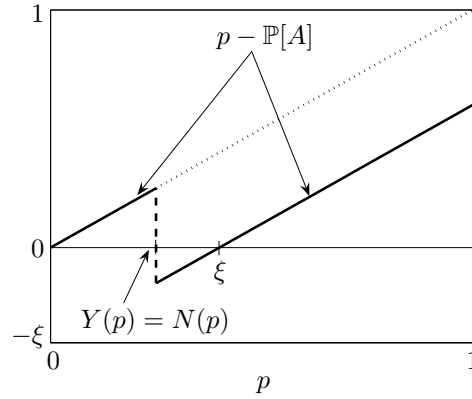


FIGURE 2. For affectionate voters the optimal quorum can also lead to an equilibrium in which the proposal is never accepted ( $\mathbb{P}[A] = 0$ ).

define the subset  $\hat{\mathcal{P}}$  of  $\mathcal{P}$  as follows

$$\hat{\mathcal{P}} = \left\{ (\bar{m}, \alpha, \gamma) \in \mathcal{P} \mid \bar{m} \in \left( -\frac{\alpha}{2} + \max\{0, -\gamma\}, \frac{\alpha}{2} + \min\{0, -\gamma\} \right) \right\}.$$

Note that this restriction resembles the assumptions on  $\bar{m}^y$  and  $\bar{m}^n$  in Proposition 1-3. In fact, for any parameters  $(\bar{m}_k, \alpha_k, \gamma_k) \in \hat{\mathcal{P}}$  the assumption in the proposition indicated by  $\gamma_k$  is satisfied for  $\bar{m}_k, \alpha_k$  and  $\gamma_k$ . Denote the distribution function of the parameters of yes-voter  $i$  by  $\Phi^y$  and of no-voter  $j$  by  $\Phi^n$ . By the law of large numbers,  $\Phi_y$  and  $\Phi^n$  are also the population distributions. Denote the density functions by  $\phi^y$  and  $\phi^n$  respectively. The first condition on the density functions is that  $\phi^y(\bar{m}_k, \alpha_k, \gamma_k) = \phi^n(\bar{m}_k, \alpha_k, \gamma_k) = 0$  if  $(\bar{m}_k, \alpha_k, \gamma_k) \notin \hat{\mathcal{P}}$ . This assures that of all the yes- or no-voters with a type  $(\bar{m}_k, \alpha_k, \gamma_k)$  that can occur, some will indeed vote while others will not. Now define the following average parameters of the yes-voters

$$\bar{m}^y = \mathbb{E}_y \left[ \frac{\bar{m}_i}{\alpha_i} \right] = \int_{\hat{\mathcal{P}}} \frac{\bar{m}_i}{\alpha_i} d\Phi^y(\bar{m}_i, \alpha_i, \gamma_i),$$

$$\gamma^y = \mathbb{E}_y \left[ \frac{\gamma_i}{\alpha_i} \right] = \int_{\hat{\mathcal{P}}} \frac{\gamma_i}{\alpha_i} d\Phi^y(\bar{m}_i, \alpha_i, \gamma_i).$$

Denote the counterparts for the no-voters by  $\bar{m}^n$  and  $\gamma^n$ . The second condition on the

density function is that  $\gamma^y = \gamma^n$ . Since this is equivalent to  $\mathbb{E}_y[\gamma_i/\alpha_i] = \mathbb{E}_n[\gamma_j/\alpha_j]$ , this condition is satisfied if for example  $\bar{m}_k$  and  $(\alpha_k, \gamma_k)$  are independently distributed and the density function for  $(\alpha_k, \gamma_k)$  is independent of being in favor or against the proposal. The common density function is the analogue of the assumption made in the previous section that  $\gamma$  is a population parameter and that the scaling parameter  $\alpha$  of the moral pressure distribution is equal for both voter groups. Although this assumption is mainly made to keep the model tractable, there are no reasons to assume that  $\gamma^y$  and  $\gamma^n$  are very different. When they are close to each other, the outcomes will be similar to when they are identical. Define  $\gamma = \gamma^y = \gamma^n$ . The second condition implies that both the average type, i.e. simple-hearted, calculating or affectionate, and the extent of the affection (or the extent to which voters are calculating) scaled by  $\alpha$  are equal among yes- and no-voters. The following proposition claims that knowledge of these average parameters together with  $\xi = \mathbb{P}[y \geq \frac{1}{2}]$  is sufficient to design a referendum that achieves the population majority outcome.

**PROPOSITION 4.** (Heterogenous Voters and the Population Majority Outcome)

*Assume that the supports of  $\Phi^y$  and  $\Phi^n$  are contained in  $\hat{\mathcal{P}}$  and that  $\mathbb{E}_y[\frac{\gamma_i}{\alpha_i}] = \mathbb{E}_n[\frac{\gamma_j}{\alpha_j}]$ . Then, the quorum, default outcome and uniqueness of the population majority outcome are as in the model with only the representative voter types defined by  $(\bar{m}^y, 1, \gamma)$  and  $(\bar{m}^n, 1, \gamma)$ .*

The proposition states that when the population consists of simple-hearted, calculating and affectionate voters and when the other parameters are allowed to vary across the voters, the quorum and default options should be set as for the population that only consists of the representative voter types  $(\bar{m}^y, 1, \gamma)$  and  $(\bar{m}^n, 1, \gamma)$ . Hence, the analysis in the first three subsections is not a simplification but instead describes models with heterogenous voter types as well. When the signs and sizes of individual  $\gamma_k$ 's can be different, an increase in  $p$  has different effects on voters with different  $\gamma_k$ 's. In case of different signs, it makes some voters more willing to vote and others less. Only the average effect counts for setting the optimal quorum. Note that the representative voter types also determine whether the optimal quorum necessarily results in the population majority outcome or that the equilibrium with  $\mathbb{P}[D] = 1$  can occur as well.

### 3.4. A Non-Optimal Quorum

In this section we analyze the consequences of a non-optimal quorum. There are two reasons why a non-optimal quorum can arise. Firstly, the quorum could have been set non-optimally due to insufficient knowledge about the relevant parameters or for political reasons. Secondly, after the quorum is set, whether optimally or not, pressure groups have incentives to affect the behavior of voters in order to make their preferred outcome more likely.

Throughout it is assumed that the proportion of yes-voters  $y$  has a uniform distribution on  $[\underline{y}, \bar{y}]$  with  $\underline{y} < \frac{1}{2} < \bar{y}$ . Let  $\phi$  denote the density, so  $\phi = (\bar{y} - \underline{y})^{-1}$ . The probability of accepting the proposal according to the population majority is then given by  $\xi = \phi(\bar{y} - \frac{1}{2})$ .

The analyses for the default outcomes  $A$  and  $R$  are symmetric. We assume  $D = R$  so the proposal can only be accepted when the referendum is valid and when a majority of the participating voters is in favor.

**3.4.1. A Not-Optimally Set Quorum.** First consider the simple-hearted voters with  $\gamma = 0$ . The outcome of the referendum does not affect the behavior of the voters so the propensities to vote  $Y$  and  $N$  are fixed. When it is known which constraints are binding, the probability of accepting the proposal can be computed in a straightforward manner using the three cases considered in Subsection 3.2.3. Denote this probability by  $p_m$  when only the majority constraint is binding, by  $p_q$  when only the quorum constraint is binding and by  $p_b$  when both constraints are binding. Let  $s$  denote the sum of the propensities to vote, so  $s = Y + N$ . The probabilities of accepting the proposal given the binding constraints are then

$$\begin{aligned} p_m &= \phi \left( \bar{y} - \frac{N}{s} \right), \\ p_q &= \phi \frac{\bar{y}Y + (1 - \bar{y})N - q}{Y - N}, \\ p_b &= \phi \frac{q - \frac{2YN}{s}}{Y - N}. \end{aligned}$$

To analyze the effect of the quorum on the probability of accepting the proposal, these equilibrium probabilities are related to the quorum in the following proposition. Instead of framing the proposition in terms of the deep parameters  $\bar{m}^y$ ,  $\bar{m}^n$ ,  $\alpha$  and  $\gamma$ , it is easier to use  $Y$  and  $N$ .

PROPOSITION 5. (Simple-Hearted Voters and a Not-Optimally Set Quorum)

Suppose  $\gamma = 0$ .

i) Suppose  $\bar{m}^y \geq \bar{m}^n$  and  $\frac{N}{s} > \underline{y}$ . Then

$$\mathbb{P}[A] = \begin{cases} p_m & \text{if } q \leq \frac{2YN}{s}, \\ p_q & \text{if } \frac{2YN}{s} < q \leq \bar{y}Y + (1 - \bar{y})N, \\ 0 & \text{if } \bar{y}Y + (1 - \bar{y})N < q. \end{cases}$$

ii) Suppose  $\bar{m}^y < \bar{m}^n$  and  $\frac{N}{s} < \bar{y}$ . Then

$$\mathbb{P}[A] = \begin{cases} p_m & \text{if } q \leq \bar{y}Y + (1 - \bar{y})N, \\ p_b & \text{if } \bar{y}Y + (1 - \bar{y})N \leq q \leq \frac{2YN}{s}, \\ 0 & \text{if } \frac{2YN}{s} \leq q. \end{cases}$$

A first observation is that for every quorum an equilibrium exists. To see why this is the case, the function  $p - \mathbb{P}[A]$  is key. Although for the optimal quorum  $q^*$  this function is discontinuous in  $p$ , it is continuous for a non-optimal quorum. Together with the fact that  $\mathbb{P}[A] \in [0, 1]$  this shows that there is at least one  $p \in [0, 1]$  for which  $p - \mathbb{P}[A] = 0$ . There thus exists an equilibrium.

