



**Department of Economics**

# **Essays in Recursive Macroeconomics**

**Pontus Rendahl**

Thesis submitted for assessment with a view to obtaining the degree of  
Doctor of Economics of the European University Institute

Florence, November, 2007

EUROPEAN UNIVERSITY INSTITUTE  
**Department of Economics**

# **Essays in Recursive Macroeconomics**

**Pontus Rendahl**

Thesis submitted for assessment with a view to obtaining the degree of  
Doctor of Economics of the European University Institute

**Examining Board:**

Prof. Stefania Albanesi, Columbia University  
Prof. Giancarlo Corsetti, EUI  
Prof. John Hassler, Stockholm University  
Prof. Morten Ravn, EUI, Supervisor

© 2007, Pontus Rendahl  
No part of this thesis may be copied, reproduced or  
transmitted without prior permission of the author

# ESSAYS IN RECURSIVE MACROECONOMICS

PONTUS RENDAHL

*European University Institute, 2007.*

## Acknowledgements

If I only was allowed to acknowledge one single person's contribution to this thesis, my choice would be as sincere as obvious: My supervisor Morten Ravn. Without ever revealing any sign of loss of faith in my academic ability, Morten has been a continuous support, a motivator, and a great role-model. Morten has taught me how to ask and answer important economic questions; how to write and present scientific results; and he has shown me how to be an honest and devoted researcher. For this I am truly indebted to him, and it is an honor and a privilege being so.

Due not to any fault of his own, Giancarlo Corsetti has probably had a comfortable time at his job as my 2<sup>nd</sup> advisor. His services as an emergency advisor - or advisor of last resort, if you wish - have, fortunately for both of us, never been needed. Yet, Giancarlo has continuously shown sincere support and care for my research. He has, even in my absence, devotedly promoted my work to others, and always made sure to introduce me to other economists from which I could benefit. For this I am truly grateful to Giancarlo.

During my first year at the EUI, I got invaluable help from classmates that is not forgotten. I am foremost indebted to Karel Mertens and Sanne Zwart, without whom I would not have survived for long. I only wish you benefited from me a fraction of what I benefited from you. My thanks from this period also goes to Michael Wycherley and Mario Mariniello. Throughout the years at the EUI, I am not sure whether my numerous coffee breaks with Karel Mertens and Aitor Erce increased or decreased my productivity. But it was fun, and I miss them, so thank you guys.

Several other people have had an impact on my research conducted at the EUI. Fearing that I might have left someone out, I wish to thank Wouter den Haan, Ramon Marimon, Salvador Ortigueira, Rick van der Ploeg, Tom Sargent, and Jonathan Skinner. I also wish to thank my external committee members Stefania Albanesi and John Hassler, for their most appreciated comments and suggestions. My thanks also

goes to the administrative staff at the EUI, and in particular to Jessica Spataro and Lucia Vigna.

From a less scientific perspective, I wish to thank my parents and my brothers for their continuous love and support throughout my doctoral studies. My thoughts goes in particular to my father who endured a serious illness during this period. This thesis is dedicated to them.

*Pontus Rendahl*

*5th of November, 2007.*

## Contents

Acknowledgements	3
Preface	7
Chapter 1. Inequality Constraints in Recursive Economies	9
1. Introduction	10
2. Theory	11
3. Examples	19
4. Concluding Remarks	26
Chapter 2. Asset Based Unemployment Insurance	29
1. Introduction	30
2. Structure of the economy	33
3. Analysis	38
4. Concluding Remarks	49
Chapter 3. Unemployment Insurance for the Liquidity Constrained	51
1. Introduction	52
2. Structure of the economy	54
3. Analysis	60
4. Quantitative Analysis	67
5. Concluding Remarks	71
Bibliography	73
Appendix	77
A. Code for Chapter 1.	78
B. Proofs of Chapter 2	79
C. Proofs of Chapter 3	82

## Preface

This thesis contains several lines of research conducted during my four years at the European University Institute. It deals with two distinct topics in the area of recursive economies, developed in three chapters.

The first chapter considers a general class of recursive models in which inequality constraints pose a challenging problem: Standard Dynamic Programming techniques often necessitate a non established differentiability of the value function, while Euler equation based techniques have problematic or unknown convergence properties. The chapter aims to resolve parts of these two concerns: An envelope theorem is presented that establishes the differentiability of any element in the convergent sequence of approximate value functions when inequality constraints may bind. As a corollary, convergence of an iterative procedure on the Euler equation, usually referred to as time iteration, is ascertained. This procedure turns out to be very convenient from a computational perspective; dynamic economic problems with inequality constraints can be solved reliably and extremely efficiently by exploiting the theoretical insights provided.

The second chapter studies a model of optimal redistribution policies in which agents face unemployment risk and in which savings may provide partial self-insurance. Moral hazard arises as job search effort is unobservable. The optimal redistribution policies provide new insights into how an unemployment insurance scheme should be designed: *First*, the unemployment insurance policy is recursive in an agent's wealth level, and thus independent of the duration of the unemployment spell. *Second*, the level of benefit payments is negatively related to the agent's asset position. The reason behind the latter result is twofold; in addition to the first-order insurance effect of wealth, an increase in non-labor income (wealth) amplifies the opportunity cost of employment and thus reduces the agent's incentive to search for a job.

During unemployment the agent decumulates assets and the sequence of benefit payments is observationally *increasing* - a result that stands in sharp contrast with previous studies.

The third chapter studies a very similar model to that explored in Chapter 2. In contrast, however, I impose a liquidity constraint that limits agents' possibility to borrow. As will be shown, this additional constraint will have salient quantitative implication on how an optimal unemployment insurance programme should be designed. As in the second chapter, the optimal unemployment insurance scheme is recursive in an agent's asset position and her past and current employment status. As a consequence, a liquidity constrained agent receives a *constant* flow of benefit payments throughout the unemployment spell. In the quantitative analysis I show that the effect of a liquidity constraint is of high importance: A constrained agent with zero liquid wealth ought to receive benefits payments three times higher than that received by an agent with wealth equal to one months labor income; twenty times higher than that received by an agent with wealth equal to three months labor income; and one hundred times higher compared to an agent with savings equal to twelve months of labor income (US median labor income to wealth ratio).

CHAPTER 1

**Inequality Constraints in Recursive Economies**

## 1. Introduction

Dynamic models with inequality constraints are of considerable interest to many economists. In microeconomics, and in particular in *consumption theory*, the importance of liquidity constraints is widely recognized (e.g. Deaton, 1991). With respect to macroeconomic models of heterogeneous agents, a debt limit is generally a necessary condition for the existence of an ergodic set (see for instance Ljungqvist and Sargent (2004), Aiyagari (1994) and Krusell and Smith (1998)), and models with limited enforcement have recently proven to provide a realistic description of international co-movements (Kehoe and Perri, 2002). Additionally, inequality constraints may convey substantial empirical relevance; for instance, employment laws may prohibit firing, lending contracts may prevent bank runs. Foreign direct investments, minimum wages, price regulations, etc. are all examples of potentially binding inequality constraints. Nonetheless, solving dynamic economic models with inequality constraints is generally perceived as challenging: Methods that can handle inequality constraints with ease, generally suffer from the curse of dimensionality, while methods that can moderate this curse have difficulties dealing with such constraints. This paper shows the conditions under which the *n-step value function* for a dynamic problem with inequality constraints is differentiable, and utilizes this result to show how a Euler equation based method can deal with inequality constraints in an easily implementable, efficient and accurate manner.<sup>1</sup>

In the context of discretized Dynamic Programming, dealing with inequality constraints is generally straightforward; the state space is trivially delimited such that any inequality constraint cannot be violated. Nevertheless, discretized Dynamic Programming severely suffers from the curse of dimensionality. To circumvent this difficulty, researchers have on many instances relied upon continuous state approximation methods.<sup>2</sup> These procedures generally work well for interior problems where it is known that the value function is differentiable, which is commonly a necessary condition to recover the equilibrium policy function. However, given that Benveniste and Scheinkman's (1979) envelope theorem assumes interiority, this result does not

---

<sup>1</sup>The "*n-step value function*" refers to any element in the sequence  $\{v_n\}_{n \in \mathbb{N}}$ .

<sup>2</sup>Or, equivalently, "*Parameterized Dynamic Programming*".

extend to models where inequality constraints may occasionally bind. In the literature, many researchers have chosen to ignore this problem and to proceed as the value function is known to be differentiable even when such constraints are present.

An appealing approach to deal with inequality constraints in dynamic models is to operate on the Euler equation. Christiano and Fisher (2000) show that such constraints can be dealt with in a straightforward way when preferably using the parameterized expectations algorithm developed by den Haan and Marcet (1990), or a version thereof.<sup>3</sup> However, when using such Euler equation based methods, convergence is far from certain and, without an educated initial guess for the equilibrium policy function, convergence may indeed often fail.<sup>4</sup>

This paper addresses these concerns. It will be shown that under certain conditions, any element of the sequence of value functions defined by *value function iteration* is differentiable when a general class of inequality constraints are considered. Moreover, analytical expressions of their respective derivatives will be presented.

By exploiting these theoretical insights, an iterative procedure on the Euler equation, commonly known as *time iteration*, is derived. Given that this procedure is equivalent to value function iteration it is, under mild initial conditions, a globally convergent method of finding the equilibrium functions for recursively defined, Pareto optimal problems. Due to the concavity of the problem, this turns out to be a very convenient and efficient technique from a computational perspective.

The outline of the paper is the following: Section 2 states and proves the paper's main propositions. Section 3 shows through three examples how the results in section 2 may be implemented in practice. Section 4 concludes.

## 2. Theory

In this section two central propositions will be presented: Proposition 1 establishes the conditions under which any element of the convergent sequence of approximate value functions,  $\{v_n\}_{n \in \mathbb{N}}$ , is differentiable. After defining *time iteration* as a particular

---

<sup>3</sup>See McGrattan (1996) for an alternative Euler equation based technique that utilizes the notion of a “*penalty function*”.

<sup>4</sup>In Christiano and Fisher (2000), a log linearized version of the model is solved and used as an initial guess for the equilibrium functions.

iterative procedure on the Euler equation, Proposition 2 will establish that the sequence of policy functions generated by this method converges to the unique solution.

This paper looks for solutions for problems that may be framed on the basis of the following Bellman equation

$$v(x, z) = \max_{y \in \Gamma(x, z)} \left\{ F(x, y, z) + \beta \int_Z v(y, z') Q(z, dz') \right\} \quad (1)$$

Where  $x \in X$  is the endogenous state,  $z \in Z$  is the exogenous state with a law of motion determined by the stationary transition function  $Q$ . The following is assumed:

- (i)  $X$  is a convex Borel set in  $\mathbb{R}^\ell$  with Borel subsets  $\mathcal{X}$ , and  $Z$  is a compact Borel set in  $\mathbb{R}^k$  with Borel subsets  $\mathcal{Z}$ . Denote the (measurable) product space of  $(X, \mathcal{X})$  and  $(Z, \mathcal{Z})$  as  $(S, \mathcal{S})$ .
- (ii) The transition function,  $Q : Z \times \mathcal{Z} \rightarrow [0, 1]$ , has the Feller property.<sup>5</sup>
- (iii) The feasibility correspondence  $\Gamma : X \times Z \rightarrow 2^X$  is, nonempty, compact-valued, and continuous. Moreover, the set  $A = \{(y, x) \in X \times X : y \in \Gamma(x, z)\}$  is convex in  $x$ , for all  $z \in Z$ .
- (iv) The return function  $F(\cdot, \cdot, z) : A \rightarrow \mathbb{R}$  is, once continuously differentiable, strictly concave and bounded on  $A$  for all  $z \in Z$ .
- (v) The discount factor,  $\beta$ , is in the interval  $(0, 1)$ .

It is important to note that the above definition of the feasibility correspondence includes the possibility of inequality constraints.

If  $v_0$  is (weakly) concave and the above assumptions hold, the following statements are true for any  $n \in \mathbb{N}$  (Section 9.2 in Stokey, Lucas and Prescott, 1989):

- (i) The sequences of functions defined by

$$v_{n+1}(x, z) = \max_{y \in \Gamma(x, z)} \left\{ F(x, y, z) + \beta \int_Z v_n(y, z') Q(z, dz') \right\}$$

$$g_{n+1}(x, z) = \operatorname{argmax}_{y \in \Gamma(x, z)} \left\{ F(x, y, z) + \beta \int_Z v_n(y, z') Q(z, dz') \right\}$$

converge pointwise (in the *sup-norm*) to the unique fixed points  $v$  and  $g$ .<sup>6</sup>

- (ii)  $v$  and  $v_n$  are strictly concave.
- (iii)  $g$  and  $g_n$  are continuous functions.

<sup>5</sup>Alternatively one may assume that  $Z$  is countable and that  $\mathcal{Z}$  is the power set of  $Z$ ;  $\mathcal{Z} = 2^Z$ .

<sup>6</sup>Where  $g$  is the argmax of (1).

For subsequent reference, the following additional assumptions will be used

ASSUMPTION 1. *The feasibility correspondence can be formulated as*

$$\Gamma(x, z) = \{y \in X : m_j(x, y, z) \leq 0, j = 1, \dots, r\}$$

and the functions  $m_j(x, y, z)$ ,  $j = 1, \dots, r$ , are, once continuously differentiable in  $x$  and  $y$ , and convex in  $y$ .

ASSUMPTION 2. *Linear Independence Constraint Qualification (LICQ): The Jacobian of the  $p$  binding constraints has full (row) rank; i.e.  $\text{rank}(\mathbf{J}_m) = p$ .*

ASSUMPTION 3. *The following hold*

- (i)  $\Gamma(x, z) \subset \text{int}(X)$  or
- (ii)  $X$  is compact and  $g_n(x, z) \in \text{int}(X)$ , for all  $n \in \mathbb{N}$ .

Note that Assumption 2 implies that there exists a  $\hat{y}$  such that  $m_j(x, \hat{y}, z) < 0$ , for all  $x, z$  and  $j$  (Slater's Condition). Moreover, part (i) in Assumption 3 implies part (ii), but the converse is generally not true.

Define the operator  $T$  on  $C^1(S)$ , the space of bounded, strictly concave once continuously differentiable functions, as

$$(Tf)(x, z) = \max_{y \in \Gamma(x, z)} \left\{ F(x, y, z) + \beta \int_{\mathcal{Z}} f(y, z') Q(z, dz') \right\} \quad (2)$$

Before moving ahead, it is important to note that under the above additional assumptions it is possible to express the problem in (2) as

$$(Tf)(x, z) = \min_{\mu \geq 0} \max_{y \in X} L(x, y, z, \mu) = \max_{y \in X} \min_{\mu \geq 0} L(x, y, z, \mu) \quad (3)$$

$$L(x, y, z, \mu) = F(x, y, z) + \beta \int_{\mathcal{Z}} f(y, z') Q(z, dz') - \sum_{j=1}^r \mu_j m_j(x, y, z)$$

where  $L(x, y, z, \mu)$  is a saddle function (see for instance Rockafellar, 1970).

The ultimate goal of this section is to show that time iteration yields a convergent sequence of policy functions. The following definition of time iteration will be used.<sup>7</sup>

<sup>7</sup>This definition covers of course the special cases of time iteration discussed in, for instance, Judd (1998), and Coleman (1990). As far as the author is aware, there has been no application of "time iteration" that has not complied with this definition.

DEFINITION 1. Denote the partial derivatives of  $F$  and  $m_j$  with respect to the  $i$ th element of  $y$  as  $F_i(x, y, z)$  and  $m_{j,i}(x, y, z)$ , respectively. Then, **time iteration** is the iterative procedure that finds the sequence  $\{h_n(x, z)\}_{n=0}^\infty$  as  $y = h_{n+1}(x, z)$  such that

$$0 = F_i(x, y, z) + \beta \int_Z [F_i(y, h_n(y, z'), z')] \\ - \sum_{j=1}^r \mu_{j,n}(y, z') m_{j,i}(y, h_n(y, z'), z') Q(z, dz') - \sum_{j=1}^r \mu_{j,n+1}(x, z) m_{j,i}(x, y, z)$$

Notwithstanding the seemingly esoteric notation, time iteration can be thought of as using the Euler equation to find today's optimal policy,  $h_{n+1}$ , given the policy of tomorrow,  $h_n$ .

In order to verify that this procedure yields a sequence of policy functions converging to  $g$ , the following will be shown: Proposition 1 ascertains that the value functions  $v_n$ , all  $n \in \mathbb{N}$ , are differentiable and, by exploiting this finding, Proposition 2 will establish the desired result.

The following lemma is necessary for Proposition 1.

LEMMA 1. The minimizer,  $\mu(x, z)$ , of (3) is a continuous function with respect to  $x$  and  $z$ .

PROOF. By the definition of a saddle function, the fact that  $\mu \geq 0$  and  $m_j(x, \hat{y}, z) < 0$ , for all  $x, z$  and  $j$ , it follows that

$$(Tf)(x, z) \geq L(x, \hat{y}, z, \mu^*) \geq F(x, \hat{y}, z) + \beta \int_Z f(\hat{y}, z') Q(z, dz') - \mu_j(x, z) m_j(x, \hat{y}, z)$$

Which further implies that

$$\mu_j(x, z) \leq \bar{\mu}_j \equiv \max_{x \in X} \frac{(Tf)(x, z) - F(x, \hat{y}, z) - \beta \int_Z f(\hat{y}, z') Q(z, dz')}{-m_j(x, \hat{y}, z)} < +\infty$$

Denote  $\tilde{g}$  as  $\tilde{g}(x, z, \mu) = \operatorname{argmax}_{y \in X} L(x, y, z, \mu)$ . By Berge's Theorem of the Maximum,  $L(x, \tilde{g}(x, z, \mu), z, \mu)$  is a continuous function in  $\mu$ . Hence, the set of minimizers  $\mu(x, z)$  that solve the dual problem

$$\min_{0 \leq \mu \leq \bar{\mu}} L(x, \tilde{g}(x, z, \mu), z, \mu)$$

is an upper hemicontinuous correspondence in  $x$  and  $z$ . Assumptions 2 and 3, ensures that  $\mu(x, z)$  is single valued and, consequently, a continuous function in  $x$  and  $z$ .  $\square$

PROPOSITION 1. *The  $n$ -step value function,  $v_n$ , is (once) continuously differentiable with respect to  $x \in \text{int}(X)$  and its partial derivatives are given by*

$$v_{i,n}(x, z) = F_i(x, g_n(x, z), z) - \sum_{j=1}^r \mu_{j,n}(x, z) m_{i,j}(x, g_n(x, z), z)$$

for  $i = 1, \dots, \ell$ .

PROOF. It is sufficient to show that  $T : C^1(S) \rightarrow C^1(S)$ .

Define the saddle function

$$\begin{aligned} L(x, g(x, z), z, \mu(x, z)) &= F(x, g(x, z), z) + \beta \int_Z f(g(x, z), z') Q(z, dz') \\ &\quad - \sum_{j=1}^r \mu_j(x, z) m_j(x, g(x, z), z) = (Tf)(x, z) \end{aligned}$$

Pick an  $x \in \text{int}(X)$  and an  $x'$  in a neighborhood,  $N_\varepsilon(x)$ , such that  $x'_i > x_i$  and  $x'_j = x_j \forall j \neq i$ . Here,  $x_i$  denotes the  $i$ th element of the vector  $x$ . For notational convenience, denote the policy and multiplier functions from (3) as  $g, \mu$  and  $g', \mu'$  for  $(x, z)$  and  $(x', z)$  respectively.