When the propensity to vote is higher for yes-voters than for no-voters,  $Y > N$ , the default outcome is correctly set. This case is discussed in the first statement of the proposition and depicted in Figure 3. The probability of acceptance is constant for a low quorum. The quorum will always be met and the majority constraint is binding. The definition of  $p_m$  shows that in this case  $p_m > \xi$ . Intuitively, for a quorum below the optimal quorum  $q^*$ , the referendum will be too often valid and the probability of acceptance is above  $\mathbb{P}[y \geq \frac{1}{2}]$ . Note that the condition  $N/s > \underline{y}$  implies that  $p_m < 1$ . When  $q$  increases, more participating voters are needed to meet the quorum. Since  $Y > N$ , the required proportion of yes-voters increases. When  $q$  increases further, the quorum constraint takes over from the majority constraint. The probability of acceptance decreases and crosses  $\xi$ . For higher  $q$  it can reach a level such that even with the highest participation rate  $\bar{y}Y + (1 - \bar{y})N$  the quorum can not be met. From here on, the probability of acceptance equals zero.

When the propensities to vote for yes- and no-voters are equal, the participation rate is constant. The quorum constraint is either always satisfied or never. According to the first statement,  $p_q$  does not occur since the two borders are equal. The probability of acceptance suddenly drops from  $p_m = \xi$  to 0 if  $q$  raises above  $\frac{1}{2}s$ .

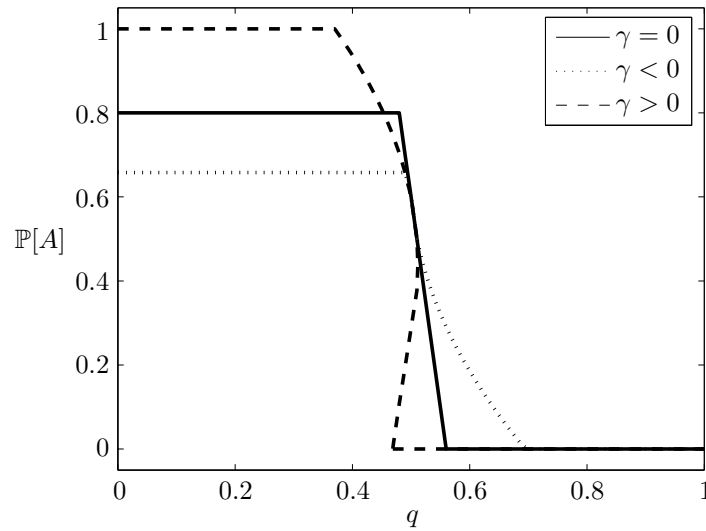


FIGURE 3. The effect of the quorum on the probability of acceptance when the default outcome is correctly set.<sup>7</sup>

The second statement assumes that the default outcome is incorrectly set. The definition of  $p_m$  shows that even when the quorum is so low that it is not affecting the referendum, the probability of acceptance is below the population majority outcome  $\xi$ . The condition  $N/s < \underline{y}$  implies that the probability of acceptance is positive. When the quorum constraint becomes binding, it imposes an upper bound on the proportion of yes-voters. Since the propensity to vote is lower for yes-voters than for no-voters, the quorum will not be met when there are too many yes-voters. When the quorum is higher than  $2YN/s$  more than half of the participating voters should be no-voters, but then the majority constraint cannot be satisfied and the probability of accepting the proposal is zero.

The proposition shows that when the quorum is lower than the optimal quorum  $q^*$ , the probability of accepting the proposal is at most  $p_m$ . It also shows that when the quorum is set higher than the optimal quorum, it can be 0. Especially when the difference between the average moral pressures  $\bar{m}^y$  and  $\bar{m}^n$  is small, so that  $Y$  and  $N$  are similar and  $p_m$  is close to  $\xi$ , it is less harmful when the quorum is set

<sup>7</sup>This figure uses  $\alpha = 2$ ,  $\gamma \in \{-0.9, 0, 0.9\}$ ,  $\underline{y} = 0.3$ ,  $\bar{y} = 0.8$  and thus  $\xi = 0.6$ . Since the range of admissible values of  $\bar{m}^y$  and  $\bar{m}^n$  is determined by  $\gamma$ , the average moral pressures need to be adjusted for different values of  $\gamma$ . Using  $\bar{m}^y = 0.2$  and  $\bar{m}^n = -0.2$  when  $\gamma = 0$ , the adjustment  $\bar{m}^y = 0.2 - \frac{1}{2}\gamma$  and  $\bar{m}^n = -0.2 - \frac{1}{2}\gamma$  achieves that the optimal quorum is the same for all  $\gamma$  and equal to  $\frac{1}{2}$ .

too low than too high. Moreover, suppose that the exact values of  $\bar{m}^y$  and  $\bar{m}^n$  are not known. When the quorum is based on their expected values, it will be as often too low as too high. But to assess the effect on the outcome, it is important that a too high quorum is more harmful. Hence, the uncertainty about the average moral pressures causes the proposal to be rejected too often.

When  $\gamma \neq 0$ , the propensities to vote depend on the probability that the proposal is accepted, which in turn depends on the propensities to vote. As in the case for the simple-hearted voters, the equilibrium probabilities can be computed if it is known which constraints are binding. We here discuss the results using Figure 3; Appendix 3.B contains the precise statements.

For the model with calculating voters, so  $\gamma < 0$ , an equilibrium exists for every quorum when  $\gamma$  is not too negative. This ensures that changes in the probability of accepting the proposal do not have too big impacts. Note that the interpretation of  $\gamma$  as the average across heterogeneous voters suggest that the value of  $\gamma$  is not that extreme. Since the calculating voters show some “balancing” behavior, changes in  $q$  effect the equilibrium probability more gradually than for the simple-hearted voters. The effects of a not-optimally set quorum are thus similar though less severe.

The model with affectionate voters,  $\gamma > 0$ , is more complicated. Here, an upper bound on  $\gamma$  is needed to limit the effect of the equilibrium probability on the voters. As was shown in the previous section, even for the optimal quorum two equilibria can exist. When the quorum is not optimally set there can be up to three equilibria.<sup>8</sup> As before, multiple equilibria can arise since the model resembles a coordination game. Voters act according to what they expect and thereby make their expectations happen, in other words, there are self-fulfilling prophecies. Changes in  $q$  thus have a larger impact than for the simple-hearted voters. Note especially that when the quorum is set already slightly too high (in the figure the optimal quorum is 0.5), a sure rejection will result. Again, setting the quorum a bit too low is less harmful than setting it a bit too high.

In case of three equilibria, the middle one only serves to separate the others. This equilibrium is unstable in the sense that when a small fraction of voters changes

---

<sup>8</sup>Although it cannot be seen from the figure, there is a hole in the graph when  $\gamma > 0$ : for the optimal quorum  $q^* = \frac{1}{2}(Y(\xi) + N(\xi))$  the equilibrium in the middle does not exist conform Proposition 3.

behavior, this would trigger changes in the behavior of other voters that would ultimately lead to one of the other equilibria. Although their instability makes them less appealing, they cannot be completely ignored in the analysis. Clearly, the properties of the stable and unstable equilibria are opposites. So, a higher quorum decreases the probability of acceptance in the stable equilibria with a positive probability, but increases it in the unstable equilibria.

**3.4.2. Pressure Groups.** After the quorum is set, pressure groups might want to affect the turnout of the voters. For example, in the Italian referendum no-voters were urged to go to the beach instead of the ballot box.<sup>9</sup> In our model, we assume that pressure groups cannot directly affect the behavior of voters of the other side: a yes-pressure group can only affect the average moral pressure of the yes-voters  $\bar{m}^y$  and a no-pressure group only the average moral pressure of no-voters  $\bar{m}^n$ . In essence, the model has become a group-based voting model of mobilization.

We still assume that the preferences of the voters are given. Although before this was already a simplification, in the face of pressure groups, it needs even more justification. Apart from affecting the participation rate of their side, these pressure groups have of course incentives to try to convert voters. For example, Neijens and van Praag (2006) discuss the dynamics of opinion formation and show that a large fraction of the voters changes their opinion in the period before the election. The assumption that voters' preferences are given thus implies that the model deals with the short period directly preceding the referendum day. Since affecting the participation rate is just a part of the pressure group strategy, we will only analyze its marginal effect. Its sign already indicates in which direction a pressure group should affect the voters. Herrera and Mattozzi (2007) discuss a referendum model where pressure groups setting the participation rates play against each other.

**3.4.2.1. Yes-Pressure Groups.** The equilibrium probabilities of accepting the proposal follow from rewriting the conditions stated in Proposition 5. The analysis of the not-optimally set quorum dealt separately with a correctly and an incorrectly set default outcome. When the effect of the average moral pressures is analyzed, it matters whether the moral propensity to vote of the other side is above or below

---

<sup>9</sup>Hanafin (2006) discusses in detail the strategic lobbying that preceded the enacting of the fertility law in 2004 and the failure of the referendum.



the quorum. Remember that the propensities to vote  $Y$  and  $N$  should be between 0 and 1.