The definition of a saddle function implies

$$L(x', g, z, \mu') \leq L(x', g', z, \mu') \leq L(x', g', z, \mu)$$

and

$$L(x, g', z, \mu) \leq L(x, g, z, \mu) \leq L(x, g, z, \mu')$$

Combine these two expressions and divide by  $x'_i - x_i$

$$\frac{L(x', g, z, \mu') - L(x, g, z, \mu')}{x'_i - x_i} \leq \frac{(Tf)(x', z) - (Tf)(x, z)}{x'_i - x_i} \leq \frac{L(x', g', z, \mu) - L(x, g', z, \mu)}{x'_i - x_i}$$

By Lemma 1 and the results on page 12, the functions  $g$  and  $\mu$  are continuous. Consequently the limits of  $g'$  and  $\mu'$  exist and equal  $\lim_{x' \rightarrow x} g' = g$ ,  $\lim_{x' \rightarrow x} \mu' = \mu$ . Hence

$$\lim_{x' \rightarrow x} \frac{L(x', g, z, \mu') - L(x, g, z, \mu')}{x'_i - x_i} = \lim_{x' \rightarrow x} \frac{L(x', g', z, \mu) - L(x, g', z, \mu)}{x'_i - x_i}$$

By the Pinching (Squeeze) Theorem

$$\lim_{x' \rightarrow x} \frac{(Tf)(x', z) - (Tf)(x, z)}{x'_i - x_i} = L_i(x, g, z, \mu)$$

Assuming, instead, that  $x'_i < x_i$ , one may repeat the same steps attaining the same result. Thus

$$\frac{\partial(Tf)(x, z)}{\partial x_i} = L_i(x, g, z, \mu) = F_i(x, g, z) - \sum_{j=1}^r \mu_j m_{j,i}(x, g, z)$$

If  $v_0$  is a weakly concave and differentiable function, the desired result is achieved.  $\square$

Note that since the space  $C^1(S)$  is not complete in the sup-norm, Proposition 1 does not imply that the limiting value function,  $v$ , is differentiable. Moreover, in the proposition above, strict concavity of the problem and full rank of  $\mathbf{J}_m$  is assumed. This simplifies the proof given in Corollary 5, p. 597, in Milgrom and Segal (2002), which essentially is equivalent for  $x \in (0, 1)$ .

The final proposition will show that the sequence of policy functions obtained by time iteration converges to the true policy function.

PROPOSITION 2. *The function  $y = h_{n+1}(x, z)$  that solves*

$$0 = F_i(x, y, z) + \beta \int_Z [F_i(y, g_n(y, z'), z')] Q(z, dz')$$

$$- \sum_{j=1}^r \mu_{j,n}(y, z') m_{j,i}(y, g_n(y, z'), z') Q(z, dz') - \sum_{j=1}^r \mu_{j,n+1}(x, z) m_{j,i}(x, y, z)$$

for  $i = 1, \dots, \ell$ , is equal to

$$g_{n+1}(x, z) = \operatorname{argmax}_{y \in \Gamma(x, z)} \{F(x, y, z) + \beta \int_Z v_n(y, z') Q(z, dz')\}$$

PROOF. Due to the stated assumptions, a sufficient condition for a maximum is a saddle point of the Lagrangian

$$L(x, y, z, \mu) = F(x, y, z) + \beta \int_Z v_n(y, z') Q(z, dz') - \sum_{j=1}^r \mu_{j,n+1} m_j(x, y, z)$$

By Proposition 1, the value function  $v_n(y, z')$  is differentiable and by Assumption 3, given minimizers  $\mu_{n+1}$ , sufficient conditions for a saddle point are thus<sup>8</sup>

$$0 = F_i(x, y, z) + \beta \int_Z v_{n,i}(y, z') Q(z, dz') - \sum_{j=1}^r \mu_{j,n+1}(x, z) m_{j,i}(x, y, z)$$

<sup>8</sup>Assuming that differentiation under the integral is legitimate.

for  $i = 1, \dots, \ell$ . By Proposition 1, this can be rewritten as

$$0 = F_i(x, y, z) + \beta \int_Z [F_i(y, g_n(y, z'), z') - \sum_{j=1}^r \mu_{j,n}(y, z') m_{j,i}(y, g_n(y, z'), z')] Q(z, dz') - \sum_{j=1}^r \mu_{j,n+1}(x, z) m_{j,i}(x, y, z)$$

Due to strict concavity the solution is unique and  $h_{n+1}(x, z) = g_{n+1}(x, z)$ , which concludes the proof.  $\square$

Since it is known that for all  $\varepsilon > 0$  there exist an  $N_s$  such that  $\sup_s |g(s) - g_n(s)| < \varepsilon$  for all  $n \geq N_s$ , Proposition 2 states that  $\sup_s |g(s) - h_n(s)| < \varepsilon$  for all  $n \geq N_s$ . Hence, the sequence  $\{h_n\}_{n \in \mathbb{N}}$  converges to the unique function  $g$ .<sup>9</sup>

Lastly, there are two additional remarks to be made: Firstly,  $g_n \rightarrow g$  implies that  $F_i(x, g_n(x, z), z) \rightarrow F_i(x, g(x, z), z)$ . As long as  $m_j(x, y, z) = m_j(y, z)$ , this further implies that  $v_{i,n}(x, z) \rightarrow F_i(x, g(x, z), z)$ .<sup>10</sup> Hence, if convergence of  $g_n$  is *uniform*, then  $v(x, z)$  is, under these additional conditions, indeed differentiable and its derivative is given by  $F_i(x, g(x, z), z)$ . In fact, this result holds under weaker assumptions than previously stated; undeniably, LICQ is dispensable.

Secondly, a *sufficient* condition for  $v(x, z)$  to be differentiable in the more general setting, is that  $\mu(s)$  is unique for each  $s \in S$ .<sup>11</sup>

**2.1. Discussion.** A natural question to ask is how the propositions above are useful in the sense of *finding* the solution to an infinite horizon problem. Indeed, what has been proven is an equivalence between value function and time iteration and, as such, neither method has any advantage over the other. From a strict theoretical viewpoint this is certainly true. However, it should be noted that very few problems actually have an analytical solution, and a numerical approximation to the solution is commonly required. When such procedures are necessary, the propositions above can be used extensively if inequality constraints are present.

<sup>9</sup>If  $X$  is compact,  $N_s$  is independent of  $s$ .

<sup>10</sup>Such constraints, (endogenous) state independent constraints, corresponds, for instance, to debt limits.

<sup>11</sup>If the dual objective function is strictly convex in  $\mu$  (it is known to be weakly convex), then  $\mu(s)$  is unique for each  $s \in S$ .

To appreciate this line of reasoning, note that in many applications Dynamic Programming relies upon a discretized state space, and such a formulation makes any inequality constraint easy to implement. Nonetheless, to achieve high accuracy the discretization must be made on a very fine grid and this causes the procedure to suffer severely from the curse of dimensionality. To avoid the curse of dimensionality, scholars have relied upon sophisticated approximation methods to enhance accuracy without markedly increasing computer time.<sup>12</sup> Generally, such approximation methods use the derivative of a numerically approximated value function to find the sequence of policy functions. Clearly, Proposition 1 confirms that such continuous state methods will converge to the true solution under a wide set of circumstances.

Moreover, when numerical approximations are used, there may be significant differences between value function- and time iteration, and on some occasions there are reasons to favor the latter: Depending on the character of the problem, the policy function might behave in a less complicated way than the value function, and hence might be more straightforward to approximate. More importantly, given that the derivative of the value function is usually needed to find the policy function, an accurate approximation of its *slope* is as important as its *level*. As a consequence, not only are more data points needed for the approximation, but the choice of approximation method is also restricted. This restriction generally causes Dynamic Programming to suffer more from the curse of dimensionality than time iteration.<sup>13</sup>

As a final remark it ought to be mentioned that time iteration can be implemented using the standard timing convention, or the timing convention defined in Carroll (2005). Hence, problems within the preceding framework can thus be solved extremely efficiently with sustained convergence features.

---

<sup>12</sup>For instance, Judd and Solnick (1994) show, in the case of the standard neoclassical growth model, that using a grid with 12 nodes and applying a shape-preserving spline performs as well as a discretized technique with 1200 nodes.

<sup>13</sup>Approximation methods that are capable of accurately approximating both the level and the slope of a function - certain classes of finite element methods - are not even theoretically developed to deal with high dimensions. Thus, time iteration is the only available technique for reliably solving high-dimensional nonlinear problems.

### 3. Examples

This section will provide three examples of problems with inequality constraints where time iteration is applicable. The examples are variations of the infinite horizon neoclassical growth model and are chosen on the basis that they represent a large class of models used in the literature. For each respective model, the underlying assumptions required for the results in section 2 will be explicitly verified. In addition, the possible caveats and violations to Assumptions 2 and 3 will be explored.

It is not the purpose of this paper to establish the accuracy or efficiency of various algorithms by solving large scale Dynamic Programming problems. However, since the first example presented below allows for a closed form solution, an accuracy verification is indeed easily carried out and will thus be presented.

The economies are comprised by an infinite number of ex ante homogenous agents of measure one. The agents maximize their utility by choosing a stochastic consumption process that has to satisfy some feasibility restrictions. In general, the problem faced by any agent can be formulated as

$$v(k, z) = \max_{k' \in \Gamma(k, z)} \left\{ u(y(k, z) - k') + \beta \int_Z v(k', z') Q(z, dz') \right\}$$

$$\Gamma(k, z) = \{k' \in K : m_j(k, k', z) \leq 0, j = 1 \dots r\}$$

Where  $y(k, z) - k'$  denotes consumption,  $k$  denotes capital,  $y$  is some function determining income and  $z$  denotes some stochastic element. Naturally, it is assumed that  $u, \beta, K, Z, Q$  and  $m$  fulfill the assumptions stated on page 12. Moreover, it is assumed that  $u(c) = \lim_{\gamma \rightarrow \sigma} \frac{c^{1-\gamma}}{1-\gamma}$ ,  $\infty > \sigma \geq 1$ , that  $y(k, z)$  is concave in  $k$  and, unless something else is specifically stated, that  $y$  is such that for all  $z \in Z$  there exist an  $\hat{k} > 0$  such that  $k \leq y(k, z) \leq \hat{k}$ , all  $0 \leq k \leq \hat{k}$ , and  $y(k, z) < k$ , all  $k > \hat{k}$ . As in most of the neoclassical literature it is assumed that  $y$  somehow depends on the function  $f(k, h, z) = zk^\alpha h^{1-\alpha}$ , for  $\alpha \in (0, 1)$ . Labor,  $h$ , is assumed throughout to be supplied inelastically and is normalized to unity.

**3.1. An analytical example.** The purpose of this example is to show how the results from Corollary 1 and Propositions 1 and 2 work in a setting with a closed form solution.

It is assumed that  $\sigma = 1$ ,  $y(k) = k^\alpha$ ,  $K = [\underline{k}, \bar{k}]$ ,  $m_1(k, k') = b - k'$ ,  $m_2(k, k') = k' - k^\alpha$  and  $\alpha \in (0, 1)$ . The economic model is hence characterized by the Bellman

equation

$$v(k) = \max_{k' \in \Gamma(k)} \{\ln(k^\alpha - k') + \beta v(k')\}$$

$$\Gamma(k) = \{k' \in K : b - k' \leq 0, k' - k^\alpha \leq 0\}$$

The model is the deterministic neoclassical growth model with full depreciation and logarithmic utility, with an additional constraint on capital holdings. As long as  $\underline{k} < b$  and  $\bar{k} > 1 = \hat{k}$ , Assumption 3 is guaranteed to hold. Note that the specific choice of utility function together with the additional assumption that  $0 < b^{1/\alpha} < \underline{k}$  will ensure that  $k' - k^\alpha \leq 0$  never is breached. Hence, without violating Assumption 3, it is possible to reduce the correspondence to

$$\Gamma(k) = \{k' \in K : b - k' \leq 0\}$$

By construction, Assumption 2 will hold. To eliminate uninteresting cases it is assumed that  $b$  is set such that  $b < (\frac{1}{\alpha\beta})^{\frac{1}{\alpha-1}}$ .

Under the above conditions the results on page 12 hold, and the problem can be solved with value function iteration. Assume for the sake of simplicity that  $(b/\beta)^{1/\alpha} < \underline{k} < b$ . Then finding

$$v_1(k) = \max_{k' \in \Gamma(k)} \{\ln(k^\alpha - k') + \beta v_0(k')\}$$

for  $v_0(k) = \frac{\alpha \ln k + \ln(1-\beta)}{1-\beta}$ , corresponds to the time iteration step of finding  $k' = g_1(k)$  such that

$$\frac{1}{k^\alpha - k'} + \mu_0(k) = \beta \frac{1}{k'^\alpha - g_0(k')} \alpha k'^{\alpha-1}$$

for  $g_0(k) = \beta k^\alpha$ .<sup>14</sup> Since, the problem itself is strictly concave, it is possible to ignore the multiplier: The policy function from solving this equation is accordingly given by  $g_1(k) = \max\{\frac{\alpha\beta}{1-\beta+\alpha\beta}k^\alpha, b\}$ . Let  $\underline{v}$  and  $\bar{v}$  denote the value functions when the agent is and is not constrained respectively. Hence

$$\bar{v}_1(k) = \alpha \frac{1-\beta+\alpha\beta}{1-\beta} \ln k + A_1, \quad \underline{v}_1(k) = \ln(k^\alpha - b) + \beta v_0(b)$$

Where  $A_1$  is some constant. The derivatives of these two functions are given by

$$\bar{v}'_1(k) = \frac{\alpha}{k} \frac{1-\beta+\alpha\beta}{1-\beta}, \quad \underline{v}'_1(k) = \frac{1}{k^\alpha - b} \alpha k^{\alpha-1}$$

<sup>14</sup>Note that  $v_0(k) = \frac{\ln(k^\alpha - g_0(k))}{1-\beta}$ . Moreover,  $g_0$  is a feasible policy for all  $k \in K$ . Feasibility of  $g_0$  is not a necessary requirement, but is merely used for the sake of simplicity.

The value function,  $v_1$ , is consequently differentiable if, and only if,  $\bar{v}'_1(k) = \underline{v}'_1(k)$  at  $k$  such that  $b = \frac{\alpha\beta}{1-\beta+\alpha\beta}k^\alpha$ . Inserting this expression for  $b$  into  $\underline{v}'_1(k)$  yields

$$\underline{v}'_1(k) = \frac{\alpha}{k} \frac{1 - \beta + \alpha\beta}{1 - \beta} = \bar{v}'_1(k)$$

Hence,  $v_1$  is differentiable and its derivative is given by<sup>15</sup>

$$v'_1(k) = \frac{1}{k^\alpha - g_1(k)} \alpha k^{\alpha-1}$$

Continuing by induction one finds that

$$\begin{aligned} g_n(k) &= \max \left\{ \alpha\beta \frac{(1-\beta)((\alpha\beta)^{n-1} - 1) + (\alpha\beta)^{n-1}(\alpha\beta - 1)}{(1-\beta)((\alpha\beta)^n - 1) + (\alpha\beta)^n(\alpha\beta - 1)} k^\alpha, b \right\} \\ \bar{v}_n(k) &= \alpha \ln k \frac{(1-\beta)((\alpha\beta)^n - 1) + (\alpha\beta)^n(\alpha\beta - 1)}{(1-\beta)(\alpha\beta - 1)} + A_n \\ \underline{v}_n(k) &= \ln(k^\alpha - b) + \beta v_{n-1}(b) \end{aligned}$$

And by the same argument,  $v_n$  is differentiable and its derivative is given by

$$v'_n(k) = \frac{1}{k^\alpha - g_n(k)} \alpha k^{\alpha-1}$$

The limiting functions are

$$\begin{aligned} g(k) &= \max \{ \alpha\beta k^\alpha, b \} \\ \bar{v}(k) &= \frac{\alpha}{1 - \alpha\beta} \ln k + \frac{\frac{\alpha\beta}{1-\alpha\beta} \ln(\alpha\beta) + \ln(1 - \alpha\beta)}{1 - \beta} \\ \underline{v}(k) &= \ln(k^\alpha - b) + \beta v(b) \end{aligned}$$

And the limiting value function is differentiable with derivative

$$v'(k) = \frac{1}{k^\alpha - g(k)} \alpha k^{\alpha-1}$$

Finally, the Lagrange multiplier can be recovered as<sup>16</sup>

$$\mu(k) = \frac{1}{k^\alpha - g(k)} - \beta \frac{\alpha g(k)^{\alpha-1}}{g(k)^\alpha - g(g(k))}$$

Since the problem allows for an analytical solution, accuracy of various numerical algorithms can be assessed straightforwardly. Table 1 lists the numerical results of

<sup>15</sup>Equivalently, one could exploit the, ex ante known, directional differentiability of  $v_1(k)$  and show that  $v'(k; -1) = -v'(k; 1)$ ,  $\forall k \in \text{int}(K)$ ; i.e. that the left and right derivative of  $v_1(k)$  coincides at all interior points of  $K$ .

<sup>16</sup>Clearly, the complete sequence of multipliers,  $\{\mu_n\}_{n=1}^\infty$ , could be recovered in a similar fashion.

TABLE 1.1. Performance of Algorithms<sup>a</sup>

Algorithm	Value Iteration		Time Iteration	
	#1	#2	#3	#4
$N$	500	1000	20	20
Accuracy	5.3e-3	3.3e-3	5.8e-4	2.9e-6
REE	4.2e-3	2.1e-3	1.2e-3	3.2e-5
CPU-time	72	295	0.01	0.02
Remark	Discrete grid		Linear	Spline

<sup>a</sup>*Accuracy* refers to the maximum absolute percentage error of the policy function in terms of capital. REE refers to the maximum relative Euler equation errors defined in Judd (1998). Computer time is denoted in seconds, *Linear* and (cubic) *Spline* refer to the interpolation method used for the equilibrium functions, and  $N$  denotes the number of nodes in the grid.

applying discretized value function iteration and time iteration to the model with  $\alpha = 0.3$ ,  $\beta = 1.03^{-1/4}$ ,  $b = 0.15$ ,  $K = [0.7k_{ss}, 1.3k_{ss}]$  and  $k_{ss} = (1/\alpha\beta)^{1/(\alpha-1)}$ . The

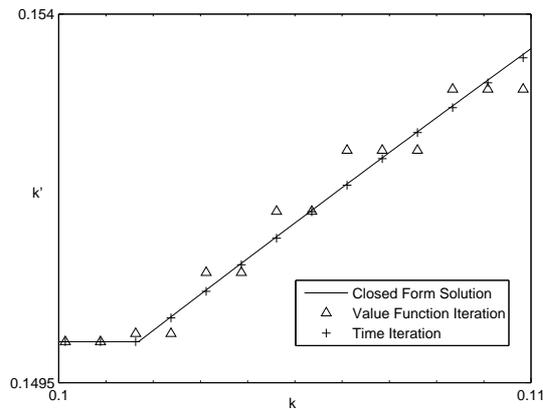


FIGURE 1.1. Policy functions for Algorithm #1 and #3.

advantage of time iteration is here quite clear; time iteration outperforms value function iteration in both norms, using a very coarse grid and in a fraction of the time. The advantage of time iteration is further illuminated by Figure 1.1 where the policy functions recovered from the procedures are graphed close to the debt limit. Even at the binding point, time iteration performs extremely well.

**3.2. Irreversible investment.** (Christiano and Fisher, 2000) Irreversibility of investment in the neoclassical growth model is an important example given that it captures the problem of *state dependent* inequality constraints.

For this economy it is assumed that  $y(k, z) = f(k, z) + (1 - \delta)k$ ,  $K = [k, \bar{k}]$ ,  $m_1(k, k', z) = (1 - \delta)k - k'$  and  $m_2(k, k', z) = k' - y(k, z)$ . Moreover, markets for idiosyncratic risks are complete. The problem is thus characterized by the following Bellman equation

$$v(k, z) = \max_{k' \in \Gamma(k, z)} \left\{ u(y(k, z) - k') + \beta \int_Z v(k', z') Q(z, dz') \right\}$$

$$\Gamma(k, z) = \{k' \in K : (1 - \delta)k - k' \leq 0, k' - y(k, z) \leq 0\}$$

In the previous example, it was possible to use an unbounded return function since the “borrowing constraint” together with restrictions on the income function generated a natural boundedness of the problem. However, in this formulation it is not possible to impose a similar (debt) constraint, since such a restriction would clearly interfere with the irreversibility constraint on investment and hence violate Assumption 2. As an alternative it will be assumed ex ante that there exist an  $\varepsilon > 0$  such that for all  $z \in Z$ ,  $n \in \mathbb{N}$ ,  $g_n(\varepsilon, z) > \varepsilon$ ; that is, a lower interiority of  $g_n(k, z)$  is ex ante assumed for all  $k, z$  and  $n$ .<sup>17</sup> By the definition of  $\hat{k}$  on page 19, the set of maintainable capital stocks are thus given by  $K = [\varepsilon, \hat{k}]$  and, given the specific choice of the utility function, the feasibility correspondence can be reformulated as  $\Gamma(k, z) = \{k' \in K : (1 - \delta)k - k' \leq 0\}$  without violating Assumption 3.

Under these restrictions it is known that

$$v_{n+1}(k, z) = \max_{k' \geq (1-\delta)k} \left\{ u(y(k, z) - k') + \beta \int_Z v_n(k', z') Q(z, dz') \right\}$$

converges to  $v$ . By Proposition 2 and for a given  $\mu_{n+1}(k, z)$ , this procedure reduces to finding  $k' = g_{n+1}(k, z)$  such that

$$u'(y(k, z) - k') - \mu_{n+1}(k, z) = \beta \int_Z [u'(y(k', z') - g_n(k', z')) y_k(k', z') - \mu_n(k', z')(1 - \delta)] Q(z, dz') \quad (4)$$

As can be seen from (4), the multiplier from the previous iteration is in the expectation term. This indicates the presence of a state dependent constraint.

<sup>17</sup>Naturally, such a conjecture needs to be verified when solving the model.

Although it is necessary to find both a policy function and a multiplier at each iteration, this is a trivial task. Since the problem itself is strictly concave, it is possible to ignore  $\mu_{n+1}$  in (4) and find the function  $\hat{g}_{n+1}$  that solves the (reduced) equation. The true policy function  $g_{n+1}$  can then be recovered as  $g_{n+1} = \max\{\hat{g}_{n+1}, (1 - \delta)k\}$  and  $\mu_{n+1}$  is merely the residual in (4) when  $g_{n+1}$  is inserted into the equation.

For a parameterization given by,  $\alpha = 0.3$ ,  $\beta = 1.03^{-1/4}$ ,  $\delta = 0.02$ ,  $\sigma = 1$ ,  $Z = \exp(\{0.23, -0.23\})$ , and  $Q(z, z') = 1/2$  for all  $(z, z')$  pairs, the solution is depicted in Figure 1.2. Figure 1.2 illustrates how distinctly the procedure captures the Kuhn-Tucker condition of  $\mu(k, z)m_1(k, k', z) = 0$ . The MATLAB program for this model,

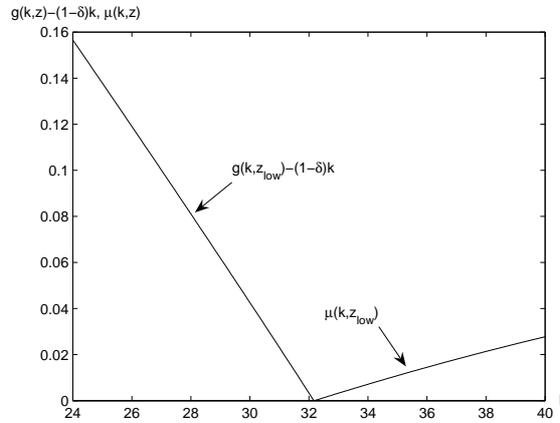


FIGURE 1.2. Investment function and multiplier for the model with irreversible investment.

presented in Appendix A, clearly illustrates the simplicity of the procedure.

**3.3. Incomplete markets.** (Aiyagari, 1994) Standard models with incomplete market are relevant for the procedure proposed in this paper since the assumption of risk-free borrowing induces a debt limit as a necessary condition for the characterization of the economy to be valid.

It is assumed that  $y(k, z) = wz + (1+r)k$ ,  $K = [\underline{k}, \bar{k}]$ ,  $Z$  is countable,  $m_1(k, k', z) = -\phi - k'$  and, as before,  $m_2(k, k', z) = k' - y(k, z)$ . Here  $z$  denotes an uninsurable idiosyncratic component; markets are incomplete. However, there is no aggregate risk in the economy. Moreover,  $w$  and  $r$  are given by  $f_h(\tilde{k}, h)$  and  $1 + f_k(\tilde{k}, h) - \delta$  respectively.  $\tilde{k}$  represents the aggregate capital stock in the economy and, as before,  $h$  represent the employment rate, normalized to unity. The problem is thus

characterized by the following equations

$$v(k, z) = \max_{k' \in \Gamma(k, z)} \left\{ u(y(k, z) - k') + \beta \int_Z v(k', z') Q(z, z') \right\}$$

$$\Gamma(k, z) = \{k' \in K : -\phi - k' \leq 0, k' - y(k, z) \leq 0\}$$

$$\tilde{k} = \sum_z \int_K k \lambda(k, z) dk$$

$$\lambda(k', z') = \sum_z \int_{\{k \in K : k' = g(k, z)\}} \lambda(k, z) Q(z, z') dk$$

Where  $\lambda(k, z)$  denotes the (stationary) distribution of asset holdings and employment status.

Note that  $y(k, z)$  does not fulfill the desired properties to ensure an upper bound on the endogenous state space (as stated on page 19). However, as noted in Aiyagari (1994), for all  $z \in Z$ , there exist a  $k^*$  such that, for all  $k \geq k^*$ ,  $k' \leq k$ . In order to ensure that Assumption 3 holds, set  $\bar{k} > k^*$  and  $\underline{k} < -\phi < w\underline{z} + \underline{k}(1+r)$ , where  $\underline{z} = \inf Z$ . By again exploiting the properties of the functional form of the return function, the feasibility correspondence can be reformulated as  $\Gamma(k, z) = \{k' \in K : -\phi - k' \leq 0\}$  and Assumption 2 will, by construction, hold.<sup>18</sup>

Under the above stated conditions, it is known that the procedure

$$v_{n+1}(k, z) = \max_{-\phi \leq k'} \left\{ u(y(k, z) - k') + \beta \int_Z v_n(k', z') Q(z, z') \right\}$$

converges to  $v$ . Given  $\mu_{n+1}(k, z)$ , Proposition 2 asserts that this procedure reduces to finding  $k' = g_{n+1}(k, z)$  such that

$$u'(y(k, z) - k') - \mu_{n+1}(k, z) = \beta \int_Z u'(y(k', z') - g_n(k', z'))(1+r)Q(z, z')$$

As in the previous example, it is possible due to the concavity of the problem, to ignore the multiplier  $\mu_{n+1}$  and solve the problem to find  $\hat{g}_{n+1}$ . Again, the true policy function  $g_{n+1}$  is recovered as  $g_{n+1} = \max\{-\phi, \hat{g}_{n+1}\}$ . The multiplier can then be obtained as a residual. Thus, except for applying a “max” operator at each iteration, such a procedure is no more difficult to solve than a model with no constraints at all.

---

<sup>18</sup>Note that  $-\phi$  in the above analysis is set strictly *higher* than what Aiyagari (1994) refers to as “the natural debt limit”. Here,  $-\phi$  is what is usually referred to as an “ad-hoc constraint”; an important feature in the current setting to ensure the boundedness of the problem. See for instance Krusell and Smith (1997) for the empirical relevance of ad-hoc constraints.

For a parameterization given by,  $\alpha = 0.3$ ,  $\beta = 0.95$ ,  $\delta = 0.1$ ,  $\sigma = 1$ ,  $\phi = -2$ ,  $Z = \{1, 0.5\}$ , and  $Q(z, z') = 1/2$  for all  $(z, z')$  pairs, the solution is depicted in Figure 1.3. Again, Figure 1.3 illustrates how ably the procedure captures the Kuhn-Tucker condition of  $\mu(k, z)m_1(k, k', z) = 0$ .

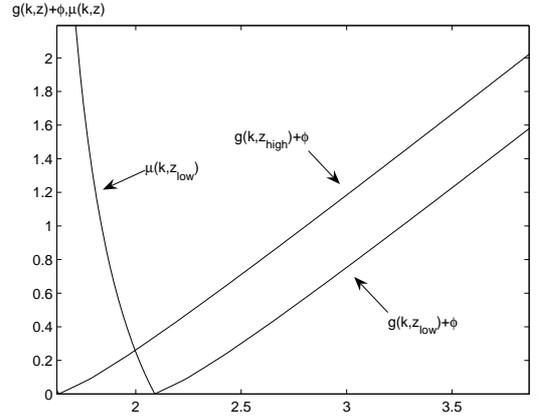


FIGURE 1.3. Policy and multiplier for an Aiygari economy with an ad hoc constraint ( $\phi = -2$ ).

#### 4. Concluding Remarks

Recursive models with inequality constraints are generally problematic to solve: Discretized Dynamic Programming suffers severely from the curse of dimensionality and Parameterized Dynamic Programming imposes a differentiability property of the value function that might be false. Furthermore, Euler equation techniques have unknown or very poor convergence properties, and are thus difficult to solve without making initial educated guesses for the equilibrium functions.

This paper has resolved parts of these problems: It has been established that under weak conditions, the  $n$ -step value function is differentiable for problems with inequality constraints. Thus, solution techniques that impose a differentiability of the value function will, at least theoretically, converge to the true solution. Moreover, through a derived analytical expression of the derivative of the value function, an iterative Euler equation based method has been shown to be convergent when inequality constraints might be present.

Moreover, as shown in section 3, time iteration proposes an iterative procedure that is appealing from a computational perspective. Firstly, high-dimensional approximation methods are applicable given that there is no need to approximate the *slope* of any equilibrium function. Secondly, policy functions possibly have a relatively uncomplicated behavior relative to the value function and are hence more accurately approximated. Thirdly, in the iterative procedure, Lagrange multipliers come out as residuals from the Euler equation and these are, in the case of state dependent constraints, merely needed to be interpolated at each iteration.

As a direction for future research, it would be desirable to establish under which additional conditions the limiting value function is differentiable when inequality constraints potentially bind. Moreover, methods for evaluating the accuracy of numerical solutions using the Euler equation residuals, are well developed for *interior* problems (Santos, 2000). However, they are not extended to deal with problems formulated in the context of this paper.



## CHAPTER 2

# Asset Based Unemployment Insurance

## 1. Introduction

Between the ages of 18 and 40, an American worker can expect to be unemployed on five different occasions.<sup>1</sup> An average spell of unemployment lasts for approximately three months. Unsurprisingly then, unemployment is perceived as one of the greatest economic risks an individual faces during her working life, and insurance against such shortfalls in labor income is of high importance. Whereas most modern economies provide unemployment insurance through a governmentally sponsored unemployment benefits programme,<sup>2</sup> several empirical studies suggest that this is not the only source of insurance available to the unemployed. Of the total fraction of unemployed eligible for benefits, Blank and Card (1991) estimate that only 67% take up unemployment insurance, indicating that many of the unemployed find insurance elsewhere. Among the group of participating individuals, Gruber (1997) finds that the consumption smoothing effect of insurance is particularly high at late stages of the unemployment spell, arguing that this occurs when financial wealth is depleted. Lastly, Gruber (1998) shows that unemployment benefits have a significant crowding-out effect on savings, not only suggesting that unemployment benefits and wealth act as close substitutes, but also that savings is an important factor to consider when designing an unemployment benefits programme.

Motivated by these issues, this paper develops a theoretical model in order to characterize an optimal unemployment benefit programme in the presence of *moral hazard* and partial *self-insurance*. An infinitely lived individual can at any date either be employed or unemployed. While working she faces an idiosyncratic exogenous risk of losing her job, and while unemployed she can devote time and effort to search for a new job. The agent enjoys consumption and leisure, and she may reallocate resources intertemporally by means of a riskless asset. A utilitarian government provides unemployment insurance. It has information on the agents' consumption

---

<sup>1</sup>According to the Bureau of Labor Statistics' National Longitudinal Survey of Youth, 1979 (NLSY79).

<sup>2</sup>As unemployment insurance reduces the opportunity cost of employment, it evokes substantial moral hazard effects in the labor market (Meyer, 1990; Moffitt, 1985). Private insurance solutions are thus unlikely to function efficiently, and may even fail to exist. As a consequence, most modern economies relies exclusively on a governmentally funded unemployment insurance programme (Oswald, 1986; Chiu and Karni, 1998).

level and preferences, but not on their search effort. The government's redistribution policy must therefore be *incentive compatible*.

In this setting, the government has full control over the agent's consumption and search effort allocations, and may thus choose these directly. Allowing the government to choose allocations, rather than policies, simplifies the problem considerably. However, it also forces the analysis to proceed in two separate steps: The first step characterizes the optimal allocations while the second implements these allocations through a tax system in a decentralized economy.

I show that the government's intertemporal first order condition must observe an inverse Euler equation (Rogerson, 1985). By Jensen's inequality, this optimality condition implies a wedge between the agent's intertemporal marginal rate of substitution and the economy-wide interest rate (the marginal rate of transformation). Said differently, in relation to a frictionless economy, the agent is saving constrained. The reason behind this result is straightforward: In order to provide incentives to exert search effort, the government wishes to generate a positive correlation between consumption and employment. When the agent's utility function is concave, higher savings weakens this correlation and thus decreases search effort. Thus, at an optimal programme, a crowding-out effect of unemployment insurance on savings is indeed desired.

Following recent developments in the dynamic public finance literature, I construct tax (or policy-) functions that implement the optimal allocations in a decentralized economy (cf. Kocherlakota (2005); Albanesi and Sleet (2006); and Golosov and Tsyvinski (2006)). By *implement*, I mean a tax system such that the solution to a decentralized maximization problem faced by an individual agent that takes the tax system as given, coincides with the government's optimal solution. The resulting tax functions are *simple*: Current taxes depend solely on the agent's current and previous employment state, and on her level of assets. These tax functions provide new insights into how an optimal unemployment insurance scheme should be designed: *First*, the unemployment insurance policy is time-invariant, and thus independent of the duration of the unemployment spell. *Second*, unemployment benefit payments relate negatively to the agent's asset position: In addition to the first-order insurance effect of wealth, a ceteris paribus increase in non-labor income (wealth) amplifies the opportunity cost of employment and thus reduces the agent's incentive to search for a job. Moreover, during unemployment the agent decumulates assets and the

sequence of benefit payments is observationally *increasing* - a result that stands in sharp contrast with previous studies (e.g. Shavell and Weiss, 1979; Hopenhayn and Nicolini, 1997; Pavoni, 2007; Pavoni and Violante, 2007).

The essential economic mechanisms in this paper are closest related to those in Shavell and Weiss (1979), Hopenhayn and Nicolini (1997), and Pavoni (2007). In their seminal study, Shavell and Weiss (1979) show that consumption ought to be decreasing with respect to the duration of the unemployment spell, a result further confirmed and strengthened in Hopenhayn and Nicolini (1997) and Pavoni (2007).<sup>3</sup> Since these studies abstract from savings, the policy recommendation is immediate; unemployment benefits are given as the difference between consumption and labor income, and should therefore decrease along the duration of the unemployment spell. I deviate from this literature by relaxing two assumptions: Firstly, I model employment and unemployment as *recurrent* states, while previous studies have assumed that employment is an absorbing state. Secondly - and more importantly - I allow for partial *self-insurance* by means of a riskless asset. This has salient implications for the optimal unemployment benefit policy. While the consumption pattern largely remains unaltered, the benefit policy does not.

In order identify the effect of savings and benefit payments on consumption, I rely on recent developments in the dynamic public finance literature. Following Kocherlakota (2005) and Albanesi and Sleet (2006), I consider tax systems that resemble modern economies' combined usage of taxes and markets to reallocate resources in the economy. Kocherlakota (2005) and Albanesi and Sleet (2006) consider dynamic versions of *Mirrleesian taxation* (Mirrlees, 1971); concisely, a utilitarian government wishes to allocate resources in an economy where skills are unobservable, but labor income is not. Although the economy explored in this paper functions under fundamentally different informational frictions, the proximity of some results should be noted. As in both Kocherlakota (2005) and Albanesi and Sleet (2006), (wealth-) taxes and marginal taxes are period-by-period expected to be zero. Moreover, whereas Kocherlakota (2005) puts no restrictions on the process governing the evolution of

---

<sup>3</sup>In fact, Pavoni (2007) finds that consumption should be *non-increasing*: By exogenously imposing a minimum lower bound on the agent's present value utility - a constraint that may be interpreted as a minimum subsistence level - the consumption sequence embeds a flat profile whenever this constraint is binding.

agents' skills, the resulting tax system admits a complex structure in which the tax in any period depends upon the full history of past labor income reports. In contrast, Albanesi and Sleet (2006) assume that the evolution of agents' skills are identically and independently distributed over time, and show that the tax system lends itself to a recursive representation in the agents' wealth. Although the evolution of employment status in this paper is endogenous and exhibits high persistence, the tax system admits a simple recursive representation in the agents' wealth and current employment status transition.

In a recent paper, Shimer and Werning (2005) consider a problem closely related to the question explored in this paper. Similar to this paper, Shimer and Werning (2005) first consider the optimal allocations, and then, by proving an equivalence result, derive the decentralized policy that implements these allocation . However, the two papers show considerable differences: Shimer and Werning (2005) consider a version of McCall's (1970) search model with hidden reservation wages. This paper considers hidden search effort decisions. More importantly, all qualitative properties explored in Shimer and Werning (2005) hinges on the assumption of CARA utility, and thus on potentially negative consumption levels.<sup>4</sup> Abstracting from some standard regulatory conditions, this paper puts no restrictions on the specific functional form of the agents' momentary utility function.

## 2. Structure of the economy

The economy is populated by a utilitarian government and a continuum of risk-averse agents. The planning horizon is infinite. Time is discrete and denoted by  $t = 0, 1, \dots$ . In any given period  $t$ , an agent can either be employed or unemployed and the agent's employment status is publicly observable.

When an agent is employed, she earns a gross wage,  $w$ . There is no on-the-job search and the probability of losing the job is exogenously given at the constant hazard rate  $1 - \gamma$ .

When unemployed, the agent receives unemployment benefits and searches for a job with effort  $e$ . The probability of finding a job, conditional on search effort, is denoted  $p(e)$ . Search effort - and thus the probability of finding a job - is considered private information, not observable by the government or by any other agent in the

---

<sup>4</sup>In Shimer and Werning (2005) it is shown that their results do not extend to a setting with CRRA utility.

economy.<sup>5</sup> The wage distribution is degenerate, and a job offer is, consequently, always accepted. The agents can save using a riskless bond that pays net pre-tax return equal to  $r > 0$ . The intertemporal price of consumption,  $1/(1+r)$ , is denoted by  $q$ . Savings are publicly observable.