PROPOSITION 6. (Simple-Hearted Voters and Yes-Pressure Groups)

Suppose  $\gamma = 0$  and  $N > \frac{y}{1-y}$ .

i) Suppose  $N \leq q$ . Then

$$\mathbb{P}[A] = \begin{cases} 0 & \text{if } Y < \frac{q-(1-\bar{y})N}{\bar{y}}, \\ p_q & \text{if } N \leq \frac{1}{2}q \quad \text{and} \quad \frac{q-(1-\bar{y})N}{\bar{y}} \leq Y, \\ & \text{or if } N > \frac{1}{2}q \quad \text{and} \quad \frac{q-(1-\bar{y})N}{\bar{y}} \leq Y \leq \frac{qN}{2N-q}, \\ p_m & \text{if } N > \frac{1}{2}q \quad \text{and} \quad \frac{qN}{2N-q} \leq Y. \end{cases}$$

ii) Suppose  $N > q$ . Then

$$\mathbb{P}[A] = \begin{cases} 0 & \text{if } Y \leq \frac{qN}{2N-q}, \\ p_b & \text{if } \frac{qN}{2N-q} \leq Y \leq \frac{q-(1-\bar{y})N}{\bar{y}}, \\ p_m & \text{if } \frac{q-(1-\bar{y})N}{\bar{y}} \leq Y. \end{cases}$$

The first statement assumes that the propensity to vote of no-voters is so low that the quorum is not met when everyone is against the proposal, when  $N < q$ , or exactly met when  $N = q$ . This case is depicted in Figure 4. Even for low values of the average moral pressure of yes-voters, the quorum will not be met. The propensity to vote needs to be higher than  $q$  before the quorum can be met to offset the low propensity of the no-voters. In this case the quorum constraint will be binding. Now suppose that  $N$  is not too low, so  $N > \frac{1}{2}q$ . When  $\bar{m}^y$  is increased further, the quorum constraint is always met and it is the majority constraint that determines the equilibrium probability. When  $N$  is below  $\frac{1}{2}q$ , the majority constraint is always satisfied if the quorum constraint is satisfied. In this case the equilibrium probability remains  $p_q$ . The condition that  $N > \frac{y}{1-y}$  guarantees that  $\mathbb{P}[A] < 1$ . Comparison between this proposition and Proposition 5 shows that increasing  $Y$  is similar to decreasing  $q$ .

The second statement assumes that when all voters are no-voters, the quorum constraint is met. In this case, the quorum can already be met for  $Y < q$ . The quorum constraint is then binding from above, so the equilibrium probability is given by  $p_b$ . When  $Y$  is increased further, the quorum constraint is always satisfied. From here on  $p_m$  determines the equilibrium probability. Again, increasing  $Y$  is similar to decreasing  $q$ .

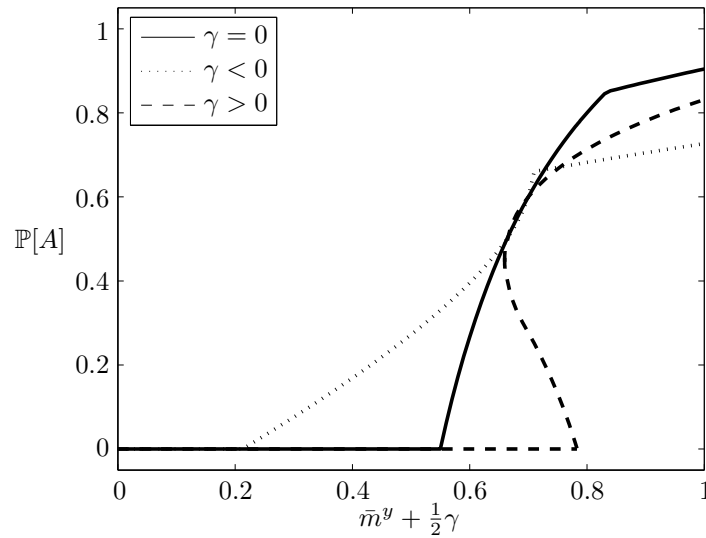


FIGURE 4. The effect of the average moral pressure of yes-voters on the probability of acceptance.<sup>10</sup>

Figure 4 also shows the equilibria for calculating and affectionate voters. In both cases the equilibrium lines are similar to the mirrored images of those in Figure 3. This reflects that increasing the propensity to vote of yes-voters is comparable to decreasing the quorum. For the calculating voters there is again a unique equilibrium. The offsetting behavior leads to positive probabilities for lower values of  $Y$  and to smoother effects of  $\bar{m}^y$  in general. Changes in the propensity to vote of yes-voters are partially undone by their own calculating attitude.

In case of affectionate voters multiple equilibria again exist for intermediate values of  $\bar{m}^y$ . The equilibria in the middle are unstable. Similar to the effect of a quorum slightly higher than the optimal quorum, a propensity to vote slightly below the value for which the quorum is optimal, which is  $2q - N$ , immediately leads to a sure rejection (in the figure the quorum is optimal for  $\bar{m}^y + \frac{1}{2}\gamma = 0.2$ ).

For all voter types, an increase in  $Y$  leads *ceteris paribus* to more participating yes-voters. The majority constraint is met for lower values of  $y$ . Since there are more participating voters also the quorum constraint is met for lower  $y$ . This shows that apart from the unstable equilibria when  $\gamma > 0$  and the equilibria with  $\mathbb{P}[A] = 0$ , an increase in  $Y$  raises the equilibrium probability of accepting the proposal. Loosely

<sup>10</sup>Figure 4 uses  $\bar{m}^n = -0.2 - \frac{1}{2}\gamma$ ,  $\alpha = 2$ ,  $\gamma \in \{-0.9, 0, 0.9\}$ ,  $q = 0.5$ ,  $\underline{y} = 0.3$  and  $\bar{y} = 0.8$ . Note that when  $\gamma = 0$ ,  $Y$  ranges from 0.25 to 0.75.

speaking, a yes-pressure group should always encourage voters to participate by increasing  $\bar{m}^y$ .

3.4.2.2. *No-Pressure Groups.* For no-pressure groups the recommendation is not that straightforward. On the one hand, an increase in  $N$  leads to more participating no-voters so that the participating no-voters are a majority for lower  $y$ . On the other hand, an increase in  $N$  leads to more participating voters so that the quorum is met for lower  $y$ . When the referendum is valid more often, this can lead to a higher probability of accepting the proposal. To analyze these opposite effects in more detail, the following proposition states the equilibrium probabilities as function of  $N$ .

PROPOSITION 7. (Simple-Hearted Voters and No-Pressure Groups)

Suppose  $\gamma = 0$  and  $Y > \frac{1-\bar{y}}{y}$ .

i) Suppose  $Y \geq q$  and  $Y < \frac{q}{2y}$ . Then

$$\mathbb{P}[A] = \begin{cases} 0 & \text{if } Y < \frac{q}{y} & \text{and } N < \frac{q-\bar{y}Y}{1-\bar{y}}, \\ p_q & \text{if } Y < \frac{q}{y} & \text{and } \frac{q-\bar{y}Y}{1-\bar{y}} \leq N \leq \frac{qY}{2Y-q}, \\ & \text{or if } Y \geq \frac{q}{y} & \text{and } N \leq \frac{qY}{2Y-q}, \\ p_m & \text{if } \frac{qY}{2Y-q} \leq N. \end{cases}$$

ii) Suppose  $Y < q$ . Then

$$\mathbb{P}[A] = \begin{cases} 0 & \text{if } Y < \frac{1}{2}q, \\ & \text{or if } Y \geq \frac{1}{2}q & \text{and } N < \frac{qY}{2Y-q}, \\ p_b & \text{if } Y \geq \frac{1}{2}q & \text{and } \frac{qY}{2Y-q} \leq N \leq \frac{q-\bar{y}Y}{1-\bar{y}}, \\ p_m & \text{if } Y \geq \frac{1}{2}q & \text{and } \frac{q-\bar{y}Y}{1-\bar{y}} \leq N. \end{cases}$$

The first statement assumes that  $Y > q$ . This case is depicted in Figure 5. When  $Y < q/\bar{y}$  the quorum is not met when  $N = 0$ . The equilibrium probability equals zero until the quorum will be met when the proportion of yes-voters equals  $\bar{y}$ . When  $Y \geq q/\bar{y}$  the quorum constraint is binding from the beginning onwards. When  $N$  is sufficiently high, the quorum constraint is always met and the majority constraint determines the equilibrium probability. Since an increase in  $N$  makes a valid referendum more likely,  $p_b$  is increasing in  $N$ . On the other hand, an increase in  $N$  makes a majority of the participating no-voters more likely, so  $p_m$  is decreasing in  $N$ . It is clear that the maximum probability of accepting the proposal is attained for  $N = qY/(2Y - q)$ . The condition  $Y < \frac{1}{2}q/\bar{y}$  implies that the maximum probability

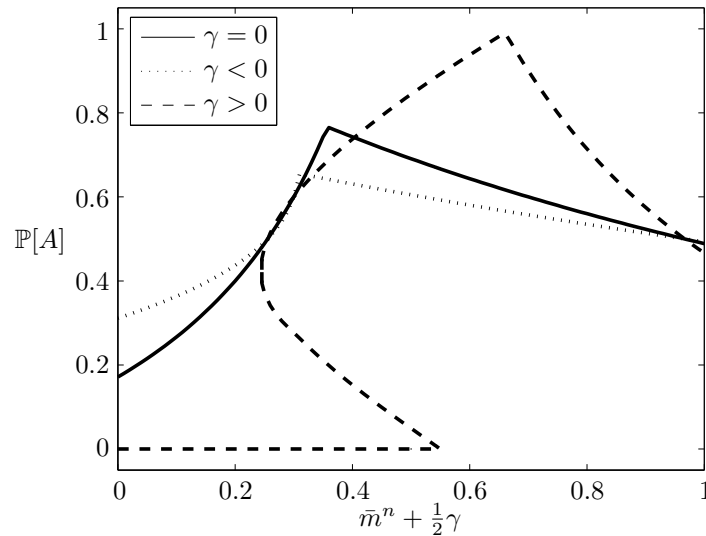


FIGURE 5. The effect of the average moral pressure of no-voters on the probability of acceptance.<sup>11</sup>

of accepting the proposal is below 1. The condition  $Y > (1 - \bar{y})/\bar{y}$  implies that even when  $N = 1$ , the yes-voters can constitute the majority of the participating voters, so that  $\mathbb{P}[A] > 0$ .