**2.1. Model.** Formally, employment status in any period  $t$  is given by  $\theta_t \in \Theta = \{0, 1\}$ . Let  $\theta_t = 1$  denote employment. The history of employment status up to period  $t$  is given by  $\theta^t = (\theta_0, \dots, \theta_t) \in \Theta^t$ , where  $\Theta^t = \{0, 1\} \times \{0, 1\} \times \dots \times \{0, 1\}$ , represent all possible histories up to period  $t$ .

At time zero, each agent is born as either employed or unemployed, and she is entitled some level of initial cash-on-hand,  $b_0$ . The initial entitlement/employment status-pair,  $(b_0, \theta_0)$ , is taken as given by each agent in the economy (the government included). The joint distribution of  $(b_0, \theta_0)$  is given by  $\psi(b_0, \theta_0)$ , with support on  $B \times \Theta$ , where  $B$  is some subset of the real numbers,  $B \subseteq \mathbb{R}$ . Thus, at every date,  $t$ , each agent is distinguished by her initial entitlements and history of employment status,  $(b_0, \theta^t)$ .

Without any loss of generality, I will henceforth formulate the problem such the agents choose  $p$  - the probability of finding a job -, rather than effort  $e$ , directly. The agent then ranks contemporaneous consumption and search effort allocations according an additively separable felicity function,  $\{u(c) - (1 - \theta)v(p)\}$ . There is no disutility from working.<sup>6</sup> The function  $u$  and  $v$  are strictly increasing and once continuously differentiable. In addition,  $u$  is strictly concave and  $v$  is strictly convex. The standard Inada conditions apply for  $u$ ;  $u'(0) = \infty$  and  $\lim_{c \rightarrow \infty} u'(c) = 0$ .

An *allocation* in this economy is denoted  $\sigma = \{c_t, p_t\}_{t=0}^{\infty}$ , where

$$\begin{aligned} c_t &: B \times \Theta^t \rightarrow \mathbb{R}_+ \\ p_t &: B \times \Theta^t \rightarrow [0, 1] \end{aligned}$$

Here,  $c_t(b_0, \theta^t)$  is the amount of consumption an  $(b_0, \theta_0)$ -agent is assigned under history  $\theta^t$ . The contemporaneous probability of finding a job,  $p_t(b_0, \theta^t)$ , is defined equivalently. Let  $\lambda(b_0, \theta^{t+1})$  denote the probability measure for history  $\theta^{t+1}$ , conditional on  $(b_0, \theta_0)$ . For notational convenience let  $p_t(b_0, \theta^t)$  be defined as  $\gamma$  if and only if  $\theta_t = 1$ .  $\lambda(b_0, \theta^{t+1})$

<sup>5</sup>This is the source of moral hazard in the model; if benefit payments would be made contingent upon search effort, the economy would reach its first best allocation.

<sup>6</sup>Including disutility from working would not change any of the results in the paper.

is then recursively given by

$$\lambda(b_0, \theta^{t+1}) = \begin{cases} p_t(b_0, \theta^t)\lambda(b_0, \theta^t), & \theta_{t+1} = 1 \\ (1 - p_t(b_0, \theta^t))\lambda(b_0, \theta^t), & \theta_{t+1} = 0 \end{cases}$$

An agent's net present value utility of an allocation  $\sigma$  is given as

$$V(\sigma, b_0, \theta_0) = \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} \{u(c_t(b_0, \theta^t)) - (1 - \theta_t)v(p_t(b_0, \theta^t))\} \lambda(b_0, \theta^t) d\theta^t \quad (5)$$

The utilitarian government wishes to find  $\sigma$  that maximizes the sum of net present value utilities

$$\hat{V}(\psi) = \max_{\sigma} \int_{B \times \Theta} \{V(\sigma, b_0, \theta_0)\} d\psi \quad (6)$$

subject to each agent's present value budget constraint

$$b_0 \geq \sum_{t=0}^{\infty} q^t \int_{\Theta^t} \{c_t(b_0, \theta^t) - \theta_t w\} \lambda(b_0, \theta^t) d\theta^t, \quad \forall (b_0, \theta_0) \in B \times \Theta \quad (7)$$

Furthermore, since the search effort allocation is private information, the optimal allocation must also respect incentive compatibility

$$\{p_t\}_{t=0}^{\infty} = \operatorname{argmax}\{V(\sigma, b_0, \theta_0)\}, \quad \forall (b_0, \theta_0) \in B \times \Theta \quad (8)$$

The motivation behind the *incentive compatibility constraint* is simple: Each agent takes the consumption allocation as given and chooses search effort to maximize her private utility. Without any loss of generality, the problem is organized such that the government directly proposes a search effort allocation that coincides with the agent's private optimal choice.

Constraint (7) ensures feasibility. It should be noted that this constraint will always hold as an equality; if it did not, the government could simply increase the agent's period zero consumption without inflicting with incentive compatibility. An allocation that is both incentive compatible and feasible will be referred to as *incentive feasible*.

Note that in (7),  $q$  is the constant intertemporal price equal to  $1/(1+r)$ . Implicitly, this assumes that there exist an exogenous financial sector, willing to borrow and lend at the intertemporal price  $q$ .

The following lemma states that maximizing (5) subject to individual incentive compatibility and feasibility, is equal to solving the more complicated problem given in (6). The result is standard and the proof is merely included for completeness.

LEMMA 2. Define  $\sigma^*$  as the allocation that maximizes (5) for each  $(b_0, \theta_0) \in B \times \Theta$ , subject to individual incentive compatibility and feasibility. Define  $\hat{\sigma}^*$  as the allocation that solves (6). Then

$$\hat{V}(\psi) = \int_{B \times \Theta} V(\sigma^*, b_0, \theta_0) d\psi$$

PROOF. By construction,  $\hat{V}(\psi) \geq \int_{B \times \Theta} V(\sigma^*, b_0, \theta_0) d\psi$ . If the inequality was strict, then there exist some  $(b_0, \theta_0)$  such that  $V(\hat{\sigma}^*, b_0, \theta_0) > V(\sigma^*, b_0, \theta_0)$ . Since  $\hat{\sigma}^*$  is incentive compatible and delivers  $b_0$ ,  $\sigma^*$  could not have attained the maximum in (5).  $\square$

**2.2. A recursive formulation.** Following the insights provided by Lemma 2, the problem of interest is given by

$$V(b_0, \theta_0) = \max_{\sigma} \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} \{u(c_t(b_0, \theta^t)) - (1 - \theta_t)v(p_t(b_0, \theta^t))\} \lambda(b_0, \theta^t) d\theta^t \quad (9)$$

$$\text{s.t. } \{p_t\}_{t=0}^{\infty} = \text{argmax}\{V(\sigma, b_0, \theta_0)\} \quad (10)$$

$$b_0 = \sum_{t=0}^{\infty} q^t \int_{\Theta^t} \{c_t(b_0, \theta^t) - \theta_t w\} \lambda(b_0, \theta^t) d\theta^t \quad (11)$$

Under an optimal allocation,  $\sigma^*$ , equations (9) and (11) can be written as

$$V(b_0, \theta_0) = u(c_0^*(b_0, \theta_0)) - (1 - \theta_0)v(p_0^*(b_0, \theta_0)) + \beta \int_{\Theta^1} V(\sigma^*, b^*(\theta_1), \theta_1) \lambda(b_0, \theta^1) d\theta^1 \quad (12)$$

$$b_0 = c_0^*(b_0, \theta_0) - \theta_0 w + q \int_{\Theta^1} b^*(\theta_1) \lambda(b_0, \theta^1) d\theta^1 \quad (13)$$

The following lemma asserts that, given the budget  $b^*(\theta_1)$ , re-optimizing the problem in period one, does not alter period zero present value utility.

LEMMA 3.  $V(\sigma^*, b^*(\theta_1), \theta_1)$  maximizes the agent's utility subject to the budget  $b^*(\theta_1)$  and incentive compatibility. That is,  $V(\sigma^*, b^*(\theta_1), \theta_1) = V(b^*(\theta_1), \theta_1)$ .

PROOF. See Appendix B.  $\square$

The result is not trivial. If  $V(b^*(\theta_1), \theta_1) > V(\sigma^*, b^*(\theta_1), \theta_1)$  for at least one  $\theta_1$ , period zero incentive compatibility is violated. The idea behind the proof lies in the fact that  $V(b_0, \theta_0)$  is strictly increasing in  $b_0$ , and that  $b^*(\theta_1)$  must therefore be resource minimizing given utility  $V(\sigma^*, b^*(\theta_1), \theta_1)$ . The Inada conditions on  $u$  then guarantees

that duality holds: If  $b^*(\theta_1)$  is resource minimizing under utility  $V(\sigma^*, b^*(\theta_1), \theta_1)$ ,  $V(\sigma^*, b^*(\theta_1), \theta_1)$  must be utility maximizing under the budget  $b^*(\theta_1)$ .

Let  $b_e$  and  $b_u$  denote period  $t+1$  contingent claims in the employed and unemployed state, respectively. Then - by exploiting the insights provided by Lemma 3 and following the arguments outlined in Spear and Srivastava (1987) - problem (9) can be made recursive as

$$V(b, \theta) = \max_{c, p, b_e, b_u} \{u(c) - (1 - \theta)v(p) + \beta(pV(b_e) + (1 - p)V(b_u))\} \quad (14)$$

subject to

$$p = \operatorname{argmax}_p \{u(c) - \theta v(p) + \beta(pV(b_e) + (1 - p)V(b_u))\} \quad (15)$$

and

$$b = c - \theta w + q(pb_e + (1 - p)b_u) \quad (16)$$

Since the function  $v$  is differentiable and strictly convex, the incentive compatibility constraint (15) can be replaced by its first order condition

$$v'(p) = \beta(V(b_e) - V(b_u))$$

The solution to (14)-(16) yields a value function,  $V(b, \theta)$ , associated with policy functions  $c(b, \theta)$ ,  $p(b, \theta)$ ,  $b_e(b, \theta)$  and  $b_u(b, \theta)$ . When there is no confusion regarding the agent's employment status, the policy functions will be addressed by their respective initial letter, and reliance on  $b$  will be left implicit.

Previous studies on optimal unemployment insurance adopt a dual formulation to the problem in (14)-(16). Specifically, the literature has, without exception, followed the cost-minimization framework commonly employed in the repeated-agency literature. Fundamentally, this approach amounts to minimize (7) such that the agent receives a pre-specified level of present value utility, and subject to incentive compatibility. Due to Spear and Srivastava (1987), this dual formulation lends itself straightforwardly to a recursive representation. In contrast, this paper adopts a *primal* approach. The reason for this is twofold: First, the primal formulation simplifies the subsequent analysis and provides an intuitive recursive representation in terms of (non-labor) cash-on-hand,  $b$ . Second, this way of formulating the problem has a quite appealing and natural interpretation: Akin to a *social planner*, the government maximizes the agent's utility by choosing current consumption, search effort, and one

period ahead Arrow securities at prices  $qp$  and  $q(1-p)$ . By respecting incentive compatibility, moral hazard is internalized through *individually* and *quantity contingently* priced assets.

### 3. Analysis

Consistent with the formulation of the problem in (14), the government chooses *allocations* rather than *policies*. While it facilitates the analysis of the governments optimal policy problem, it also restricts the subsequent analysis to proceed in two separate steps. The first step concerns the optimal allocations. The second step considers the tax functions that implement these allocations in a decentralized *bond economy*.

Although the two steps presented above may appear distinctly separate, they are, in effect, intimately related. Thus, as a third step, Section 3.3 will show how the shape of the derived tax functions are closely tied to the incentive compatibility constraint, and how a quite esoteric optimality condition, commonly known as the inverse Euler equation, relate to a more familiar form of the standard Euler equation.

**3.1. Allocations.** Analogous to the definition of  $b_e$  and  $b_u$ , let  $c_e$  and  $c_u$  denote period  $t+1$  consumption at the associated employment states. During employment, moral hazard is absent and the first order necessary conditions from (14) (together with the envelope condition) gives

$$u'(c) = \frac{\beta}{q}u'(c_e) = \frac{\beta}{q}u'(c_u) \quad (17)$$

When  $\beta = q$ , condition (17) implies that consumption is constant for any two consecutive periods; on a period-by-period basis, the agent is fully insured.

The equivalent optimality conditions for an unemployed agent gives

$$\frac{1}{u'(c)} = \frac{q}{\beta} \left( p \frac{1}{u'(c_e)} + (1-p) \frac{1}{u'(c_u)} \right) \quad (18)$$

$$\mu v''(p) = \lambda q (b_e - b_u) \quad (19)$$

$$\frac{\mu}{\lambda} = p(1-p) \left( \frac{1}{u'(c_u)} - \frac{1}{u'(c_e)} \right) \quad (20)$$

Where  $\lambda$  and  $\mu$  are the Lagrange multipliers on the budget- and the incentive compatibility constraint, respectively.

Equation (18) is commonly known as the “inverse Euler equation” (Rogerson, 1985). When  $c_e \neq c_u$ , Jensen’s inequality implies

$$u'(c) < \frac{\beta}{q}(pu'(c_e) + (1-p)u'(c_u)) \quad (21)$$

Rearranging terms, equation (21) infers that there is a wedge between the agent’s marginal rate of substitution and the economy’s marginal rate of transformation. In particular, (21) implies that current marginal utility of consumption is lower than the expected future marginal utility. In other words, the agent is savings constrained relative to an economy with no private information. Golosov, Kocherlakota and Tsyvinski (2003) interpret this wedge as an “implicit tax”.

According to the *standard* Euler equation, an optimal intertemporal plan has the property that any marginal, temporary and feasible change in behavior equates marginal benefits to marginal costs in the present and in the future. The inverse Euler equation *appears* to violate this logic. For a given value of  $p$ , consider the choice of reallocating resources from period  $t$  to period  $t + 1$ . If an increase in savings would bring about a proportional increase in  $b_e$  as well as  $b_u$ , equation (21) reveals that, at least on the margin, such a policy would increase overall utility. However, the incentive compatibility constraint in (15) does generally not permit a proportional increase in  $b_e$  and  $b_u$ . To keep the choice of  $p$  unaltered, the incentive compatibility constraint forces the increase in resources to be relatively low in future states where the marginal utility of resources is relatively high, and vice versa. Period  $t+1$  marginal utilities will thus be “weighted” by their respective incentive compatible inflow of state contingent resources. In contrast, utility maximization implies relatively high weights of resource inflow to states in which the marginal benefit of resources is relatively high. Since incentive compatibility inflicts with period  $t + 1$  resources only, it is thus optimal to relegate a high degree of resources to period  $t$  consumption. As a result, the agent appears savings constrained. The inverse Euler equation is simply the resulting expression when these conflicting forces are internalized. Section 3.3 will more algebraically confirm the validity of this interpretation of the inverse Euler equation.

LEMMA 4. *If  $V(b, \theta)$  is concave and  $q = \beta$ , then*

- (i)  $c_e(b, 0) > c(b, 0) > c_u(b, 0)$ .
- (ii)  $c(b, 1) > c(b, 0)$ .

(iii)  $b > b_u(b, 0) > b_e(b, 0)$  and  $b_u(b, 1) > b = b_e(b, 1)$ .

PROOF. (i) Assume that  $c_u(b, 0) \geq c_e(b, 0)$ . Then from equation (19),  $b_e(b, 0) \geq b_u(b, 0)$ . From (18) it is immediate that  $c \in (c_e, c_u)$  and thus that  $b_u(b, 0) \geq b$ . By concavity of  $V$ ,  $c(b, \theta)$  is non-decreasing, and thus  $c(b, 0) \geq c_e(b, 0) \geq c(b, 1)$ , where the last inequality follows from  $b_e(b, 0) \geq b_u(b, 0) \geq b$ . When  $\theta = 1$ , we have that  $b = b_e(b, 1)$ . Moreover, since  $c(b, 0) \geq c(b, 1) = c_u(b, 1)$ ,  $b \geq b_u(b, 1)$ . Collecting inequalities yield

$$b_e(b, 0) \geq b_u(b, 0) \geq b = b_e(b, 1) \geq b_u(b, 1)$$

From the budget constraint, and using the fact that  $w > 0$ , this implies that  $c(b, 1) > c(b, 0)$ , which contradicts  $c(b, 1) \leq c(b, 0)$ . Since  $c(b, 1) \leq c(b, 0)$  was a corollary of  $c_u(b, 0) \geq c_e(b, 0)$ , we must have  $c_u(b, 0) < c_e(b, 0)$ .

Claims (ii) and (iii) are immediate consequences of the proof of (i).  $\square$

The mechanisms underlying the proof can be seen from equation (19), in which the utility gain/cost from a marginal increase in  $p$  is equalized. If  $c_u > c_e$ , the left-hand side in equation (19) states the utility *gained* through a marginal increase in  $p$ . It is a gain since a small increase in  $c_e$ , accompanied with a decrease in  $c_u$ , attains the marginal change in the right-hand side of the incentive compatibility constraint (15) necessary to accompany the change in  $p$ . Such a change provides more insurance and thus *increases* utility. However, due to interiority, there is an associated utility *cost*;  $b_e$  must be larger than  $b_u$ , and an increase in  $p$  thus increase the share of the budget spent on period  $t + 1$  resources. The proof then proceeds by showing that  $c_u > c_e$  together with  $b_u < b_e$ , cannot be budget feasible since the wage when employed is strictly positive.

In a two period setting, the terms  $b_e$  and  $b_u$  in equation (19) may be replaced by  $c_e - w$  and  $c_u$ , respectively. The intuition behind the result in Lemma 4 is then straightforward: To provide incentives to exert search effort, the government generates a positive correlation between employment and consumption,  $c_e > c_u$ . Insurance is provided by a low intertemporal variance,  $c_e > c > c_u$ . Concavity then ensures that this logic extends to a setting with an infinite planning horizon.

*Remarks.* The notion of Lemma 4 is equivalent to Proposition 1 in Hopenhayn and Nicolini (1997). The proof is however substantially different: Here, employment is not an absorbing state and the problem is primal rather than dual.

In Lemma 4, concavity of  $V(b, \theta)$  is assumed.<sup>7</sup> The assumption is common in the literature and is indispensable for the analysis (Hopenhayn and Nicolini, 1997; Ljungqvist and Sargent, 2004). The difficulty in proving concavity lies in the fact that the choice set in (14) is not necessarily convex, and that (functions of) some choice variables does not enter the Bellman equation additively.<sup>8</sup>

Previous studies on optimal unemployment insurance abstract from self-insurance (e.g. Shavell and Weiss (1979), Hopenhayn and Nicolini (1997) and Pavoni (2007)). In the absence of savings, the policy implication from Lemma 4 is lucid; the tax/subsidy policy is defined as the difference between consumption and labor income, and benefit payments should therefore decrease along the duration of an unemployment spell. While Lemma 4 reveals that the consumption pattern remains unaltered in the current setting with self-insurance, the unemployment benefit policy does not: Most theoretical models of self-insurance (e.g. Aiyagari (1994)) display a decreasing consumption profile even in the absence of *any* unemployment benefit programme. It is thus the aim of the subsequent section to characterize the policy that can implement the optimal allocations in an economy with self-insurance.

### 3.2. Decentralization.

3.2.1. *A fiscal implementation.* The previous section characterized the constrained Pareto-optimal allocations attainable in the economy. This section will demonstrate *how* these allocations may be attained in a setting in which *the agents* choose consumption, search effort, and savings, taking the government's policy as given. The ultimate task of this section is thus to find the tax policy such that the agents' private choices corresponds to the optimal allocations derived above.

The agents in the decentralized economy have access to a riskless bond,  $a$ , that pays net (pre-tax) return equal to  $r$ . At time zero, the agents enter a market economy with a given level of cash-on-hand equal to  $b_0$ . For a given tax policy, the agents maximize their utility by choosing consumption, savings, and search processes that fulfill their intertemporal budget constraint. If there is a one-to-one correspondence

---

<sup>7</sup>Indeed, conditions (17)-(19) are derived using Benveniste and Scheinkman's (1979) envelope theorem - a theorem that requires concavity.

<sup>8</sup>Note that these are sufficient, but not necessary conditions for concavity. All numerical solutions in, for instance, Hopenhayn and Nicolini (1997) and Ljungqvist and Sargent (2004) display a strictly concave value function (or, equivalently, a strictly convex cost function).

between the chosen processes and the optimal allocation,  $\sigma^*$ , the tax allocation is called a *fiscal implementation* of  $\sigma^*$ .

Formally,

DEFINITION 2. Let  $b_0 = a_0 - T_0$  be given. If there exist a tax allocation  $\hat{T} = \{T_t\}_{t=0}^\infty$ ,  $T_t : \Theta^t \times \mathbb{R}^t \rightarrow \mathbb{R}$ , such that  $\{c_t, a_{t+1}, p_t\}_{t=0}^\infty$  solves

$$V(b_0, \theta_0) = \max_{\{c_t, a_{t+1}, p_t\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t \int_{\Theta^{t+1}} \{u(c_t(b_0, \theta^t)) - (1 - \theta_t)v(p_t(b_0, \theta^t))\} \lambda(b_0, \theta^t) d\theta^t \quad (22)$$

subject to

$$w\theta_t + a_t(b_0, \theta^{t-1}) - T_t(\theta^t, a(b_0, \theta^t)) = c_t(b_0, \theta^t) + qa_{t+1}(b_0, \theta^t) \quad \text{for } t = 0, 1, \dots \quad (23)$$

and  $\{c_t, p_t\}_{t=0}^\infty$  equals the optimal allocation  $\sigma^*$ , then  $\hat{T}$  is said to be a **fiscal implementation** of  $\sigma^*$ .

Note that the tax allocation has a very general form. Taxes in any period  $t$  may depend on the full history of employment as well on the full history of asset positions. The motivation underlying this formulation is not obvious; since the agents choose  $t + 1$  assets using information available up to period  $t$ , it is plausible to conjecture that taxes in  $t + 1$  will themselves only depend on information available up to period  $t$ . However, as shown by Kocherlakota (2005), this intuition may fail; when actions are hidden there might not exist a fiscal implementation limited to this information set. Section 3.3 will explore the underlying reason behind this conclusion further.

The following proposition shows that a fiscal implementation exists and that the resulting tax functions are *simple*: The tax level is recursive and contingent on the agent's current transition and her level of wealth.

PROPOSITION 3. There exist a time invariant tax function,  $T_t = T(a_t, \theta_t, \theta_{t-1})$ , that implements  $\sigma^*$ .

PROOF. The proof is direct and establishes a one-to-one relationship between the government's and the agent's problem.

By Bellman's Principle of Optimality, the government's problem in (14)-(16) can be split up as

$$V(b, \theta) = \max_{c, \zeta} \{u(c) + X(\zeta, \theta)\}$$

$$\text{s.t. } b = c - \theta w + q\zeta$$

$$X(\zeta, \theta) = \max_{p, b_e, b_u} \{-(1 - \theta)v(p) + \beta(pV(b_e) + (1 - p)V(b_u))\}$$

$$\text{s.t. } v'(p) = \beta(V(b_e) - V(b_u))$$

$$\zeta = pb_e + (1 - p)b_u$$

Define functions  $T_e$  and  $T_u$  as  $T_e(\zeta, \theta) = \zeta - b_e(\zeta, \theta)$  and  $T_u(\zeta, \theta) = \zeta - b_u(\zeta, \theta)$ , respectively. By definition,

$$X(\zeta, \theta) = \max_p \{-(1 - \theta)v(p) + \beta(pV(\zeta - T_e(\zeta, \theta)) + (1 - p)V(\zeta - T_u(\zeta, \theta)))\}$$

Thus,

$$V(b, \theta) = \max_{c, \zeta} \{u(c) + \max_p \{-(1 - \theta)v(p) + \beta(pV(\zeta - T_e(\zeta, \theta)) + (1 - p)V(\zeta - T_u(\zeta, \theta)))\}\}$$

$$= \max_{c, \zeta, p} \{u(c) - (1 - \theta)v(p) + \beta(pV(\zeta - T_e(\zeta, \theta)) + (1 - p)V(\zeta - T_u(\zeta, \theta)))\}$$

$$\text{s.t. } b = c - \theta w + q\zeta$$

Where the last equality follows, again, from the Principle of Optimality. By construction, if  $a' = \zeta$ , the above Bellman equation is the recursive formulation of the decentralized problem given in Definition 2.  $\square$

The above proposition hinges upon an important assumption: As in Kocherlakota (2005) and Albanesi and Sleet (2006), I assume that the fiscal implementation is such that the optimal allocation is "*affordable*". Affordability means that if the agent had the possibility to buy the optimal allocation, she would period-by-period afford it. That is,

$$w\theta_t + a_t - T_t = c_t + q(ptb_{e,t+1} + (1 - pt)b_{u,t+1})$$

This restriction is crucial for separating the effect of savings and taxes on consumption. Affordability implies that the government's state variable,  $b_t$ , must equal the

agent's non-labor cash-on-hand  $a_t - T_t$ . As a consequence, taxes are strictly redistributive

$$a_{t+1} = (p_t(a_{t+1} - T_{e,t+1}) + (1 - p_t)(a_{t+1} - T_{u,t+1})) \quad (24)$$

By Lemma 4, it is thus immediate that  $b_{u,t+1} > a_{t+1} > b_{e,t+1}$ . The agent is consequently positively taxed when employed and negatively taxed when unemployed (or equivalently, receiving an unemployment benefit).

When savings and taxes are identified as above, the intuition underlying Proposition 3 is quite straightforward. Bellman's Principle of Optimality reveals that savings,  $a'$ , is a sufficient state variable for the choice of  $b_e$ ,  $b_u$  and  $p$ . The tax functions are then defined as the difference between savings and the optimal  $t + 1$  non-labor cash-on-hand,  $b_e$  and  $b_u$ . By the design of the tax function, the agent can always choose the assigned allocation. Any other feasible choice amounts to imitating the  $t + 1$  allocation of some other agent. By construction, imitating someone else is incentive compatible and budget feasible. Thus, since the allocation is optimal under incentive compatibility and budget feasibility, imitation cannot be optimal.

The tax functions in Proposition 3 are recursive in an agent's wealth, her current and previous employment state. Akin to the tax functions that map *savings* to state contingent cash-on-hand, functions  $b_e(b, \theta)$  and  $b_u(b, \theta)$  map period  $t$  resources to period  $t + 1$  state contingent cash-on-hand. Why, then, could the tax functions not be recursive in  $(b, \theta)$ ? Inasmuch the optimal allocation still would be attainable for an agent operating in the decentralized economy, choosing the allocation would no longer be optimal: Imitating someone else is feasible, but *not* incentive compatible. By the same logic underlying the inverse Euler equation, the agent would, then, increase savings to equalize equation (21), violating the incentive compatibility of the optimal allocation.

*Remarks.* There is a continuum of tax systems that may implement any incentive feasible allocation. To appreciate this, consider an arbitrary incentive feasible allocation at time  $t$ . The agent consumes  $c$  and she exerted search effort in the previous period inducing  $p_{-1}$ . Her asset position and unemployment benefit handouts equal  $a$  and  $\tau$ , respectively. Then another allocation with  $a' = a + \epsilon$ ,  $\tau' = \tau - \epsilon$  and  $c' = c$ , is still incentive compatible, feasible, and generates the same level of utility to the agent *for any real value of  $\epsilon$* . At one extreme, 100% wealth- and labor taxes with lump-sum transfers equal to consumption, would indeed implement *any* allocation. Arguably,

such a tax system is quite draconian and does not resemble the combined usage of taxes and markets to reallocate resources observed in most current economies. At another extreme, zero taxes and individually and quantitatively-contingently priced Arrow securities could be designed to exactly mimic the problem in (14)-(16). While perhaps elegant, and by construction optimal, such a market arrangement requires an elaborate pricing system relying on common knowledge of individual asset positions and preferences.

Ruling out such elaborate asset structures and focusing on the one bond scenario, one may, alternatively, view the problem of indeterminacy as a question regarding savings. Specifically, it is a question regarding whether it is the government, or the agent (or any combination of the two), that carries out the intertemporal allocations of resources. Of course, inasmuch there are a continuum of possible arrangements of storage, one may legitimately wonder on what basis one can rationally choose between those arrangements. As in Kocherlakota (2005) and Albanesi and Sleet (2006), this paper imposes two assumptions in order to identify the effect of self-insurance from taxes/benefits on consumption. First, agents save using a riskless bond. The presence of a riskless bond can be thought of as a parsimonious representation of a more elaborate underlying diversified portfolio choice (at the intertemporal price  $q$ ). Second, the optimal allocation is assumed to be period-by-period affordable. Fundamentally this assumes that all intertemporal transfers of resources are actualized by the agents' savings. This identification scheme guarantees to attain the optimal allocation with minimal governmental interference.<sup>9</sup>

*3.2.2. Characterization.* While taxes has been shown to have a simple recursive representation, so far little has been shown regarding their properties. Examining the qualitative properties of the tax function  $T$  corresponds to examine how  $T = a - b$  responds to a change in  $a$ . To this end, I will derive and exploit the properties of the *marginal tax functions*.

This section will state the main results, supported by brief comments. In the subsequent section, I will relate the results presented here to properties of a “weighted” Euler equation, and, in turn, relate this equation to the inverse Euler equation. For clarity of exposition, focus is put on the case (of interest) at  $\theta = 0$ . To facilitate

---

<sup>9</sup>Allowing the government to intertemporally allocate resources using her own storage technology, however subject to some “iceberg cost”, would endogenously identify savings, and thus taxes, as in the current setting.

notation, let  $T_e(a')$  and  $T_u(a')$  denote period  $t+1$  taxes at the associated employment states at  $\theta = 0$ .

PROPOSITION 4. *If  $V(b, \theta)$  is concave, there exist marginal tax functions given by*

$$T_e'(a') = 1 - \frac{u'(c_u)}{pu'(c_u) + (1-p)u'(c_e)}, \quad T_u'(a') = 1 - \frac{u'(c_e)}{pu'(c_u) + (1-p)u'(c_e)}$$

PROOF. See Appendix B. □

The idea behind the proof is to consider an infinitesimal change in  $a'$ . The resulting marginal change in taxes must be such that the government's first order conditions hold, incentive compatibility is preserved and the budget balances. In addition, the agent's decentralized first order condition must hold

$$u'(c) = \frac{\beta}{q}(pu'(c_e)(1 - T_e'(a')) + (1-p)u'(c_u)(1 - T_u'(a')))$$

Combining the marginal taxes in Proposition 4 with the inverse Euler equation in (18) gives

$$T_e'(a') = 1 - \frac{qu'(c)}{\beta u'(c_e)}, \quad T_u'(a') = 1 - \frac{qu'(c)}{\beta u'(c_u)}$$

If  $\beta = q$ , and since  $c_e > c > c_u$ , it is evident that  $T_e'(a') < 0$  and  $1 > T_u'(a') > 0$ . Thus, *both unemployment benefits and "reemployment taxes" are decreasing with the agents asset position.*

COROLLARY 1. *Marginal taxes are expected to be zero.*

PROOF. When the agent is unemployed Proposition 4 together with the inverse Euler equation (18), gives the result.

When the agent is employed, taxes satisfies  $a' = \gamma(a' - T_e(a')) + (1-\gamma)(a' - T_u(a'))$ . If taxes are differentiable, the derivative of this expression with respect to  $a'$  gives the result. □

Zero expected marginal taxes are not particularly surprising in this setting; by the construction of the tax functions, taxes are always expected to be zero. A ceteris paribus change in savings mimics the action taken by some other agent and taxes respond accordingly.

The main part of the literature on optimal unemployment insurance has concluded that benefit payments ought to decrease along the duration of unemployment. The result is intuitive; in the absence of savings, a decreasing benefit profile induces a

decreasing consumption profile, providing both insurance as well as sufficient search effort incentives. Abstracting from savings, Lemma 4 confirms this result. Nevertheless, Proposition 3 shows that this result does not immediately generalize to a setting in which partial self-insurance is present: The tax policy is time-invariant and thus independent of the duration of the unemployment spell. In addition, the following proposition reveals that the intuition supporting a decreasing benefit profile fails in the current setting. Indeed, along the duration of the unemployment spell, the agent will decumulate assets and the sequence of unemployment benefits will observationally be *increasing*.

PROPOSITION 5. *If  $V(b, \theta)$  is concave and  $\beta = q$ , then (i)  $a > a'$ , (ii)  $T_u(a) > T_u(a')$ , and (iii)  $T_e(a) < T_e(a')$ .*

PROOF. By Proposition 4,  $1 > T'_u(a') > 0$ . Thus for any  $a_1$  and  $a_2$ , such that  $a_1 > a_2$ ,  $T_u(a_1) > T_u(a_2)$ . If  $a' \geq a$ ,  $1 > T'_u(a')$  implies that  $b_u \geq b$ , which contradicts Lemma 4, part (iii). Thus  $a > a'$ ,  $T_u(a) > T_u(a')$  and, by Proposition 4,  $T_e(a) < T_e(a')$ .  $\square$

The result is intuitive. During unemployment, the agent exploits the insurance effect of savings by decumulating assets. Proposition 4 infers that unemployment taxes are positively related to the agent's asset position. Thus, as the agent's level of assets decline, so does the level of the tax. Since unemployment taxes are negative this implies that unemployment benefits will increase.

Accompanied with the inverse Euler equation, Proposition 5 has an intuitive explanation. First, wealth has a first order insurance effect. The higher is an agent's wealth, the less she needs to worry about loss of consumption if she loses her job. Second, in order to provide incentives to exert search effort, the government wishes to generate a positive correlation between consumption and employment. When the agent's utility function is concave, a higher level of savings makes it costlier for the government to induce such a correlation and the agent's search effort decreases. By generating a negative correlation between savings and unemployment benefits, the government manages to mitigate the distortionary effect of savings on search.

**3.3. The Euler equation, taxes, and the inverse Euler equation.** I now provide a deeper intuition underlying some of the results presented in the preceding sections. To this end I will consider an equivalent version of the government's problem

in which the sole choice is strictly intertemporal, and not state contingent. It will be shown how this problem formulation leads to a “weighted Euler equation”, and further how these weights relate to marginal taxes. At the optimum, the weighted Euler equation implies the inverse Euler equation.

The inverse Euler equation can be thought of as the outcome when savings are chosen to balance two conflicting forces: To maximize utility, resources should be allocated to where the marginal benefit of resources is relatively high. For incentive compatibility, resources should be allocated to states in which the marginal benefit of resources is relatively low. Since incentive compatibility inflicts with period  $t + 1$  resources only, it is thus optimal to relegate a relatively high degree of resources to period  $t$  consumption. As a result, the agent *appears* savings constrained.

For a given value of savings, it is instructive to think of the optimal division of period  $t + 1$  resources across employment states as *functions* fulfilling two restrictions: The incentive compatibility constraint and the budget constraint. Similar to the tax functions explored in the previous section, these functions then allocate, for a given level of savings, resources to the different employment states. Let the government choose savings,  $a'$ , and let the functions  $\delta_e(a')$  and  $\delta_u(a')$  allocate resources between employment states such that the budget is balanced and incentive compatibility holds. That is, for a given  $p$ ,  $a' = p\delta_e(a') + (1 - p)\delta_u(a')$  and  $v'(p) = \beta(V(\delta_e(a')) - V(\delta_u(a')))$ .

The government then faces the following intertemporal maximization problem

$$V(b) = \max_{a'} \{u(b - qa') + \beta(pV(\delta_e(a')) + (1 - p)V(\delta_u(a')))\}$$

The first order condition to the above problem, evaluated at the optimal solution, is given by

$$u'(c) = \frac{\beta}{q}(pV'(b_e)\delta'_e(a') + (1 - p)V'(b_u)\delta'_u(a')) \quad (25)$$

Equation (25) resembles a standard Euler equation, and has an interpretation in terms of marginal intertemporal trade-offs: The utility cost of an marginal increase in savings equals its feasible marginal utility gain. As with standard intertemporal problems, the  $t + 1$  feasible marginal utility gain is determined by the feasible inflow of resources in period  $t + 1$  - a marginal decrease of period  $t$  consumption is accompanied by a proportional marginal increase of period  $t + 1$  resources, weighted by the interest rate:  $1 = p\delta'_e(a') + (1 - p)\delta'_u(a')$ . In addition, however, there is a further restriction on how the period  $t + 1$  resources must be divided between employment states. In

order to leave  $p$  unaltered, a *marginal* incentive compatibility constraint must hold

$$V'(\delta_e(a'))\delta'_e(a') = V'(\delta_u(a'))\delta'_u(a') \quad (26)$$

One can combine this marginal incentive compatibility constraint with the “marginal budget constraint” above, to solve for the weights  $\delta'(a')$

$$\delta'_e(a') = \frac{V'(b_u)}{pV'(b_u) + (1-p)V'(b_e)}, \quad \delta'_u(a') = \frac{V'(b_e)}{pV'(b_u) + (1-p)V'(b_e)} \quad (27)$$

The expressions above reveals an important feature: Whenever  $V'(b_u) > V'(b_e)$ ,  $\delta'_e(a') > \delta'_u(a')$ , and vice versa. That is, for states in which the marginal value of resources is relatively high, the marginal inflow of resources should be relatively low. Substituting the relationship in (27) into (25) gives the inverse Euler equation.

It is important to note that the functions in (27) are directly related to the marginal taxes derived in Proposition 4. In particular,  $\delta'(a') = 1 - T'(a')$ . The intuition underlying the shape of the tax function then becomes evident: For a certain choice of  $p$  to remain incentive compatible, an increase in savings must be divided between employment states such that the incentive compatibility constraint holds. That is, the inflow of resources must be relatively high at states in which the marginal value of resources is relatively low. By Lemma 4, the marginal value of resources is high in the unemployed state, and the additional inflow must therefore be low. Since the optimal policy is recursive in an agent’s wealth, a higher level of assets must induce a lower level of unemployment benefits.

Additionally, the marginal incentive compatibility constraint in (26) illuminates the answer to a further inquiry explored in the literature (e.g. Kocherlakota (2005), Section 3): As savings are chosen on the basis of information available in period  $t$ , could period  $t + 1$  taxes be a function of period  $t$  information only? That is, could  $\delta'_e(a')$  equal  $\delta'_u(a')$ ? From equation (26) it is straightforward to see that this cannot be the case. In order for incentive compatibility to hold, period  $t + 1$  taxes can only be a function of period  $t$  information if (and only if)  $V'(b_e) = V'(b_u)$ , or, equivalently, if  $c_e = c_u$ . Under all other circumstances, a tax contingent on period  $t$  information only would, with certainty, violate the incentive compatibility constraint.