The second statement assumes that the propensity to vote of the yes-voters is below the quorum. When the propensity is below  $\frac{1}{2}q$ , the quorum constraint and the majority constraint cannot be simultaneously met and the probability of accepting the proposal is 0. When  $Y \geq \frac{1}{2}q$ , the equilibrium probability is also zero for low  $N$ . Only for higher  $N$  it becomes positive. Note that in this case  $N > q > Y$ , so that both constraints are binding. The equilibrium probability is determined by  $p_b$  until  $N$  is so high that the quorum is always satisfied. From here on the majority constraint is binding.

For the calculating and the affectionate voters similar reasonings hold. It should not come as a surprise that the equilibrium for the calculating voters is unique and as function of  $\bar{m}^n$  flatter than for the simple-hearted voters. For the affectionate voters there are multiple equilibria possible as before. Again, when  $N$  is slightly below the value implied by the quorum, which is  $2q - Y$ , the only equilibrium has  $\mathbb{P}[A] = 0$  (in the figure the quorum is optimal for  $\bar{m}^n + \frac{1}{2}\gamma = -0.2$ ).

<sup>11</sup>Figure 5 uses  $\bar{m}^y = 0.2 + \frac{1}{2}\gamma$ ,  $\alpha = 2$ ,  $\gamma \in \{-0.9, 0, 0.9\}$ ,  $q = 0.5$ ,  $\underline{y} = 0.3$  and  $\bar{y} = 0.8$ . Note that when  $\gamma = 0$ ,  $N$  ranges from 0.25 to 0.75.

It is clear than in all stable equilibria with  $\mathbb{P}[A] > 0$ , the probability of acceptance is increasing for low  $N$  and decreasing for high  $N$ . There thus exists a value of  $N$  for which  $\mathbb{P}[A]$  attains its maximum. Denote this value by  $\hat{N}$ . Under the conditions of the proposition,  $\hat{N}$  for the simple-hearted voters is given by

$$\hat{N} = \begin{cases} \frac{qY}{2Y-q} & \text{if } Y \geq q, \\ \min \left\{ \frac{qY+2Y^2\sqrt{\frac{q}{Y}-1}}{2Y-q}, \frac{q-\bar{y}Y}{1-\bar{y}}, 1 \right\} & \text{if } Y < q. \end{cases}$$

The expression in the second line follows from setting the derivative of  $p_b$  to zero and noting that the maximum should be attained before the majority constraint takes over or the propensity to vote exceeds 1. Loosely speaking, a no-pressure group should *decrease*  $\bar{m}^n$  when  $N$  is below  $\hat{N}$  and *increase*  $\bar{m}^n$  when  $N$  is higher than  $\hat{N}$ . This is in line with intuition: when the propensity to vote of no-voters is rather high, the quorum is likely to be met. To ensure that the participating no-voters form the majority, a no-pressure group should encourage no-voters to vote. When on contrary the propensity to vote is rather low, the quorum will probably not be met. A no-pressure group should now lower the propensity to vote even further to decrease the probability that the quorum is met.

### 3.5. Conclusion

In this paper we studied the impact of the quorum on referendum outcomes. Although a quorum is potentially useful to attain the population majority outcome, this crucially depends on the ability of setting the quorum at the appropriate level. Insufficient knowledge or a lack of political power to do so tend to favor the status quo. Moreover, when voters care more about the outcome when they are participating, there can be a second equilibrium in which the referendum is always invalid. Pressure groups opposing the proposal should also strategically aim for an invalid outcome when turnout is expected to be low.

This paper thus adds another critique concerning the use of referenda to the list of Nurmi (1998). Without resorting to compulsory voting, the choice is between imposing a quorum and accepting its possible distortions on the one hand and not imposing a quorum and accepting the possible non-representativeness of the participating voters on the other. Clearly, if a low turnout is expected, a referendum is not the ideal tool for decision making. Also topics for which minority groups have

some strong opinions should be excluded from opinions. When the turnout on both sides is expected to be at least moderate a referendum can be appropriate. The results of this paper suggest that in this case imposing a quorum is more harmful than not imposing one. This argument for abolishing the quorum complements the arguments of Felsenthal and Machover (1997) who show that the highest degree of democratic participation is achieved, i.e. the opinion of the average voter achieves its maximum impact, in the absence of a quorum. Without a quorum, each side can only reach its aim by convincing voters of its position and of the necessity to vote. This is clearly more in line with democratic principles than giving one side the possibility to abuse the rules of the game.

However, in a recent referendum in Portugal about easing restrictions on abortion, the Catholic Church did *not* urge voters to stay at home. Interestingly, late polls suggested a significant majority of proponents, with as only doubt “whether enough voters will turn out for the result to be constitutionally binding” (The Economist 2007). This would have been the ideal case to discourage opponents from participating. Although this would just have been strategically exploiting the referendum rules, reactions on their campaign in Italy might have made the Catholic Church to act closer in line with the democratic principles underlying referenda.

### Bibliography

- Ashworth, J., Geys, B. and Heyndels, B. (2006), ‘Everyone likes a winner: An empirical test of the effect of electoral closeness on turnout in a context of expressive voting’, *Public Choice* **128**(3-4), 383–405.
- Bartels, L. M. (1988), *Presidential primaries and the dynamics of public choice*, Princeton University Press, Princeton, NJ.
- Blais, A. (2000), *To Vote or Not to Vote*, University of Pittsburgh Press, Pittsburgh, PA.
- Blais, A., Young, R. and Lapp, M. (2000), ‘The calculus of voting: An empirical test’, *European Journal of Political Research* **37**, 181–201.
- Clausen, A. R. (1968), ‘Response Validity: Vote Report’, *The Public Opinion Quarterly* **32**(4), 588–606.
- Côrte-Real, P. P. and Pereira, P. T. (2004), ‘The voter who wasn’t there: Referenda, representation and abstention’, *Social Choice and Welfare* **22**, 349–369.
- Downs, A. (1957), *An Economic Theory of Democracy*, University of Pittsburgh Press, Pittsburgh.
- Feddersen, T. J. (2004), ‘Rational Choice Theory and the Paradox of Not Voting’, *Journal of Economic Perspectives* **18**(1), 99–112.

- Felsenthal, D. S. and Machover, M. (1997), 'Ternary Voting Games', *International Journal of Game Theory* **26**, 335–351.
- Fishburn, P. and Brahm, S. (1983), 'Paradoxes of preferential voting', *Math Magazine* **56**, 207–214.
- Fort, R. and Bunn, D. N. (1998), 'Whether one votes and how one votes', *Public Choice* **95**(1-2), 51–62.
- Franklin, M. N. (1999), 'Electoral Engineering and Cross-National Turnout Differences: What Role for Compulsory Voting?', *British Journal of Political Science* **29**(1), 205–216.
- Haan, M. A. and Kooreman, P. (2003), 'How majorities can lose the election. Another voting paradox', *Social Choice and Welfare* **20**, 509–522.
- Hanafin, P. (2006), 'Gender, citizenship and human reproduction in contemporary Italy', *Feminist Legal Studies* **14**, 329–352.
- Herrera, H. and Mattozzi, A. (2007), 'Quorum and turnout in referenda', *mimeo*.
- Jakee, K. and Sun, G.-Z. (2006), 'Is compulsory voting more democratic?', *Public Choice* **129**(1-2), 61–75.
- Judd, K. L. (1985), 'The law of large numbers with a continuum of i.i.d. random variables', *Journal of Economic Theory* **35**, 19–25.
- Kanazawa, S. (1998), 'A Possible Solution to the Paradox of Voter Turnout', *The Journal of Politics* **60**(4), 974–995.
- Levine, D. K. and Palfrey, T. R. (2007), 'The Paradox of Voter Participation? A Laboratory Study', *American Political Science Review* **101**(1), 143–158.
- Marquette, J. F. and Hinckley, K. A. (1988), 'Voter Turnout and Candidate Choice: A Merged Theory', *Political Behavior* **10**(1), 52–76.
- Matsusaka, J. G. (2005), 'The eclipse of legislatures: Direct democracy in the 21st century', *Public Choice* **124**(1-2), 157–177.
- May, K. O. (1952), 'A Set of Independent Necessary and Sufficient Conditions for Simple Majority Voting', *Econometrica* **20**(4), 680–684.
- Myatt, D. P. (2007), 'On the Theory of Strategic Voting', *Review of Economic Studies* **74**(1), 255–281.
- Myerson, R. B. (2000), 'Large Poisson Games', *Journal of Economic Theory* **94**(1), 7–45.
- Neijens, P. and van Praag, P. (2006), 'The dynamics of opinion formation in local popular referendums: why the Dutch always say no', *International Journal of Public Opinion Research* **18**(4), 445–462.
- Nurmi, H. (1998), 'Voting Paradoxes and Referenda', *Social Choice and Welfare* **15**(3), 333–350.
- Riker, W. and Ordeshook, P. (1968), 'A Theory of the Calculus of Voting', *American Political Science Review* **62**, 25–42.
- Rosema, M. (2004), *The Sincere Vote. A Psychological Study of Voting*, PhD thesis, University of Leiden.
- Schuessler, A. A. (2000), 'Expressive Voting', *Rationality and Society* **12**(1), 87–119.
- The Economist (2007), 'A painful choice', **February 10**.