#### 4. Concluding Remarks

This paper has studied a model of optimal redistribution policies in which the foremost risk in an agent’s life is unemployment. Moral hazard arises as job search

effort is unobservable. The model permits agents to self-insure by means of a riskless bond.

In contrast with previous studies in the literature, it is shown that the optimal unemployment insurance *policy* does not display any duration dependence. Whereas wealth encodes the agents' relevant employment status history, the insurance policy is time-invariant and, instead, contingent on the agents' asset position. In order to induce job search effort, the government wishes to provide a positive correlation between consumption and employment status. Since a higher level of savings reduces the correlation, unemployment benefits relate negatively to wealth. The agents decumulates assets over the unemployment spell in order to exploit the intrinsic insurance effect of wealth. Thus, the sequence of benefit payments is, observationally, increasing with the duration of unemployment.

The policy implications from the analysis are stark; unemployment benefits should be asset based and relate negatively to wealth. As wealth itself encodes insurance possibilities, the negative relation between wealth and unemployment benefits is intuitive. However, asset based approaches have commonly been criticized for its distortive, and negative, effect on savings (e.g. Hubbard, Skinner and Zeldes (1995), Gruber (1998)). Although undesirable per se, this paper has revealed an additional effect of wealth; a higher level of savings reduces the opportunity cost of being employed and thus increases the unemployment duration. Together, the net distortive effect of an asset based scheme appears to be favorable.

There are several ways in which an asset based unemployment insurance programme could be accomplished. As with Medicaid, food stamps, and until recently, Aid to Families with Dependent Children (AFDC), to mention a few social policies in the United States, unemployment benefits may be asset based means tested; that is, unemployment benefits are paid only if an agent has assets below a specified maximum amount. Alternatively, and obviously, schemes may be more elaborate with a continuous decline in benefit payments as assets increases.

CHAPTER 3

**Unemployment Insurance for the Liquidity Constrained**

## 1. Introduction

This paper examines how the presence of liquidity constraints affects the properties of optimal unemployment insurance provisions in a model of job-search, moral hazard and partial self-insurance. I show that the optimal unemployment insurance scheme is recursive in an agent's asset position and her current employment transition. Unemployment benefits are decreasing with the agent's wealth level, and they are constant whenever the liquidity constraint is binding - a result markedly in contrast with previous studies in which benefit payments displays a declining pattern along the duration of the unemployment spell (e.g. Shavell and Weiss (1979), and Hopenhayn and Nicolini (1997)). In calibrated version of the model it is shown that the effect of the liquidity constraint is quantitatively important: A constrained agent with zero liquid wealth ought to receive benefits payments three times higher than that received by an agent with wealth equal to one months labor income; twenty times higher than that received by an agent with wealth equal to three months labor income; and one hundred times higher compared to an agent with savings equal to twelve months of labor income (US median labor income to wealth ratio). The reason behind this swift increase in benefit payment along the wealth dimension is that optimal insurance provision should replace a missing market - the market for credits.

A large and growing literature has studied optimal unemployment insurance policies in models in which agents' search efforts are private information (e.g. Shavell and Weiss (1979), Hopenhayn and Nicolini (1997), and Pavoni (2007)). The conclusion from this literature is that unemployment benefits should be a decreasing function of the duration of unemployment, as this declining benefit profile allows the government to provide agents with incentives to undertake job search. It is common in this literature to neglect means of self-insurance and, therefore, the presence of liquidity constraints. Yet, several empirical studies have documented that self-insurance, and indeed, liquidity constraints, are important factors to consider when designing an unemployment benefits programme. Blank and Card (1991), for instance, calculate that only 67% of those eligible for unemployment insurance indeed take up unemployment benefits, indicating that a large fraction of the unemployed find insurance elsewhere. Gruber (1998) finds that unemployment benefits have a significant crowding-out effect on savings, suggesting that wealth and unemployment benefits act as close substitute. Browning and Crossley (2001) concludes that nearly half of job losers in the United

States report zero liquid wealth at the time of job loss, suggesting that liquidity is a concern for many of the unemployed, and Gruber (1997) finds that the consumption smoothing effect of insurance is particularly high at late stages of the unemployment spell, arguing that this occurs when financial wealth is depleted. Moreover, Chetty (2007) divides the unemployed into subgroups based on how likely they are to be liquidity constrained. He finds that while the effect of unemployment benefits on the hazard rate of reemployment is extremely small for the unconstrained, the corresponding measure for the constrained group is severe.<sup>1</sup> Chetty (2007) concludes that for liquidity constrained individuals, unemployment benefits replaces a missing credit market, and thus conveys a substantial, and *undistortionary*, wealth effect on the reemployment hazard rate.

The model adopted in this paper follows closely Shavell and Weiss (1979), and Hopenhayn and Nicolini (1997), extended with limited self-insurance: An infinitely lived individual can at any date either be employed or unemployed. While working she faces an idiosyncratic exogenous risk of losing her job, and while unemployed she can devote time and effort to search for a new job. The agent enjoys consumption and leisure, and she may reallocate resources intertemporally by means of a riskless asset. When doing so, the agent is subject to a liquidity constraint.

The utilitarian government has information on the agents' consumption level and preferences, but not on their search effort. The redistribution policy must therefore be *incentive compatible*. Akin to a social planner that respects incentive compatibility, the government has full control over the agents' consumption and search effort allocations, and may choose these directly. However - following Kocherlakota (2005) and Albanesi and Sleet (2006) - the government does not command any storage technology of her own.<sup>2</sup> Thus, any intertemporal reallocation of resources is obtained through the agents' savings, and must therefore respect the liquidity constraint. By

---

<sup>1</sup>Specifically, Chetty (2007) measure the elasticity of the hazard rate of reemployment on unemployment benefits. He finds that this elasticity is close to zero for the wealthiest 50% of the unemployed, while the corresponding elasticity for the 50% poorest equals approximately  $-0.7$ ; indicating that roughly 70% of the unemployment insurance/duration link is caused by a wealth effect, due to the presence of a liquidity constraint.

<sup>2</sup>Kocherlakota (2005), and Albanesi and Sleet (2006) assume that *all* transfers of resources across time is actualized by the agents' private savings. This is equivalent to assuming that the government operates no storage technology of her own.

carefully “buying and selling” contingent claims, however, the government Pareto-improves on the intertemporal allocation by intratemporally reallocating resources across employment states.

For an unemployed and unconstrained agent, the government’s intertemporal optimality conditions, the inverse Euler equation, implies a wedge between the agent’s intertemporal marginal rate of substitution and the economy-wide real interest rate (the marginal rate of transformation). The reason is that in order to provide incentives to exert search effort, the government wishes to generate a positive correlation between consumption and employment. When the agent’s utility function is concave, higher savings weakens this correlation and thus decreases search effort. In contrast, when the agent is liquidity constrained, a Lagrange multiplier enters into the intertemporal optimality condition. The presence of the multiplier implies that savings and consumption are constant between any two consecutive periods of unemployment.

Following recent developments in the dynamic public finance literature, I construct tax functions that implement the optimal allocations in a decentralized economy (cf. Kocherlakota (2005), and Albanesi and Sleet (2006)). By *implement*, I mean a tax system such that when taken as given, the solution to a decentralized maximization problem faced by an individual coincides with the government’s optimal solution. In addition to an agent’s current and previous employment state, the resulting tax policy is recursive in the agent’s wealth. As a consequence, a liquidity constrained agent’s unemployment benefits are constant over the course of unemployment.

## 2. Structure of the economy

The economy is populated by a utilitarian government and a continuum of risk-averse agents. The planning horizon is infinite. Time is discrete and denoted by  $t = 0, 1, \dots$ . At any given period  $t$ , an agent can either be employed or unemployed and the agent’s employment status is publicly observable.

When an agent is employed, she earns a gross wage,  $w$ . There is no on-the-job search and the probability of being fired is exogenously given at the constant hazard rate  $1 - \gamma$ .

When unemployed, the agent receives unemployment benefits and searches for a job with effort  $e$ . The probability of finding a job, conditional on search effort, is denoted  $p(e)$ . Search effort - and thus the probability of finding a job - is considered private information, not observable by the government or by any other agent in the

economy.<sup>3</sup> The wage distribution is degenerate, and a job offer is, consequently, always accepted. The agents save using a riskless bond that pays net pre-tax return equal to  $r > 0$ . The intertemporal price of consumption,  $1/(1+r)$ , is denoted  $q$ . Bond holdings,  $a$ , are subject to a liquidity constraint,  $\phi$ , such that for any  $t$ ,  $a_t \geq \phi$ . It should be noted that this liquidity constraint is imposed rather than derived from any additional assumptions on private information in the credit market. Indeed, savings are publicly observable. Several studies have found wide empirical support on the view that restricted borrowing is a reality for the vast majority of household, and has, ever since the seminal paper by Deaton (1991), been a standard ingredient in several theoretical models explaining consumption and savings behavior.<sup>4</sup>

**2.1. Model.** An agent's employment status in any period  $t$  is given by  $\theta_t \in \Theta = \{0, 1\}$ . Let  $\theta_t = 1$  denote employment. The history of employment status up to period  $t$  is given by  $\theta^t = (\theta_0, \dots, \theta_t) \in \Theta^t$ , where  $\Theta^t = \{0, 1\} \times \{0, 1\} \times \dots \times \{0, 1\}$ , represent all possible histories up to period  $t$ .

At time zero, each agent is born as either employed or unemployed, and she is entitled some level of initial cash-on-hand,  $b_0$ . The initial entitlement/employment status-pair,  $(b_0, \theta_0)$ , is taken as given by each agent in the economy (the government included). The joint distribution of  $(b_0, \theta_0)$  is given by  $\psi(b_0, \theta_0)$ , with support on  $B \times \Theta$ , where  $B$  is some subset of the real numbers,  $B \subseteq \mathbb{R}$ . Thus, at every date,  $t$ , each agent is distinguished by her initial entitlements and history of employment status,  $(b_0, \theta^t)$ .

Without any loss of generality, I will henceforth formulate the problem such the agents choose  $p$  - the probability of finding a job - rather than effort  $e$ . The agent then ranks contemporaneous consumption and search effort allocations according an additively separable felicity function,  $\{u(c) - (1 - \theta)v(p)\}$ . There is no disutility from working.<sup>5</sup> The function  $u$  is strictly concave, strictly increasing, and once continuously differentiable. The function  $v$  is strictly convex, strictly increasing, and twice continuously differentiable. In addition,  $\lim_{p \rightarrow 0} v'(p) = 0$  and  $\lim_{p \rightarrow 1} v'(p) = \infty$ .

---

<sup>3</sup>This is the source of moral hazard in the model; if benefit payments would be made contingent upon search effort, the economy would reach its first best allocation.

<sup>4</sup>For empirical evidence, see, for instance, Zeldes (1989).

<sup>5</sup>Including disutility from working would not change any of the results in the paper.

An *allocation* in this economy is denoted  $\sigma = \{c_t, p_t\}_{t=0}^\infty$ , where

$$\begin{aligned} c_t &: B \times \Theta^t \rightarrow \mathbb{R}_+ \\ p_t &: B \times \Theta^t \rightarrow [0, 1] \end{aligned}$$

Here,  $c_t(b_0, \theta^t)$  is the amount of consumption an  $(b_0, \theta_0)$ -agent is assigned under history  $\theta^t$ . The contemporaneous probability of finding a job,  $p_t(b_0, \theta^t)$ , is defined equivalently. Let  $\lambda(b_0, \theta^{t+1})$  denote the probability measure for history  $\theta^{t+1}$ , conditional on  $(b_0, \theta_0)$ . For notational convenience let  $p_t(b_0, \theta^t)$  be defined as  $\gamma$  if and only if  $\theta_t = 1$ .  $\lambda(b_0, \theta^{t+1})$  is then recursively given by

$$\lambda(b_0, \theta^{t+1}) = \begin{cases} p_t(b_0, \theta^t)\lambda(b_0, \theta^t), & \theta_{t+1} = 1 \\ (1 - p_t(b_0, \theta^t))\lambda(b_0, \theta^t), & \theta_{t+1} = 0 \end{cases}$$

An agent's net present value utility of an allocation  $\sigma$  is given as

$$V(\sigma, b_0, \theta_0) = \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} \{u(c_t(b_0, \theta^t)) - (1 - \theta_t)v(p_t(b_0, \theta^t))\} \lambda(b_0, \theta^t) d\theta^t \quad (28)$$

The utilitarian government wishes to find  $\sigma$  that maximizes the sum of net present value utilities

$$\hat{V}(\psi) = \max_{\sigma} \int_{B \times \Theta} \{V(\sigma, b_0, \theta_0)\} d\psi \quad (29)$$

subject to each agent's present value budget constraint

$$b_0 \geq \sum_{t=0}^{\infty} q^t \int_{\Theta^t} \{c_t(b_0, \theta^t) - \theta_t w\} \lambda(b_0, \theta^t) d\theta^t, \quad \forall (b_0, \theta_0) \in B \times \Theta \quad (30)$$

Furthermore, since the search effort allocation is private information, the optimal allocation must respect incentive compatibility

$$\{p_t\}_{t=0}^\infty = \operatorname{argmax}\{V(\sigma, b_0, \theta_0)\}, \quad \forall (b_0, \theta_0) \in B \times \Theta \quad (31)$$

The motivation behind the *incentive compatibility constraint* is simple: Each agent takes the consumption allocation as given and chooses search effort to maximize her private utility. Without any loss of generality, the problem is formulated such that the government directly proposes a search effort allocation that coincides with the agent's private optimal choice.

Before formally introducing the agents' liquidity constraint into the government's problem, it is useful to discuss the relationship between allocations and bond holdings. It is instructive, for the time being, to consider an allocation in a two period version of the above problem: In period zero, an  $(b_0, \theta_0)$ -type agent will exert search effort  $p$

and consume  $c$ . In period one she consumes  $c(1)$  if employed and  $c(0)$  if unemployed. The measure of  $(b_0, \theta_0)$ -agents in state  $\theta_1 = 1$  and  $\theta_1 = 0$  must then equal  $p$  and  $1 - p$ , respectively. The total amount of resources transferred *from* period zero *to* period one thus equals  $p(c(1) - w) + (1 - p)c(0)$ . As in Albanesi and Sleet (2006) and Kocherlakota (2005), I will henceforth assume that the total amount of resources transferred between period zero and period one for an  $(b_0, \theta_0)$ -agent equals that agent's savings.

Generalizing the above discussion to an infinite horizon setting gives the following assumption

ASSUMPTION 4. *All transfers of resources across time equals private savings. That is*

$$a_{t+1}(b_0, \theta^t) = \sum_{s=1}^{\infty} q^{s-1} \int_{\Theta^{t+s}} \{c_{t+s}(b_0, \theta^{t+s}) - \theta_{t+s}w\} \frac{\lambda(b_0, \theta^{t+s})}{\lambda(b_0, \theta^t)} d\theta^{t+s}$$

Note that the budget constraint in (30) above may be rewritten as

$$\begin{aligned} b_0 &\geq c_0(b_0, \theta^0) - \theta_0 w + q \sum_{t=1}^{\infty} q^{t-1} \int_{\Theta^t} \{c_t(b_0, \theta^t) - \theta_t w\} \lambda(b_0, \theta^t) d\theta^t \\ &= c_0(b_0, \theta^0) - \theta_0 w + qa_1(b_0, \theta^0) \end{aligned}$$

That is, at period  $t$ , the agent spends his resources  $b_t$  on current consumption,  $c_t$ , and reallocates resources to period  $t + 1$  by means of the riskfree bond,  $a_{t+1}$ . The government then faces the additional liquidity constraint

$$a_{t+1}(b_0, \theta^t) \geq \phi, \quad \forall (b_0, \theta^t) \in B \times \Theta^t \quad (32)$$

The liquidity constraint in (32) is exogenously imposed, and represents a reduced form presumption that agents are unwilling to lend out resources to asset poor individuals. Exogenous liquidity constraints is common in the literature of optimal social policies, and is deployed since it captures the effect of credit market imperfections in a parsimonious manner.<sup>6</sup>

It should be noted that constraint (30) together with the liquidity constraint ensures feasibility. Constraint (30) will always hold as an equality; if it did not, the

<sup>6</sup>See for instance Hansen and Imrohorglu (1992), Gomes, Greenwood and Rebelo (2001), Abdulkadiroğlu, Kuruşçu and Sahin (2002), and Wang and Williamson (2002).

government could simply increase the agent's period zero consumption without interfering with neither incentive compatibility nor the liquidity constraint. An allocation that is both incentive compatible and feasible will be referred to as *incentive feasible*.

The following lemma states that maximizing (28) subject to the individual budget constraint, incentive compatibility and the liquidity constraint, is equal to solving the more complicated problem given in (29). The result is standard and the proof is merely included for completeness.

LEMMA 5. *Define  $\sigma^*$  as the allocation that maximizes (28) for each  $(b_0, \theta_0) \in B \times \Theta$ , subject to individual incentive compatibility, feasibility and the liquidity constraint. Define  $\hat{\sigma}^*$  as the allocation that solves (29). Then*

$$\hat{V}(\psi) = \int_{B \times \Theta} V(\sigma^*, b_0, \theta_0) d\psi$$

PROOF. By construction,  $\hat{V}(\psi) \geq \int_{B \times \Theta} V(\sigma^*, b_0, \theta_0) d\psi$ . If the inequality was strict, then there exist some  $(b_0, \theta_0)$  such that  $V(\hat{\sigma}^*, b_0, \theta_0) > V(\sigma^*, b_0, \theta_0)$ . Since  $\hat{\sigma}^*$  is incentive compatible, delivers  $b_0$ , and satisfies the liquidity constraint,  $\sigma^*$  could not have attained the maximum in (28).  $\square$

**2.2. A recursive formulation.** Following the insights provided by Lemma 5, the problem of interest is given by

$$V(b_0, \theta_0) = \max_{\sigma} \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} \{u(c_t(b_0, \theta^t)) - (1 - \theta_t)v(p_t(b_0, \theta^t))\} \lambda(b_0, \theta^t) d\theta^t \quad (33)$$

$$\text{s.t. } \{p_t\}_{t=0}^{\infty} = \text{argmax}\{V(\sigma, b_0, \theta_0)\} \quad (34)$$

$$b_0 = \sum_{t=0}^{\infty} q^t \int_{\Theta^t} \{c_t(b_0, \theta^t) - \theta_t w\} \lambda(b_0, \theta^t) d\theta^t \quad (35)$$

$$0 \geq \phi - \sum_{s=1}^{\infty} q^{s-1} \int_{\Theta^{t+s}} \{c_{t+s}(b_0, \theta^{t+s}) - \theta_{t+s} w\} \frac{\lambda(b_0, \theta^{t+s})}{\lambda(b_0, \theta^t)} d\theta^{t+s}, \quad \forall t \quad (36)$$

Under an optimal allocation,  $\sigma^*$ , equations (33), (35) and (36) can be written as

$$V(b_0, \theta_0) = u(c_0^*(b_0, \theta_0)) - (1 - \theta_0)v(p_0^*(b_0, \theta_0)) + \beta \int_{\Theta^1} V(\sigma^*, b^*(\theta_1), \theta_1) \lambda(b_0, \theta^1) d\theta^1 \quad (37)$$

$$b_0 = c_0^*(b_0, \theta_0) - \theta_0 w + q \int_{\Theta^1} b^*(\theta_1) \lambda(b_0, \theta^1) d\theta^1 \quad (38)$$

$$0 \geq \phi - q \int_{\Theta^1} b^*(\theta_1) \lambda(b_0, \theta^1) d\theta^1 \quad (39)$$

The following lemma asserts that, given the budget  $b^*(\theta_1)$ , re-optimizing the problem in period one, does not alter period zero present value utility.