The Initiative & Referendum Institute (n.d.), *web sites*, [www.iandrinstitute.org](http://www.iandrinstitute.org) and [www.iri-europe.org](http://www.iri-europe.org).

The New York Times (2005), ‘Vote on Fertility Law Fires Passions in Italy’, **June 11**.

Waters, M. D. (2003), *Initiative and Referendum Almanac*, Carolina Academic Press, Durham, NC.

### Appendix 3.A. Different Intensities of Voters’ Preferences

Suppose that the proposal should only be accepted if at least a fraction  $\hat{y}$  of the population is in favor. We call this the *optimal outcome*. The referendum design is broadened by also allowing for a qualified majority among the referendum participants. Let the qualified majority  $\theta$  denote the required fraction of participating voters in favor of the non-default outcome. Define  $q^* = \hat{y}Y + (1 - \hat{y})N$  and  $\theta^* = \hat{y}Y/q^*$ . The following proposition considers simple-hearted voters, analogue results hold for calculating or affectionate voters.

PROPOSITION 8. (Intensities of Voters’ Preferences)

Assume that  $\gamma = 0$  and  $\bar{m}^y, \bar{m}^n \in (-\frac{\alpha}{2}, \frac{\alpha}{2})$ .

i) When  $\bar{m}^y = \bar{m}^n$ , the optimal outcome is only achieved in the unique equilibrium of the referendum with a required majority of  $\theta^*$ , a quorum of at most  $q^*$  and default outcome  $D \in \{A, R\}$ .

ii) When  $\bar{m}^y \neq \bar{m}^n$ , the optimal outcome is only achieved in the unique equilibrium of a referendum with either a qualified majority of at most  $\theta^*$  and quorum  $q^*$  or a referendum with qualified majority  $\theta^*$  and a quorum of at most  $q^*$ . In both cases the default outcome is  $D = R$  if  $\bar{m}^y > \bar{m}^n$  and  $D = A$  otherwise.

Statement i) follows by noting that  $\theta^* = \hat{y}$  and that the participation rate equals  $q^*$ . Statement ii) follows by noting that the participation constraint or the (qualified) majority constraint (or both) should be exactly binding when a fraction  $\hat{y}$  of the population is in favor. A sufficiently low quorum or qualified majority is always met when the other constraint is satisfied.

Note that in the paper the required majority among referendum participants is set at 50%. Although allowing for a qualified majority would introduce other referendum designs with the same outcome, focussing on a majority of 50% is the most neutral from a political point of view.



**Appendix 3.B. A Non-Optimal Quorum when  $\gamma \neq 0$** 

First consider the model with calculating voters, so  $\gamma < 0$ . Define  $\hat{p}$  as the probability for which the propensities to vote of yes-voters and no-voters are equal. Using the definitions of  $Y(\hat{p})$  and  $N(\hat{p})$  gives

$$\hat{p} = \frac{\alpha}{2\gamma}(N(0) - Y(0)).$$

The uniform distribution of the moral pressures has the convenient property that the sum of the propensities to vote  $s$  is constant

$$s = Y(p) + N(p) = \frac{1}{2} + \frac{\bar{m}^y + \gamma p}{\alpha} + \frac{1}{2} + \frac{\bar{m}^n + \gamma - \gamma p}{\alpha} = 1 + \frac{\bar{m}^y + \bar{m}^n - \gamma}{\alpha}.$$

When only the majority constraint is binding, the equilibrium condition  $p_m - \phi(\bar{y} - N(p_m)/s) = 0$  gives

$$p_m = \frac{\phi(\bar{y} - \frac{N(0)}{s})}{1 - \frac{\phi\gamma}{\alpha s}}.$$

When only the quorum constraint is binding, the equilibrium condition  $p_q - \phi(\bar{y} - (q - N(p_q))/(Y(p_q) - N(p_q))) = 0$  defines a second order polynomial equation in  $p_q$  with solutions

$$p_q^\pm = \frac{1}{2}(\hat{p} + \xi) \pm \sqrt{\frac{1}{4}(\hat{p} + \xi)^2 + \frac{\phi\alpha}{2\gamma}(\bar{y}Y(0) + (1 - \bar{y})N(0) - q)}.$$

Similarly, when both conditions are binding, the equilibrium condition  $p - \phi((q - N(p_b))/(Y(p_b) - N(p_b)) - N(p_b)/s) = 0$  defines a second order polynomial in  $p_b$  with solutions

$$p_b^\pm = \frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} \pm \sqrt{\left(\frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\right)^2 \hat{p}^2 + \frac{\phi\alpha}{2\gamma} \frac{q - \frac{2Y(0)N(0)}{s}}{1 - \frac{\phi\gamma}{\alpha s}}}.$$

The equilibrium probabilities are related to the quorum in the following proposition.

PROPOSITION 9. (Calculating Voters and a Not-Optimally Set Quorum)

Suppose  $\gamma < 0$  with  $1 + \frac{\phi\gamma}{\alpha s} > 0$ .

i) Suppose  $\bar{m}^y \geq \bar{m}^n + \gamma(1 - 2\xi)$  and  $\frac{N(1)}{s} > \underline{y}$ . Then

$$\mathbb{P}[A] = \begin{cases} p_m & \text{if } q \leq \frac{2Y(p_m)N(p_m)}{s}, \\ p_q^- & \text{if } \frac{2Y(p_m)N(p_m)}{s} \leq q \leq \bar{y}Y(0) + (1 - \bar{y})N(0), \\ 0 & \text{if } \bar{y}Y(0) + (1 - \bar{y})N(0) \leq q. \end{cases}$$

ii) Suppose  $\bar{m}^y < \bar{m}^n + \gamma(1 - 2\xi)$  and  $N(0) < Y(0)$ . Then

$$\mathbb{P}[A] = \begin{cases} p_m & \text{if } q \leq \bar{y}Y(p_m) + (1 - \bar{y})N(p_m), \\ p_b^+ & \text{if } \bar{y}Y(p_m) + (1 - \bar{y})N(p_m) \leq q \leq \frac{1}{2}s, \\ p_q^- & \text{if } \frac{1}{2}s \leq q \leq \bar{y}Y(0) + (1 - \bar{y})N(0), \\ 0 & \text{if } \bar{y}Y(0) + (1 - \bar{y})N(0) \leq q. \end{cases}$$

The proposition requires  $\gamma > -\alpha s/\phi$ . The first statement assumes that the quorum is correctly set. This case is depicted in Figure 3. The condition  $N(1)/s > \underline{y}$  implies that  $p_m < 1$ . Similar to the model with simple-hearted voters,  $p_m > \xi$  (this is made formal in the proof of the proposition). The second statement assumes that the default outcome is incorrectly set. Although the equilibrium probability  $p_m$  is positive, it is below  $\xi$ .

Now consider the model with affectionate voters,  $\gamma > 0$ . The only candidates for the equilibrium probabilities are again  $p_m$ ,  $p_q^\pm$  and  $p_b^\pm$ . Before stating the proposition that relates these probabilities with the quorum, two critical values of the quorum  $q$  are needed

$$q_q = \frac{\gamma}{2\phi\alpha}(\hat{p} + \xi)^2 + \bar{y}Y(0) + (1 - \bar{y})N(0),$$

$$q_b = \frac{2Y(0)N(0)}{s} - \frac{\gamma}{\phi\alpha} \frac{\left(1 - 2\frac{\phi\gamma}{\alpha s}\right)^2}{2 - 2\frac{\phi\gamma}{\alpha s}} \hat{p}^2.$$

From the definition of  $p_q^\pm$  it can be seen that  $p_q^+$  and  $p_q^-$  only exist for  $q \leq q_q$ . Similarly,  $p_b^+$  and  $p_b^-$  only exist for  $q \geq q_b$ .