LEMMA 6. *If  $u(0) = -\infty$ , then  $V(\sigma^*, b^*(\theta_1), \theta_1)$  in (37), maximizes the agent's utility subject to the budget  $b^*(\theta_1)$ , the liquidity constraint, and incentive compatibility. That is,  $V(\sigma^*, b^*(\theta_1), \theta_1) = V(b^*(\theta_1), \theta_1)$ .*

PROOF. See Appendix C. □

In a companion paper, Rendahl (2007), the above problem is analyzed in the absence of liquidity constraints. While the presence of the constraint complicates the proof, and indeed necessitates additional assumptions, the underlying idea is similar: Since  $V(b_0, \theta_0)$  is strictly increasing in  $b_0$ ,  $b^*(\theta_1)$  can be shown to be resource minimizing given utility  $V(\sigma^*, b^*(\theta_1), \theta_1)$ . The proof then proceed by showing that duality holds: If  $b^*(\theta_1)$  is resource minimizing under utility  $V(\sigma^*, b^*(\theta_1), \theta_1)$ ,  $V(\sigma^*, b^*(\theta_1), \theta_1)$  must be utility maximizing under the budget  $b^*(\theta_1)$ .

Let  $b_e$  and  $b_u$  denote period  $t+1$  contingent claims in the employed and unemployed state, respectively. Then - by exploiting the insights provided by Lemma 6 and following the arguments outlined in Spear and Srivastava (1987) - problem (33) can be made recursive as

$$V(b, \theta) = \max_{c, p, b_e, b_u} \{u(c) - (1 - \theta)v(p) + \beta(pV(b_e) + (1 - p)V(b_u))\} \quad (40)$$

subject to

$$p = \operatorname{argmax}_p \{u(c) - \theta v(p) + \beta(pV(b_e) + (1 - p)V(b_u))\} \quad (41)$$

and

$$b = c - \theta w + q(pb_e + (1 - p)b_u) \quad (42)$$

and

$$0 \geq \phi - pb_e - (1 - p)b_u \quad (43)$$

In accordance with Assumption 4,  $pb_e + (1 - p)b_u$  equals savings.

Since the function  $v$  is differentiable and strictly convex, the incentive compatibility constraint (41) can be replaced by its first order condition

$$v'(p) = \beta(V(b_e) - V(b_u))$$

The solution to (40)-(42) yields a value function,  $V(b, \theta)$ , associated with policy functions  $c(b, \theta)$ ,  $p(b, \theta)$ ,  $b_e(b, \theta)$  and  $b_u(b, \theta)$ . When there is no confusion regarding the agent's employment status, the policy functions will be addressed by their respective initial letter, and reliance on  $b$  will be left implicit.

Previous studies on optimal unemployment insurance adopt a dual formulation to the problem in (40)-(42). Specifically, the literature has, without exception, followed the cost-minimization framework commonly employed in the repeated-agency literature. Fundamentally, this approach amounts to minimize (30) such that the agent receives a pre-specified level of present value utility, and subject to incentive compatibility. Due to Spear and Srivastava (1987), this dual formulation lends itself straightforwardly to a recursive representation. In contrast, this paper adopts a *primal* approach. The reason for this is twofold: First, the primal formulation simplifies the subsequent analysis and provides an intuitive recursive representation in terms of (non-labor) cash-on-hand,  $b$ . Second, this way of formulating the problem has a quite appealing and natural interpretation: Akin to a *social planner*, the government maximizes the agent's utility by choosing current consumption, search effort, and one period ahead Arrow securities at prices  $qp$  and  $q(1 - p)$ . By respecting incentive compatibility, moral hazard is internalized through *individually* and *quantity contingently* priced assets.

### 3. Analysis

Consistent with the formulation of the problem in (40), the government chooses allocations rather than policies. While it facilitates the analysis of the governments optimal policy, it also restricts the subsequent analysis to proceed in two separate steps. The first step concerns the optimal allocations. The second step considers tax functions that implement these allocations.

**3.1. Allocations.** Analogous to the definition of  $b_e$  and  $b_u$ , let  $c_e$  and  $c_u$  denote period  $t + 1$  consumption at the associated employment states. During employment,

moral hazard is absent and the first order necessary conditions of (40) (together with the envelope condition) gives

$$u'(c) - \mu = \frac{\beta}{q}u'(c_e) = \frac{\beta}{q}u'(c_u), \quad \mu \geq 0 \quad (44)$$

Condition (44) implies that when  $\mu = 0$ , consumption is constant for any two consecutive periods; on a period to period basis, the agent is fully insured. When  $\mu > 0$ , consumption at  $t + 1$  is higher than consumption at  $t$ . However, consumption is still constant across states,  $c_e = c_u$ .

For an unemployed agent, it is instructive to consider the following partition of the problem given in (40)-(43)

$$V(b, \theta) = \max_{c, a'} \{u(c) + X(a', \theta)\} \quad (45)$$

$$\text{s.t. } b = c - \theta w + qa' \quad (46)$$

$$0 \geq \phi - a' \quad (47)$$

$$X(a', \theta) = \max_{p, b_e, b_u} \{-v(p) + \beta(pV(b_e) + (1-p)V(b_u))\} \quad (48)$$

$$\text{s.t. } v'(p) = \beta(V(b_e) - V(b_u)) \quad (49)$$

$$a' \geq pb_e + (1-p)b_u \quad (50)$$

As previously mentioned,  $V$  is strictly increasing. It is important to note that  $X$  is strictly increasing as well. This is formally proved in the appendix, but can more easily be seen from the following argument (Goloso et al., 2003): Suppose that (50) is an inequality. Then for some (sufficiently small)  $\varepsilon$ , the planner may increase  $b_e$  and  $b_u$  with  $\varepsilon/V'(c_e)$  and  $\varepsilon/V'(c_u)$ , respectively. At the resulting allocation,  $p$  is unchanged and lifetime utility has increased by  $\beta\varepsilon$ .

The optimality conditions for an unemployed agent give

$$\frac{1}{u'(c) - \mu} = \frac{q}{\beta} \left( p \frac{1}{u'(c_e)} + (1-p) \frac{1}{u'(c_u)} \right), \quad \mu \geq 0 \quad (51)$$

$$q(b_e - b_u) = v''(p)p(1-p) \left( \frac{1}{u'(c_u)} - \frac{1}{u'(c_e)} \right) \quad (52)$$

When  $\mu = 0$ , equation (51) is the commonly known as the “inverse Euler equation” (Rogerson, 1985). When  $c_e \neq c_u$ , Jensen’s inequality implies the following inequality

$$u'(c) < \frac{\beta}{q}(pu'(c_e) + (1 - p)u'(c_u)) \quad (53)$$

Rearranging terms, equation (53) suggests that there is a wedge between the agent’s marginal rate of substitution and the economy’s marginal rate of transformation. Note that the inequality is such that, absent agency problems, the agent would wish to increase future expected consumption at the expense of lower current consumption. In other words, the optimal allocation implies that the agent is savings constrained.

According to the standard Euler equation, an optimal plan has the property that any marginal, temporary and feasible change in behavior equates marginal benefits to marginal costs in the present and future. The inverse Euler equation *appears* to violate this logic. For a given value of  $p$ , consider the optimal choice of reallocating resources from period  $t$  to period  $t + 1$ . If an increase in savings would bring about a proportional increase in  $b_e$  as well as  $b_u$ , equation (53) reveals that, at least on the margin, such a policy would increase overall utility. However, the incentive compatibility constraint in (41) does generally not permit a proportional increase in  $b_e$  and  $b_u$ . To keep the choice of  $p$  unaltered, this constraint forces the increase in resources to be relatively low in states where the momentum (or for small changes, marginal utility) of resources is high, and vice versa. The period  $t + 1$  marginal utilities will thus be “weighted” by their respective inflow of state contingent resources such that the incentive compatibility constraint holds. These weights are high at states in which the marginal utility is low. In contrast, utility maximization implies relatively high weights in states where the marginal benefits of resources is relatively high. Since incentive compatibility inflicts with period  $t + 1$  resources only, it is thus optimal to relegate a relatively high degree of resources to period  $t$  consumption. As a result, the agent appears “savings constrained”. The inverse Euler equation is simply the resulting expression when these conflicting forces are internalized (see Rendahl (2007), Section 3.3, for an algebraic argument revealing the same logic).

The proof for the following lemma may be found in Rendahl (2007), and is therefore omitted.

LEMMA 7. *If  $X(a, \theta)$  is concave,  $q = \beta$ , and  $\mu = 0$ , then*

$$(i) \ c(b, 1) > c(b, 0)$$

- (ii)  $c_e(b, 0) > c(b, 0) > c_u(b, 0)$   
 (iii)  $b > b_u(b, 0) > b_e(b, 0)$  and  $b_u(b, 1) > b = b_e(b, 1)$

The lemma states two important facts: First, for a given level of cash on hand, consumption when employed is always strictly higher than consumption when unemployed. Second, consumption is decreasing between any two consecutive periods of unemployment. In a two period setting, the intuition underlying part (ii) in Lemma 7 is lucid. The terms  $b_e$  and  $b_u$  in equation (52) may then be replaced by  $c_e - w$  and  $c_u$ , respectively. In order to provide incentives to exert search effort, the government then generates a positive correlation between employment and consumption,  $c_e > c_u$ . Insurance is provided by a low intertemporal variance,  $c_e > c > c_u$ . Concavity ensures that this logic extends to a setting with an infinite planning horizon.

PROPOSITION 6. *If  $X(a, \theta)$  is concave and  $q = \beta$  there exist an interval  $[\bar{b}, \underline{b}]$ , such that for any  $b \in [\bar{b}, \underline{b}]$ ,  $a'(b, 0) = \phi$  and  $b_u(\phi, 0) = \underline{b}$ .*

PROOF. Let  $\theta = 0$  be implicit throughout the proof. Note that concavity of  $X$  implies strict concavity of  $V$ . From the first order conditions of (45)-(47),  $a'(b)$  is strictly increasing in  $b$  when  $\mu = 0$ . By the Maximum Theorem,  $a'(b)$  is a continuous function (Stokey et al., 1989). Thus there exist a  $\bar{b}$  such that  $a'(\bar{b}) = \phi$  and  $\mu = 0$ . By Lemma 7,  $\bar{b} > b_u(\phi)$ . Now, consider a  $b \in (\bar{b}, b_u(\phi)]$ . The proposition claims that  $a'(b) = \phi$  and that  $\mu > 0$ . Assume the opposite; that is,  $a'(b) \geq \phi$  and  $\mu = 0$ . Then by the first order conditions of (45)-(47) and concavity of  $X$ ,  $a'(b) \geq a'(\bar{b})$  and  $c(b) \geq c(\bar{b})$ . By the budget constraint in (46), this implies that  $b \geq \bar{b}$  which contradicts  $b \in (\bar{b}, b_u(\phi)]$ . Thus for any two  $b, b' \in [\bar{b}, b_u(\phi)]$ ,  $a'(b) = a'(b') = \phi$  and  $\underline{b} = b_u(\phi)$ .  $\square$

The intuition underlying the proposition is straightforward: If the constraint is binding at a certain  $b$ , then it is binding for any  $b' < b$ . The policy function from (48)-(50) is denoted  $b_u(a')$ . Since for any binding  $b$ ,  $a'$  is by definition equal to  $\phi$ . As long as  $b$  is a binding value,  $b_u$  is independent of  $b$ . Thus,  $b_u(\phi)$  is the lowest possible value of  $b$  and  $a'(b) = \phi$  at  $b = b_u(\phi)$ .

*Remarks.* In Lemma 7 and Proposition 6, concavity of  $X(a, \theta)$  is assumed.<sup>7</sup> The assumption is quite drastic but indispensable for the analysis. The difficulty in proving concavity lies in the fact that the choice set in (48)-(50) is not necessarily convex, and that (functions of) some choice variables do not enter the Bellman equation additively.<sup>8</sup>

**3.2. A fiscal implementation.** Assumption 4 asserts that an agent's private savings equals the intertemporal transfer of resources between any two consecutive periods. The assumption identifies how the agent's liquidity constraint can be introduced into a problem where a government chooses allocations rather than policies. Thus, as Assumption 4 identifies savings, it also identifies the government's policy: Taxes (and, obviously, unemployment benefits) are simply given as the residual between the agent's savings and her cash on hand

$$T_e(a', \theta) = a' - b_e(a', \theta), \quad \text{and} \quad T_u(a', \theta) = a' - b_u(a', \theta) \quad (54)$$

It will in this section be shown that given these taxes, the optimal allocation will, in fact, be chosen by an agent operating in a decentralized economy.

Each agent in the decentralized economy have access to a riskless bond,  $a$ , that pays net (pre-tax) return equal to  $r$ . At time zero, the agents enter a market economy with a given stock of non-contingent claims equal to  $b_0$ . Treating the tax system in (54) as given, the agents maximize their lifetime utility by choosing consumption, savings, and search processes respecting the liquidity and the intertemporal budget constraint.

Formally,

DEFINITION 3. *Let the tax allocation  $T : \mathbb{R} \times \{0, 1\} \times \{0, 1\} \rightarrow \mathbb{R}$ , and initial cash-on-hand  $b_0$  be given. The decentralized economy is then given by*

$$V(b_0, \theta_0) = \max_{\{c_t, a_{t+1}, p_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \int_{\Theta^{t+1}} \{u(c_t(b_0, \theta^t)) - (1 - \theta_t)v(p_t(b_0, \theta^t))\} \lambda(b_0, \theta^t) d\theta^t \quad (55)$$

<sup>7</sup>Indeed, conditions (44), (51) and (52) are derived using Benveniste and Scheinkman's (1979) envelope theorem - a theorem that requires concavity of  $V$ . Concavity of  $X$  is sufficient for this.

<sup>8</sup>Note that these are sufficient, but not necessary conditions for concavity. All numerical solutions in this paper as well as in Hopenhayn and Nicolini (1997) displays a strictly concave value function.

subject to

$$w\theta_t + a_t(b_0, \theta^{t-1}) - T(a_t(b_0, \theta^{t-1}), \theta_{t-1}, \theta_t) = c_t(b_0, \theta^t) + qa_{t+1}(b_0, \theta^t) \quad (56)$$

and

$$a_{t+1}(b_0, \theta^t) \geq \phi, \quad \text{for } t = 0, 1, 2, \dots \quad (57)$$

The following proposition shows that given the tax functions in (54), the solution to (55)-(57) coincides with the optimal allocation

**PROPOSITION 7.** *The solution to (55)-(57) coincides with the solution to (40)-(43).*

**PROOF.** The proof is direct and establishes a one-to-one relationship between the government's and the agent's problem.

By the construction of the tax function in (54), the solution to (48) may be formulated as

$$X(a', \theta) = \max_p \{ -(1 - \theta)v(p) + \beta(pV(a' - T_e(a', \theta)) + (1 - p)V(a' - T_u(a', \theta))) \}$$

Thus,

$$\begin{aligned} V(b, \theta) &= \max_{c, a'} \{ u(c) + \max_p \{ -(1 - \theta)v(p) + \beta(pV(a' - T_e(a', \theta)) + (1 - p)V(a' - T_u(a', \theta))) \} \} \\ &= \max_{c, a', p} \{ u(c) - (1 - \theta)v(p) + \beta(pV(a' - T_e(a', \theta)) + (1 - p)V(a' - T_u(a', \theta))) \} \end{aligned}$$

$$\text{s.t. } b = c - \theta w + qa', \quad \text{and } a' \geq \phi$$

Which is the recursive form of the problem given in (55)-(57).  $\square$

The intuition behind this result is immediate. By the design of the tax function, the agent can always choose the assigned allocation. Any other choice is equal to imitate the allocation of some other agent. By construction, imitating someone else is incentive compatible and budget feasible. Thus, since the allocation is optimal under incentive compatibility and budget feasibility, imitation cannot be optimal.

**3.3. Unemployment benefits and the duration of unemployment.** In this section I will analyze the qualitative properties of an optimal unemployment insurance program during the course of unemployment. When an unemployed agent is unconstrained, consumption is decreasing, the agent decumulates assets, and the sequence of unemployment benefits displays an increasing profile. As was shown in Lemma 7,

consumption should decrease along the duration of the unemployment spell in order to provide incentives to search. The agent chooses to decumulate assets along the unemployment spell in order to exploit the intrinsic insurance effect of wealth. Unemployment benefits displays an increasing profile for two reasons: *First*, a higher level of non-labor income (from wealth) reduces the correlation between employment and consumption. An increasing benefit payments structure thus intentionally crowds out savings in order to render search incentives. *Second*, an increasing benefits profile enhance the role of benefits as insurance for those who deplete their wealth as a consequence of being unemployed for long periods. Moreover, when the agent is liquidity constrained, consumption, savings and unemployment benefits are all constant. By Proposition 6,  $a'(\underline{b}) = \phi$  and  $b_u(\phi) = \underline{b}$ . By the construction of the tax function it is immediate that cash-on-hand, as well as unemployment benefits, are constant. Since cash-on-hand stays constant, so does consumption.

Examining the qualitative properties of the unemployment benefits corresponds to examine how the tax function in (54) responds to a change in  $a'$ . This is a non-trivial task; taxes and wealth must interact in a course such that the solutions to the problems (33) and (55) coincide. To this end, I will examine the properties of the marginal tax functions.

Let  $T_e(a')$  and  $T_u(a')$  denote period  $t + 1$  taxes at the associated employment states, and at  $\theta = 0$ . The following proposition reveals that the tax functions are differentiable

PROPOSITION 8. *If  $V(b, \theta)$  is concave, there exist marginal tax functions given by*

$$T'_e(a') = 1 - \frac{q}{\beta} \frac{u'(c) - \mu}{u'(c_e)}, \quad T'_u(a') = 1 - \frac{q}{\beta} \frac{u'(c) - \mu}{u'(c_u)}$$

PROOF. See Appendix C. □

The idea behind the proof is to consider an infinitesimal change in  $a'$ . The resulting marginal change in taxes must be such that the government's first order conditions hold, incentive compatibility is preserved, the budget balances, and the definition of the tax functions hold. In addition, the agent's decentralized first order conditions must hold:

$$u'(c) - \mu = \frac{\beta}{q} (pu'(c_e)(1 - T'_e(a')) + (1 - p)u'(c_u)(1 - T'_u(a')))$$

The qualitative features of the unemployment insurance program depends on the sign of these marginal tax functions. When the liquidity constraint is not binding, the result is straightforward; if  $\beta = q$ , then  $c_e > c > c_u$  and  $T'_e < 0$  and  $T'_u > 0$ . Thus, for an unconstrained agent, unemployment benefits are *decreasing* with wealth. Define  $\bar{c}$  as  $c(\bar{b}, 0)$ . For any  $b$  in the binding interval  $[\bar{b}, \underline{b}]$ , define  $c$  as  $c(b, 0)$ . Then from the first order conditions,  $\mu(b, 0) = u'(c) - u'(\bar{c})$ . Thus, for all  $b \in [\bar{b}, \underline{b}]$ , the marginal tax functions are given by

$$T'_e(a') = 1 - \frac{q}{\beta} \frac{u'(\bar{c})}{u'(c_e)}, \quad T'_u(a') = 1 - \frac{q}{\beta} \frac{u'(\bar{c})}{u'(c_u)}$$

Since  $c_e$  and  $c_u$  was optimal when  $\bar{c}$  was chosen,  $c_e > \bar{c} > c_u$ . Again it follows that  $T'_e < 0$  and  $T'_u > 0$ .