PROPOSITION 10. (Affectionate Voters and a Not-Optimally Set Quorum)

Suppose  $\gamma > 0$  with  $1 - \frac{\phi\gamma}{\alpha s} > 0$ .

i) Suppose  $\bar{m}^y \geq \bar{m}^n + \gamma(1 - 2\xi)$  and  $Y(0) < N(0)$ . Then

$$\mathbb{P}[A] = \begin{cases} 1 & \text{if } \frac{N(1)}{s} \leq \underline{y} & \text{and } q \leq \underline{y}Y(1) + (1 - \underline{y})N(1), \\ p_m & \text{if } \frac{N(1)}{s} \geq \underline{y} & \text{and } q \leq \frac{2N(p_m)Y(p_m)}{s}, \\ p_q^+ & \text{if } \frac{N(1)}{s} \leq \underline{y} & \text{and } \underline{y}Y(1) + (1 - \underline{y})N(1) \leq q \leq q_q, \\ & \text{or if } \frac{N(1)}{s} \geq \underline{y} & \text{and } \frac{2N(p_m)Y(p_m)}{s} \leq q \leq q_q, \\ p_q^- & \text{if } \frac{1}{2}s < q \leq q_q, \\ p_b^+ & \text{if } 1 - 2\frac{\phi\gamma}{\alpha s} \geq 0 & \text{and } q_b \leq q < \frac{1}{2}s, \\ & \text{or if } 1 - 2\frac{\phi\gamma}{\alpha s} \leq 0 & \text{and } \frac{2Y(0)N(0)}{s} \leq q < \frac{1}{2}s, \\ p_b^- & \text{if } 1 - 2\frac{\phi\gamma}{\alpha s} \geq 0 & \text{and } q_b \leq q \leq \frac{2Y(0)N(0)}{s}, \\ 0 & \text{if } \frac{2Y(0)N(0)}{s} \leq q. \end{cases}$$

ii) Suppose  $\bar{m}^y < \bar{m}^n + \gamma(1 - 2\xi)$  and  $\frac{N(0)}{s} < \bar{y}$ . Then

$$\mathbb{P}[A] = \begin{cases} p_m & \text{if } q \leq \bar{y}Y(p_m) + (1 - \bar{y})N(p_m), \\ p_b^+ & \text{if } Y(\xi) - N(\xi) \geq -2\frac{\gamma}{\alpha}\xi \\ & \text{and } q_b \leq q \leq \bar{y}Y(p_m) + (1 - \bar{y})N(p_m), \\ p_b^- & \text{if } Y(\xi) - N(\xi) \geq -2\frac{\gamma}{\alpha}\xi \text{ and } q_b \leq q \leq \frac{2Y(0)N(0)}{s}, \\ & \text{or if } Y(\xi) - N(\xi) \leq -2\frac{\gamma}{\alpha}\xi \\ & \text{and } \bar{y}Y(p_m) + (1 - \bar{y})N(p_m) \leq q \leq \frac{2Y(0)N(0)}{s}, \\ 0 & \text{if } \frac{2Y(0)N(0)}{s} \leq q. \end{cases}$$

The proposition requires  $\gamma < \alpha s / \phi$ . The first statement assumes that the default outcome is correctly set. This case is depicted in Figure 3. The condition  $Y(0) < N(0)$  excludes the case where yes-voters have always the highest propensity to vote. When  $N(1)/s < \underline{y}$ , the majority constraint is always satisfied for a low quorum. Otherwise the equilibrium probability  $p_m$  is below 1 though above  $\xi$ . For both cases, the quorum constraint becomes binding when the  $q$  increases. There are two possible equilibria,  $p_q^+$  and  $p_q^-$ . A necessary condition for their existence is  $Y > N$ , so they should be higher than  $\hat{p}$ . They should be lower than  $p_m$ , since equilibria with a higher probability are not possible. It follows that  $p_q^+$  exists from the point where it equals  $\min\{1, p_m\}$  until  $q_q$ , while  $p_q^-$  exists when the quorum is higher than  $\frac{1}{2}s$  but at most  $q_q$ . When the probability of acceptance is below  $\hat{p}$ , it follows that  $Y < N$ . This shows that both constraints are binding. The equilibrium with  $p_b^+$  exists until

$\frac{1}{2}s$ , since it then equals  $\hat{p}$ . When it starts from  $p_b^+ = 0$ , the  $p_b^-$  equilibrium does not exist. When  $p_b^+$  exists from  $q_b$  onwards,  $p_b^+ > 0$  and the  $p_b^-$  equilibrium exists between  $q_b$  and  $2Y(0)N(0)/s$ . For a higher quorum the equilibrium with  $\mathbb{P}[A] = 0$  exists.

The second statement assumes that the default outcome is incorrectly set. Similar to the simple-hearted voters,  $p_m$  is below  $\xi$ . There exists a range with three equilibria when  $\gamma$  is not too small.

When pressure groups can affect the turnout of voters, the equilibria are found by using Propositions 9 and 10 and rearranging the conditions.

### Appendix 3.C. Proofs

#### PROOF OF PROPOSITION 1.

This proposition is proved in the main text. □

#### PROOF OF PROPOSITION 2.

Assume that  $\bar{m}^y \geq \bar{m}^n + \gamma(1 - 2\xi)$  (the proof of statement ii) with  $\bar{m}^y < \bar{m}^n + \gamma(1 - 2\xi)$  follows in the same way). An equilibrium is characterized by  $p - \mathbb{P}[A] = 0$  and the analysis can be confined to  $p \in [0, 1]$ . Note that  $Y(p) = \frac{1}{2} + (\bar{m}^y + \gamma p)/\alpha$  is strictly decreasing in  $p$  and  $N(p) = \frac{1}{2} + (\bar{m}^n + \gamma - \gamma p)/\alpha$  strictly increasing. The participation rate equals

$$yY(p) + (1 - y)N(p) = 1 + \frac{y(\bar{m}^y + \gamma p) + (1 - y)(\bar{m}^n + \gamma - \gamma p)}{\alpha}.$$

The first step is to determine the quorum values for which the population majority outcome can occur. When the proposal should be accepted if and only if  $y \geq 0$ , it follows that  $\mathbb{P}[A] = \xi$ . When  $\bar{m}^y > \bar{m}^n + \gamma(1 - 2\xi)$ , so that  $Y(\xi) > N(\xi)$ , there is already a majority of yes-voters for  $y < \frac{1}{2}$ . To ensure that the proposal is only accepted for  $y \geq \frac{1}{2}$ , the quorum constraint should be exactly binding for  $y = \frac{1}{2}$ . This implies that the quorum should be  $q^* = \frac{1}{2}Y(\xi) + \frac{1}{2}N(\xi)$ . When  $\bar{m}^y = \bar{m}^n + \gamma(1 - 2\xi)$ , so that  $Y(\xi) = N(\xi)$ , the fractions of participating voters in favor and against are identical to the population fraction. Any quorum below  $q^*$  is always met and the majority constraint correctly determines the outcome.

The second step is to establish that only the population majority outcome can occur for the found quorum values. Suppose first that the quorum is  $q^*$ .

When  $Y(p) > N(p)$  the participation rate is increasing in  $y$ . Since for  $y = \frac{1}{2}$  it equals  $q^*$ , the quorum constraint is only met when  $y \geq \frac{1}{2}$ . Since in this case also the majority constraint is met, the probability of accepting the proposal is  $\xi$ .

When  $Y(p) = N(p)$  the participation rate is constant and equal to  $q^*$ . The fractions of participating voters in favor and against are identical to the population fractions. The quorum is always met and the majority constraint only when  $y \geq \frac{1}{2}$ , so  $\mathbb{P}[A] = \frac{1}{2}$ .

When  $Y(p) < N(p)$  the participation rate is decreasing in  $y$ . Since for  $y = \frac{1}{2}$  it equals  $q^*$ , this means that the quorum constraint can only be met for  $y < \frac{1}{2}$ . However, for these cases the majority constraint is violated and  $\mathbb{P}[A] = 0$ .

To summarize  $p - \mathbb{P}[A] = p - \xi \mathbf{1}_{\{Y(p) \geq N(p)\}}$  (see also Figure 1). Remember that  $Y$  is decreasing in  $p$  while  $N$  is increasing and that  $Y(0) \geq N(0)$ , hence any solution  $p^*$  of  $p - \mathbb{P}[A] = 0$  thus satisfies  $Y(p^*) \geq N(p^*)$ . Since  $p - \mathbb{P}[A]$  is strictly increasing on  $[0, 1]$ , any solution is necessarily unique. The claim in statement i) now follows by noting that  $p^* = \xi$  is a solution with  $Y(p^*) = N(p^*)$ . The claim in statement ii) follows by noting that  $p^* = \xi$  is a solution with  $Y(p^*) > N(p^*)$ .

Now suppose that  $Y(\xi) = N(\xi)$  and that the quorum is below  $q^*$ . Since  $Y(p) > N(p)$  for  $p < \xi$ , the quorum will be met for  $y < \frac{1}{2}$  and  $\mathbb{P}[A] > \frac{1}{2}$ . This shows that  $p - \mathbb{P}[A] < 0$  for  $p < \xi$ . Likewise it follows that  $p - \mathbb{P}[A] > 0$  for  $p > \xi$ . The equilibrium found above is thus unique.  $\square$

### PROOF OF PROPOSITION 3.

Assume that  $\bar{m}^y \geq \bar{m}^n + \gamma(1 - 2\xi)$  (the proof with  $\bar{m}^y < \bar{m}^n + \gamma(1 - 2\xi)$  follows in the same way). In the same way as in the proof of Proposition 2, the quorum values for which the population majority outcome occur are found. In an identical way it also follows that  $p - \mathbb{P}[A] = p - \xi \mathbf{1}_{\{Y(p) \geq N(p)\}}$ . However, now  $Y$  is increasing in  $p$  while  $N$  is decreasing (see also Figure 2). The proof of statement i) follows by noting that  $p^* = \xi$  is a solution with  $Y(p^*) = N(p^*)$  when  $\bar{m}^y = \bar{m}^n + \gamma(1 - 2\xi)$ , and a solution with  $Y(p^*) > N(p^*)$  when  $\bar{m}^y > \bar{m}^n + \gamma(1 - 2\xi)$ .

The proof of statement ii) follows by noting that since  $p - \mathbb{P}[A]$  is strictly increasing for  $p$  such that  $Y(p) \geq N(p)$ , any other equilibrium should satisfy  $Y(p) < N(p)$ . But for these  $p$  the probability of acceptance  $\mathbb{P}[A]$  is zero, so that  $p - \mathbb{P}[A] = p$ . This shows that  $p = 0$  is the only candidate for a solution. This is only possible if

$Y(0) < N(0)$ , so if  $Y(0) - N(0) = (\bar{m}^y - \bar{m}^n - \gamma)/\alpha < 0$ . This gives the condition for uniqueness.  $\square$

PROOF OF PROPOSITION 4.

In equilibrium  $p = \mathbb{P}[A]$  for all voters. The propensity to vote of a yes-voter  $i$  with parameters  $(\bar{m}_i, \alpha_i, \gamma_i)$  is  $\frac{1}{2} + (\bar{m}_i + \gamma_i p)/\alpha_i$ , which follows from the assumption that  $(\bar{m}_i, \alpha_i, \gamma_i) \in \hat{\mathcal{P}}$ . The average propensity to vote is given by

$$Y = \int_{\hat{\mathcal{P}}} \frac{1}{2} + \frac{\bar{m}_i + \gamma_i p}{\alpha_i} d\Phi^y(\bar{m}_i, \alpha_i, \gamma_i) = \frac{1}{2} + \bar{m}^y + \gamma p.$$

Similarly,  $N = \frac{1}{2} + \bar{m}^n + \gamma - \gamma p$ . The proofs of Propositions 1-3 go through with the found expressions for  $Y$  and  $N$  when  $\alpha$  is taken to be 1.  $\square$

PROOF OF PROPOSITION 5.

i) In this case  $Y \geq N$ . First consider  $Y > N$ . When the quorum is sufficiently small, the probability of acceptance is determined by the majority constraint. Since  $N/s < \frac{1}{2}$  and by assumption  $N/s > \underline{y}$ , it follows that  $p_m \in (0, 1)$ . The majority constraint and the quorum constraint coincide for  $q = 2YN/s$ . The quorum constraint is the only binding constraint until it can never be satisfied, so until  $q = \bar{y}Y + (1 - \bar{y})N$ . For a higher quorum the probability of acceptance is 0. Now consider  $Y = N$ . This implies that  $p_m = \xi$ . The quorum is always satisfied as long as  $q \leq Y = N = \frac{1}{2}s$ . A higher quorum can never be satisfied.

ii) In this case  $Y < N$ . Since  $N/s > \frac{1}{2}$  and by assumption  $N/s < \underline{y}$ , it follows that  $p_m \in (0, 1)$ . When the quorum is so low that it is always satisfied, i.e. below  $\bar{y}Y + (1 - \bar{y})N$ , the probability of acceptance is determined by the majority constraint. When  $q$  increases, both constraints are binding until the majority and the quorum constraint can not be simultaneously met. This happens when  $(q - N)/(Y - N) = N/s$ , which is identical to  $q = 2YN/s$ . For a higher quorum the probability of acceptance is 0.  $\square$

PROOF OF PROPOSITION 6.

The proof follows by similar reasoning as the proof of Proposition 5. The only technical detail is that for  $N \leq \frac{1}{2}q$  a proportion  $y$  that satisfies the quorum constraint also satisfies the majority constraint. Clearly  $(1 - y)N \leq \frac{1}{2}(1 - y)q$ . That  $\frac{1}{2}(1 - y)q \leq yY$  follows from  $q \leq yY + (1 - y)N \leq yY + \frac{1}{2}(1 - y)q \leq yY + \frac{1}{2}(1 - y)q + yq$  and moving all terms involving  $q$  to the left hand side.  $\square$

PROOF OF PROPOSITION 7.

The arguments of the proof are similar to previous ones. In case  $Y \geq q$ ,  $p_m < 1$  if  $N/s > \underline{y}$  for  $N = qY/(2Y - q)$ . This is implied by  $Y > \frac{1}{2}q/\bar{y}$ .

When  $Y < \frac{1}{2}q$ , the majority constraint and the quorum constraint cannot be met simultaneously. It is clear that  $yY \leq \frac{1}{2}yq$ . That  $\frac{1}{2}yq \leq (1 - y)N$  follows from  $q \leq yY + (1 - y)N < \frac{1}{2}yq + (1 - y)N \leq \frac{1}{2}yq + (1 - y)q + (1 - y)N$  and moving all terms involving  $q$  to the left hand side.  $\square$

PROOF OF PROPOSITION 8.

This proposition is proved in the text.  $\square$

PROOF OF PROPOSITION 9.

In proving the proposition, the following relations between  $\hat{p}$ ,  $\xi$  and  $p_m$  are used

$$\begin{aligned} \hat{p} - \xi &= -\frac{\bar{m}^y - \bar{m}^n - \gamma(1 - 2\xi)}{2\gamma}, \\ p_m - \hat{p} &= \frac{\bar{m}^y - \bar{m}^n - \gamma(1 - 2\xi)}{2\gamma(1 - \frac{\phi\gamma}{\alpha s})}, \\ p_m - \xi &= \frac{\bar{m}^y - \bar{m}^n - \gamma(1 - 2\xi)}{2\frac{\alpha s}{\phi}(1 - \frac{\phi\gamma}{\alpha s})}. \end{aligned}$$

i) First consider  $\bar{m}^y > \bar{m}^n + \gamma(1 - 2\xi)0$ . The derived relations above show that  $\hat{p} > p_m > \xi$ , so  $Y(p_m) > N(p_m)$  and  $Y(\xi) > N(\xi)$ . Note that no equilibria with  $p^* > p_m$  can occur. The condition that  $N(1)/s \geq \underline{y}$  guarantees that  $p_m < 1$ . The majority constraint is the only binding constraint until it crosses with the quorum constraint, which happens for  $q = 2Y(p_m)N(p_m)/s$ . When the quorum constraint takes over, it does so until it can never be satisfied, which happens for  $q = \bar{y}Y(0) + (1 - \bar{y})N(0)$ . When  $\frac{1}{2}(\hat{p} + \xi) > p_m$  it is clear that  $p_q^+$  cannot be an equilibrium. That this is the case follows by using  $\frac{1}{2}(\hat{p} + \xi) - p_m = \frac{1}{2}(\hat{p} - p_m) + \frac{1}{2}(\xi - p_m)$  and the derived relations above so that

$$\frac{1}{2}(\hat{p} + \xi) - p_m = -\frac{1}{4\gamma} \frac{\bar{m}^y - \bar{m}^n - \gamma(1 - 2\xi)}{1 - \frac{\phi\gamma}{\alpha s}} \left(1 + \frac{\phi\gamma}{\alpha s}\right) > 0.$$

When the quorum is above  $\bar{y}Y(0) + (1 - \bar{y})N(0)$ , the quorum constraint can never be satisfied and  $p^* = 0$ .

Now suppose  $\bar{m}^y = \bar{m}^n + \gamma(1 - 2\xi)$ . Then  $\hat{p} = p_m = \xi$ , and  $2Y(p_m)N(p_m)/s = \frac{1}{2}s$ . So,  $p^* = \xi$  for  $q \leq \frac{1}{2}s$ . Since  $\frac{1}{2}(\hat{p} + \xi) - p_m = 0$ , the equilibrium with  $p_q^+$  does not

exist for a higher quorum. The equilibrium probability is  $p_q^-$  until  $q$  is raised so high that it becomes 0.

ii) The relations derived above show that  $\xi > p_m > \hat{p}$ , so  $Y(p_m) < N(p_m)$  and  $Y(\xi) < N(\xi)$ . The condition  $N(0) < Y(0)$  implies  $\hat{p} > 0$  so that  $p_m > 0$ . When the quorum constraint is sufficiently small  $p^* = p_m$  is the equilibrium. The quorum constraint becomes binding when  $(q - N(p_m))/(Y(p_m) - N(p_m)) = \bar{y}$ . Since  $((1 - 2\phi\gamma/\alpha s)/(2 - 2\phi\gamma/\alpha s))\hat{p} < \hat{p} < p_m$  it is clear that  $p_b^+$  is the equilibrium that takes over from  $p_m$  and that  $p_b^-$  does not exist. For  $q = \frac{1}{2}s$ , it holds that  $q = \frac{1}{2}s^2/s$  and thus

$$\begin{aligned} \frac{1}{2}(\phi\alpha/\gamma)(q - 2Y(0)N(0)/s) &= \frac{1}{2}(\phi\alpha/\gamma s)\left(\frac{1}{2}(Y(0) + N(0))^2 - 2Y(0)N(0)\right) \\ &= \frac{1}{4}(\phi\alpha/\gamma s)(Y(0) - N(0))^2 = (\phi\gamma/\alpha s)\hat{p}^2. \end{aligned}$$

The value of  $p_b^+$  in  $q = \frac{1}{2}s$  equals

$$\begin{aligned} \frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} + \sqrt{\left(\frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\right)^2\hat{p}^2 + \frac{\phi\alpha}{2\gamma}\frac{q - \frac{2Y(0)N(0)}{s}}{1 - \frac{\phi\gamma}{\alpha s}}} \\ = \frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} + \sqrt{\frac{1}{4}\frac{\left(1 - \frac{\phi\gamma}{\alpha s} - \frac{\phi\gamma}{\alpha s}\right)^2}{\left(1 - \frac{\phi\gamma}{\alpha s}\right)^2}\hat{p}^2 + \frac{\left(1 - \frac{\phi\gamma}{\alpha s}\right)\frac{\phi\gamma}{\alpha s}}{\left(1 - \frac{\phi\gamma}{\alpha s}\right)^2}\hat{p}^2} \\ = \frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} + \sqrt{\frac{1}{4}\frac{1}{\left(1 - \frac{\phi\gamma}{\alpha s}\right)^2}\hat{p}^2} = \frac{1 - 2\frac{\phi\gamma}{\alpha s}}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} + \frac{1}{2 - 2\frac{\phi\gamma}{\alpha s}}\hat{p} = \hat{p}. \end{aligned}$$

Since  $\hat{p} = \frac{1}{2}(\alpha/\gamma)(N(0) - Y(0)) > 0$  the  $p_b^+$  equilibrium exists when  $q \leq \frac{1}{2}s$ . For a higher quorum  $Y > N$  and the quorum constraint is the only binding constraint. Since  $\frac{1}{2}(\hat{p} + \xi) > \hat{p}$ , only  $p_q^-$  can be an equilibrium. To find  $p_q^-$  in  $q = \frac{1}{2}s$ , first rewrite  $q = \frac{1}{2}s = -\frac{1}{2}(N(0) - Y(0)) + N(0) = -(N(0) - Y(0))(\bar{y} - \xi/\phi) + N(0)$ , then  $\frac{1}{2}(\phi\alpha/\gamma)(\bar{y}Y(0) + (1 - \bar{y})N(0) - q) = -\frac{1}{2}(\alpha/\gamma)(N(0) - Y(0))\xi = -\hat{p}\xi$  so that

$$\begin{aligned} \frac{1}{2}(\hat{p} + \xi) - \sqrt{\frac{1}{4}(\hat{p} + \xi)^2 + \frac{\phi\alpha}{2\gamma}\left(\bar{y}Y(0) + (1 - \bar{y})N(0) - q\right)} \\ = \frac{1}{2}(\hat{p} + \xi) - \sqrt{\frac{1}{4}(\hat{p} + \xi)^2 - \hat{p}\xi} = \frac{1}{2}(\hat{p} + \xi) - \frac{1}{2}(\xi - \hat{p}) = \hat{p}. \end{aligned}$$

The  $p_q^-$  equilibrium exists until the quorum can never be satisfied, which is the case for  $q = \bar{y}Y(0) + (1 - \bar{y})N(0)$ . For a higher quorum  $p^* = 0$ .  $\square$



PROOF OF PROPOSITION 10.

From  $2N(0)Y(0)/s \leq 2(\frac{1}{2}s\frac{1}{2}s)/s = \frac{1}{2}s$  it follows that  $q_b < \frac{1}{2}s$ . That  $q_q \geq \frac{1}{2}s$  follows from

$$\begin{aligned} \frac{\gamma}{2\phi\alpha}(\hat{p} + \xi)^2 &= \frac{\gamma}{2\phi\alpha}(\hat{p} - \xi)^2 + 4\frac{\gamma}{2\phi\alpha}\hat{p}\xi \\ &= \frac{\gamma}{2\phi\alpha}(\hat{p} - \xi)^2 + (N(0) - Y(0))(\bar{y} - \frac{1}{2}) \end{aligned}$$

so that  $q_q = \frac{1}{2}(\gamma/\phi\alpha)(\hat{p} - \xi)^2 + \frac{1}{2}(Y(0) + N(0)) \geq \frac{1}{2}s$ . The inequality is strict when  $\hat{p} \neq \xi$ , so when  $\bar{m}^y \neq \bar{m}^n + \gamma(1 - 2\xi)$ .

i) First consider  $\bar{m}^y > \bar{m}^n + \gamma(1 - 2\xi)$ . The relations derived at the beginning of the previous proof show that  $p_m > \xi > \hat{p}$ , so that  $Y(p_m) > N(p_m)$  and  $Y(\xi) > N(\xi)$ .

When  $N(1)/s \leq \underline{y}$ , the majority constraint is always satisfied and the equilibrium probability is 1 until the quorum constraint is crossed for the quorum  $\underline{y}Y(1) + (1 - \underline{y})N(0)$ . When  $N(1)/s \geq \underline{y}$  it follows that  $p_m < 1$ . The majority constraint is binding until  $(q - N(p_m))/(Y(p_m) - N(p_m)) = N(p_m)/s$ , which is the stated condition. When the quorum constraint becomes binding  $p_q^+$  is the equilibrium since  $\frac{1}{2}(\hat{p} + \xi) < p_m$  implies that  $p_q^-$  only exists for lower probabilities than  $\frac{1}{2}(\hat{p} + \xi)$ . So,  $p_q^+$  stops to exist at  $q_q$ . Note that for this quorum the minimum of  $p_q^+$  is achieved which equals  $\frac{1}{2}(\hat{p} + \xi)$ . Since this is bigger than  $\hat{p}$ , indeed  $Y > N$ . By assumption  $N(0) > Y(0)$  so that  $\hat{p} > 0$  and  $p_q^+$  exists until  $q_q$ . From here  $p_q^-$  decreases when  $q$  decreases. In the previous proof it was shown that  $p_q^- = \hat{p}$  for  $q = \frac{1}{2}s$ . This shows that  $Y > N$  so that  $p_q^-$  exists for  $q > \frac{1}{2}s$ .

Note that the equilibrium  $p^* = \hat{p}$  does not exist! The only quorum candidate would be  $q = \frac{1}{2}s$ . But for this quorum  $Y(\hat{p}) = N(\hat{p}) = \frac{1}{2}s$ , so the quorum is always met. But, if only the quorum constraint binds,  $p_m$  is the only equilibrium candidate, but  $p_m > \hat{p}$ .

When the quorum decreases from  $\frac{1}{2}s$ , both constraints are binding. When  $p < \hat{p}$  it follows that  $Y < N$ , hence only  $p_b^+$  and  $p_b^-$  are equilibrium candidates. In the previous proof it was shown that  $p_b^+ = \hat{p}$  for  $q = \frac{1}{2}s$ , so that  $Y < N$ . The minimum value of  $p_b^+$  is attained in  $q_b$  and equals  $((1 - 2\phi\gamma/\alpha s)/(2 - 2\phi\gamma/\alpha s))\hat{p}$ . The equilibrium with  $p_b^+$  does not exist on the whole interval from  $q_b$  to  $\frac{1}{2}s$  if  $1 - 2\phi\gamma/\alpha s < 0$ . In this case it only exists when  $q > 2Y(0)N(0)/s$ . When it does exist on the whole interval,  $p_b^-$  exists from  $q_b$  to  $N(0)Y(0)/s$ . In both cases,  $p^* = 0$  when  $q$  is

so big that the majority constraint and the quorum constraint cannot be satisfied simultaneously. This is the case for  $q \geq 2Y(0)N(0)/s$ .

Now consider  $\bar{m}^y = \bar{m}^n + \gamma(1 - 2\xi)$ . The relations derived in the previous proof show that  $\hat{p} = \xi = p_m$ . Note also that  $2N(p_m)Y(p_m)/s = q_q = \frac{1}{2}s$  (see the expression for  $q_q$  derived at the beginning of this proof), so the  $p_q^\pm$  part does not exist. Note also that  $q_b < \frac{1}{2}s$ , which shows that the  $p_b^-$  arm does exist.

ii) The relations derived in the previous proof show that  $\hat{p} > \xi > p_m$ , so that  $Y(p_m) < N(p_m)$  and  $Y(\xi) < N(\xi)$ .

Since by assumption  $N(0)/s < \bar{y}$ , it follows that  $p_m > 0$ . This is the only equilibrium until the quorum constraint becomes binding in  $q = \bar{y}Y(p_m) + (1 - \bar{y})N(p_m)$ . The equilibrium with  $p^* = p_b^+$  can only exist when  $((1 - 2\phi\gamma/\alpha s)/(2 - 2\phi\gamma/\alpha s))\hat{p} < p_m$ , so when

$$\begin{aligned} \frac{1}{2} \left( 1 - 2\frac{\phi\gamma}{\alpha s} \right) \hat{p} - \phi \left( \bar{y} - \frac{N(0)}{s} \right) &= \frac{1}{2} \left( 1 - 2\frac{\phi\gamma}{\alpha s} \right) \hat{p} - \xi + \frac{\phi\gamma}{\alpha s} \hat{p} \\ &= \frac{1}{2} \hat{p} - \xi < 0. \end{aligned}$$

When this is the case, the  $p_b^*$  equilibrium exists from  $q_b$  until  $\bar{y}Y(p_m) + (1 - \bar{y})N(p_m)$ . Note that  $p_b^+ > 0$  since  $\hat{p} > 0$ . When the  $p_b^+$  equilibrium exists, the  $p_b^-$  equilibrium takes over from  $q_b$ , otherwise directly from  $\bar{y}Y(p_m) + (1 - \bar{y})N(p_m)$ . It exists until  $p_b^-$  is zero, which happens at  $2Y(0)N(0)/s$ . For a higher quorum the majority and the quorum constraint are mutually exclusive and  $p^* = 0$ .  $\square$