**PROPOSITION 9.** *If  $X(a', \theta)$  is concave,  $\beta = q$ , and  $\mu = 0$ , then (i)  $a \geq a'$ , (ii)  $T_u(a) > T_u(a')$ , and (iii)  $T_e(a) < T_e(a')$ .*

**PROOF.** By Proposition 8 and Lemma 7,  $1 > T'_u(a') > 0$ . Thus for any  $a_1$  and  $a_2$ , such that  $a_1 > a_2$ ,  $T_u(a_1) > T_u(a_2)$ . If  $a' > a$ , then  $1 > T'_u(a')$  implies that  $b_u > b$ , thus contradicting Lemma 7 and Proposition 6. Thus  $a > a'$ ,  $T_u(a) > T_u(a')$ , and  $T_e(a) < T_e(a')$ .  $\square$

Thus, when  $\mu = 0$ , Lemma 7 gives that  $c > c_u$ . Proposition 9 reveals that  $a > a'$  and that  $T_u(a) > T_u(a')$ .

When  $\mu > 0$  and  $b = \underline{b}$ , Proposition 6 reveals that  $a'(\underline{b}) = \phi$  and  $b_u(\phi) = \underline{b}$ . Thus  $a = a' = \phi$  and  $b = b_u = \underline{b}$ . By the construction of the tax functions in (54), unemployment benefits are constant and equal  $\underline{b} - \phi$ .

#### 4. Quantitative Analysis

To shed further light on the properties of the optimal unemployment insurance program, I turn to a calibrated version of the model. The aim of this section is to answer the following question: Should a liquidity constrained agent be treated significantly different from an unconstrained agent?

As will be shown, an unconstrained agent with savings equal to three months of labor income (sufficient to sustain a labor income loss equal to an average unemployment spell), ought to receive a first-period replacement rate of 1.5%, to be compared with the 30% received by a liquidity constrained agent with zero liquid wealth. This result gives support for a asset based means tested unemployment insurance scheme.

**4.1. Calibration.** Following the main macroeconomic literature the function  $u$  is chosen to be of the type constant relative risk aversion

$$u(c) = \lim_{\rho \rightarrow \sigma} \frac{c^{1-\rho}}{1-\rho}$$

The coefficient of risk-aversion  $\sigma$  is set to 2.<sup>9</sup> As in Pavoni (2007) and Pavoni and Violante (2007), the length of each period is assumed to be one month. The yearly interest rate is set at 5% and the intertemporal discount factor  $\beta$  is thus  $1.05^{-1/12}$ . In order for the results to be comparable with the previous (contractual) literature on unemployment insurance, the hazard rate of unemployment,  $1 - \gamma$ , is set to zero.<sup>10</sup> Employment is thus an absorbing state. The net wage,  $w$ , is normalized to 1.

The function governing the disutility of search effort,  $v$ , is assumed to have the following functional form

$$v(p) = -\frac{\ln(1-p)}{\alpha} - \frac{p}{\alpha}$$

Note that  $v$  is strictly convex on  $[0, 1]$  and that  $v(0) = 0$ ,  $v'(0) = 0$ ,  $v(1) = \infty$  and  $v'(1) = \infty$ . Several articles on optimal unemployment insurance (e.g. Hopenhayn and Nicolini (1997), Young (2004) and Wang and Williamson (2002)), assume that  $p(e) = 1 - \exp(-\alpha e)$  and that the disutility of search equals  $e$ . A choice consistent with the literature would thus be  $v(p) = -\ln(1-p)/\alpha$ . To ensure interiority, however, the above simple modification is employed.

In line with previous research on unemployment insurance, the liquidity constraint  $\phi$  is set to zero (e.g. Hansen and Imrohoroglu (1992), Wang and Williamson (2002), Abdulkadiroğlu et al. (2002), and Young (2004)). Borrowing is thus not permitted.

To calibrate the parameter  $\alpha$  in the function  $v$ , an auxiliary economy is used. The auxiliary economy is given as the problem in equations (55)-(57), but in which the government's policy,  $T$ , is exogenously specified. As in Wang and Williamson (2002) and Young (2004), the fixed unemployment insurance policy delivers unemployment benefits equal to 50% of labor income for the first six months of unemployment, and 17% thereafter. Taxes when employed are assumed to be constant and levied on labor income. The income tax is endogenous as to balance the government's budget. The parameter  $\alpha$  is then set to match the elasticity of the hazard rate of

<sup>9</sup>Estimates show that this parameter is generally within the range  $[1, 3]$  (Mehra and Prescott, 1985).

<sup>10</sup>That is, Shavell and Weiss (1979), Hopenhayn and Nicolini (1997), Pavoni (2007) and Pavoni and Violante (2007).

employment with respect to unemployment benefits given in Chetty (2007). Under this calibration  $\alpha$  is set to 0.4 and generates an elasticity equal to  $-0.46$  for liquidity constrained individuals, and  $-0.25$  for individuals with savings equal to one year of labor income.<sup>11</sup>

Table 1 summarizes the baseline parameter calibration.

TABLE 3.1. Calibration of Parameters

Parameter	$\beta$	$\sigma$	$r$	$1 - \gamma$	$w$	$\alpha$	$\phi$
Value	$1.04^{-\frac{1}{12}}$	2	5%	0	1	.4	0

**4.2. Numerical Results.** Figure 3.1 depicts how the level of unemployment benefits are related to an agent's asset position. The agent's wealth is featured on the horizontal axis. Wealth ranges from zero to the US median labor income to wealth ratio (which is, on yearly basis, equal to one). The vertical axis displays the level of unemployment benefits as a fraction of the wage. The figure reveals an

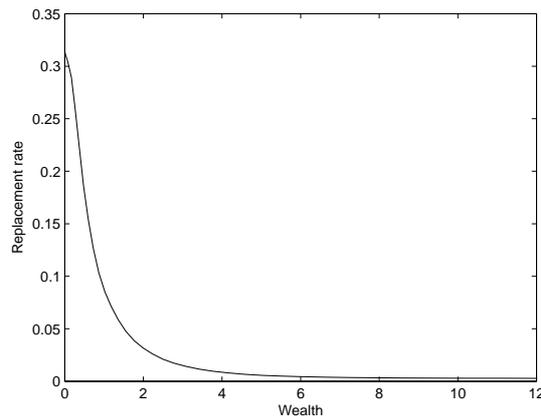


FIGURE 3.1. Unemployment benefits and wealth.

illuminating pattern; unemployment benefits for the asset poor ought to be several orders of magnitude of that of a wealthy agent. For instance, unemployment benefits paid to an agent with wealth equal to three month labor income (enough to sustain

<sup>11</sup>The corresponding numbers in Chetty (2007) roughly equal  $-0.7$  and  $+0.2$ . However, since it is impossible for the current model to generate a positive elasticity, these numbers cannot be targeted exactly. The numbers generated in this calibration are thus a compromise somewhere in between Chetty's (2007) estimates.

an average unemployment spell) is 5% of that paid to a borrowing constrained agent. The equivalent figure for an agent with wealth equal to the US median labor income to wealth ratio (i.e 12) is .6%.

The relationship featured in Figure 3.1 is sensitive to the calibration of the coefficient of relative risk aversion,  $\sigma$ , and the degree of moral hazard,  $\alpha$ . For instance, at  $\sigma = 3$  and  $\alpha = .8$  (high risk aversion, moderate moral hazard), a borrowing constrained agent ought to receive a replacement rate of 50%. At  $\sigma = .5$  and  $\alpha = .2$  (low risk aversion, severe moral hazard) the corresponding number drops to 7%. The intuition is not far fetched; a higher degree of risk aversion unequivocally generates a higher provision of insurance. When moral hazard is moderate, high insurance comes at a low cost (in terms of distortion to incentives). However, Table 3.2 reveals that the *relationship* between unemployment benefits paid at different wealth level conveys a more robust pattern. Even at a high degree of risk aversion and a modest degree of moral hazard, unemployment benefits should decrease swiftly with the agent's wealth level; at wealth equal to one month of labor income, unemployment benefits ought to be 40% of the benefits received by a constrained agent with zero wealth.

TABLE 3.2. Relative Unemployment Benefits<sup>a</sup>

Wealth	Values of Parameters $\sigma$ and $\alpha$		
	{3, .8}	{1.5, .4}	{.5, .2}
0	100%	100%	100%
1	40%	27%	20%
2	15%	10%	9%
3	7%	5%	4%
12	3%	1%	0%

<sup>a</sup>The ratios are calculated as the unemployment benefits received at the respective wealth level, divided by benefits paid to a liquidity constrained individual at zero wealth.

Figure 3.2 corresponds to Proposition 9 and depicts the sequence of consumption (upper solid line), decumulation of assets (dashed line), and unemployment benefits received along the course of unemployment. Decumulation of assets is defined as  $a_{t+1} - a_t$ . As in Proposition 9, unemployment benefits are increasing when the agent is unconstrained. This increasing profile is targeted at crowding out savings in order to provide incentives to exert search effort. Recall that the government provides incentives by letting consumption covary positively with employment. A higher level of savings reduces this correlation and thus aggravates the duration of

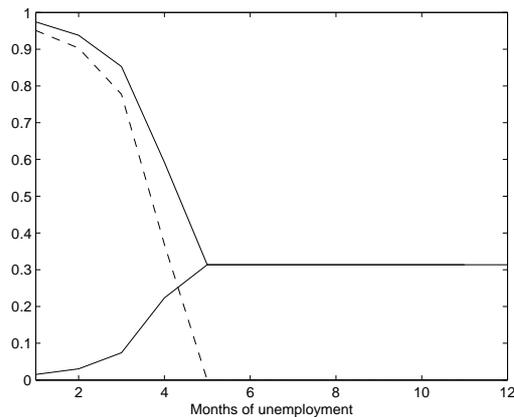


FIGURE 3.2. The sequence of consumption (upper solid line), decumulation of assets (dashed line), and unemployment benefits along the duration of unemployment.

unemployment. The net effect is a decreasing sequence of cash-on-hand and, thus, a decreasing sequence of consumption. When the liquidity constraint binds, the level of unemployment insurance peaks, and stays constant.

## 5. Concluding Remarks

This paper has studied a model of optimal redistribution policies in which the foremost risk in an agent's life is unemployment. Moral hazard arises as job search effort is unobservable. Whereas the model permits agents to self insure by means of a riskless bond, borrowing is exogenously restricted.

Previous studies on unemployment insurance - e.g. Shavell and Weiss (1979), Hopenhayn and Nicolini (1997), and Pavoni (2007) - assume that whereas each agent is borrowing constrained, the government operates her own storage technology where any such restriction is absent. As a consequence, the government saves (and borrows) in an agent's name, effectively relieving the agent of any impediment caused by the liquidity constraint. In contrast, this paper has taken a different position: A liquidity constraint is here an exogenously imposed presumption that agents are unwilling to lend resources to agents with savings lower than a pre-specified threshold. Although the government may function as a financial intermediary, she is not able to fend off the lending loath imposed by the liquidity constraint.

The optimal unemployment insurance scheme reveals two illuminating features: *First*, unemployment benefits ought to be constant for liquidity constrained agents. Since a liquidity constrained agent's situation (or *state*) does not change over the course of unemployment, the optimal program does not embed any duration dependence. *Second*, in a calibrated version of the model it was shown that the liquidity constraint conveys important quantitative implications. A constrained agent ought to receive benefits payments three times higher than that received by an agent with wealth equal to one months labor income; twenty times higher than that received by an agent with wealth equal to three months labor income; and one hundred times higher compared to an agent with savings equal to twelve months of labor income (US median labor income to wealth ratio).

The policy implications from the analysis are stark; unemployment benefits should be asset based and depend negatively on the agent's asset position. As wealth itself encodes insurance, the negative relation between wealth and unemployment benefits is intuitive. However, asset based approaches have commonly been criticized for its distortive, and negative, effect on savings (e.g. Hubbard et al. (1995)). Although undesirable per se, this paper has revealed an additional effect of wealth; a higher level of savings reduces the opportunity cost of being employed and thus increases the unemployment duration. Together, the net distortive effect of an asset based scheme appears to be favorable.

The optimal program in this paper displays a continuous, and negative, relationship between assets and unemployment benefits. Such a continuous relationship may be very costly to implement in practice. Nevertheless, the (very) steep decline in benefits along the wealth dimension does indicate that an asset based means tested insurance program may be close to optimal: As with Medicaid, food stamps, and until recently, Aid to Families with Dependent Children (AFDC), to mention a few social policies in the United States, unemployment benefits may be paid only if an agent has assets below some threshold.

## Bibliography

- Abdulkadiroğlu, A., Kuruşçu, B. and Sahin, A.: 2002, Unemployment insurance and the role of self-insurance, *The Review of Economic Dynamics* **5**, 681–703.
- Aiyagari, R. S.: 1994, Uninsured idiosyncratic risk and aggregate saving, *Quarterly Journal of Economics* **109**(3).
- Albanesi, S. and Sleet, C.: 2006, Dynamic optimal taxation with private information, *The Review of Economic Studies* **73**(1), 1–30.
- Aliprantis, C. D. and Border, K. C.: 1999, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 2 edn, Springer Verlag.
- Benveniste, L. M. and Scheinkman, J. A.: 1979, On the differentiability of the value function in dynamic models of economics, *Econometrica* **47**(3), 727–732.
- Blank, R. and Card, D.: 1991, Recent trends in insured and uninsured unemployment: Is there an explanation, *Quarterly Journal of Economics* **106**(4), 1157–1190.
- Browning, M. and Crossley, T. F.: 2001, Unemployment insurance benefit levels and consumption changes, *Journal of Public Economics* **80**, 1–23.
- Carroll, C. D.: 2005, The method of endogenous gridpoints for solving dynamic stochastic optimization problems, *NBER Technical Working Paper* (309).
- Chetty, R.: 2007, Moral hazard vs. liquidity in unemployment insurance, *Mimeo, UC Berkeley*.
- Chiu, H. W. and Karni, E.: 1998, Endogenous adverse selection and unemployment insurance, *Journal of Political Economy* **106**(4), 806–827.
- Christiano, L. J. and Fisher, J. D. M.: 2000, Algorithms for solving dynamic models with occasionally binding constraints, *Journal of Economic Dynamics and Control* **24**(8).
- Coleman, W. J.: 1990, Solving the stochastic growth model by policy-function iteration, *Journal of Business and Economic Statistics* **8**(1).
- Deaton, A.: 1991, Savings and liquidity constraints, *Econometrica* **59**(5), 1221–1248.
- den Haan, W. J. and Marcet, A.: 1990, Solving the stochastic growth model by parameterizing expectations, *Journal of Business and Economic Statistics* **8**(1).
- Golosov, M., Kocherlakota, N. and Tsyvinski, A.: 2003, Optimal indirect and capital taxation, *The Review of Economic Studies* **70**(3), 569–587.
- Golosov, M. and Tsyvinski, A.: 2006, Designing optimal disability insurance: A case for asset testing, *Journal of Political Economy* **114**(2), 257–269.
- Gomes, J., Greenwood, J. and Rebelo, S.: 2001, Equilibrium unemployment, *Journal of Monetary Economics* **48**(1), 109–152.

- Gruber, J.: 1997, The consumption smoothing benefits of unemployment insurance, *American Economic Review* **87**(1), 192–205.
- Gruber, J.: 1998, Unemployment insurance, consumption smoothing, and private insurance: Evidence from the psid and cex, *Research in Employment Policy* **1**, 3–32.
- Hansen, G. D. and Imrohoroglu, A.: 1992, The role of unemployment insurance in an economy with liquidity constraints and moral hazard, *Journal of Political Economy* **100**(1), 118–142.
- Hopenhayn, H. A. and Nicolini, J. P.: 1997, Optimal unemployment insurance, *Journal of Political Economy* **105**(2), 412–438.
- Hubbard, G. R., Skinner, J. and Zeldes, S. P.: 1995, Precautionary saving and social insurance, *Journal of Political Economy* **103**(2), 360–399.
- Judd, K. L.: 1998, *Numerical Methods in Economics*, The MIT Press.
- Judd, K. L. and Solnick, A.: 1994, Numerical dynamic programming with shape-preserving splines, *Mimeo* .
- Kehoe, P. J. and Perri, F.: 2002, International business cycles with endogenous incomplete markets, *Econometrica* **70**(3).
- Kocherlakota, N.: 2005, Zero expected wealth taxes: A mirrlees approach to dynamic optimal taxation, *Econometrica* **73**(5), 1587–1621.
- Krusell, P. and Smith, A. A.: 1997, Income and wealth heterogeneity, portfolio choice, and equilibrium asset returns, *Macroeconomic Dynamics* **1**(2).
- Krusell, P. and Smith, A. A.: 1998, Income and wealth heterogeneity in the macroeconomy, *Journal of Political Economy* **106**(5).
- Ljungqvist, L. and Sargent, T. J.: 2004, *Recursive Macroeconomic Theory*, 2 edn, The MIT Press, Cambridge, Massachusetts.
- McCall, J.: 1970, Economics of information and job search, *Quarterly Journal of Economics* **84**(1), 757–782.
- McGrattan, E. R.: 1996, Solving the stochastic growth model with a finite element method, *Journal of Economic Dynamics and Control* **20**.
- Mehra, R. and Prescott, E. C.: 1985, The equity premium: A puzzle, *Journal of Monetary Economics* **15**(2), 145–161.
- Meyer, B. D.: 1990, Unemployment insurance and unemployment spells, *Econometrica* **58**(4), 757–782.
- Milgrom, P. and Segal, I.: 2002, Envelope theorems for arbitrary choice sets, *Econometrica* **70**(2).
- Mirrlees, J. A.: 1971, An exploration in the theory of optimum income taxation, *The Review of Economic Studies* **38**(2), 175–208.
- Moffitt, R.: 1985, Unemployment insurance and the distribution of unemployment spells, *Journal of Econometrics* **28**(1), 85–101.
- Oswald, A. J.: 1986, Unemployment insurance and labor contracts under asymmetric information: Theory and facts, *American Economic Review* **76**(3), 365–377.
- Pavoni, N.: 2007, On optimal unemployment compensation, *Journal of Monetary Economics* .

- Pavoni, N. and Violante, G. L.: 2007, Optimal welfare-to-work programs, *The Review of Economic Studies* **74**(1), 283–318.
- Rendahl, P.: 2007, Asset based unemployment insurance, *Mimeo, European University Institute*.
- Rockafellar, R. T.: 1970, *Convex Analysis*, Princeton University Press.
- Rogerson, W.: 1985, Repeated moral hazard, *Econometrica* **53**(1), 69–76.
- Santos, M.: 2000, Accuracy of numerical solutions using the euler equation residuals, *Econometrica* **68**(6).
- Shavell, S. and Weiss, L.: 1979, The optimal payment of unemployment insurance benefits over time, *Journal of Political Economy* **87**(6), 1347–1362.
- Shimer, R. and Werning, I.: 2005, Liquidity and insurance for the unemployed, *Federal Reserve Bank of Minneapolis, Research Department Staff Report* **366**.
- Spear, S. E. and Srivastava, S.: 1987, On repeated moral hazard with discounting, *The Review of Economic Studies* **54**(4), 599–617.
- Stokey, N. L., Lucas, R. E. and Prescott, E. C.: 1989, *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge, Massachusetts.
- Wang, C. and Williamson, S. D.: 2002, Moral hazard, optimal unemployment insurance, and experience rating, *Journal of Monetary Economics* **49**, 1337–1371.
- Young, E. R.: 2004, Unemployment insurance and capital accumulation, *Journal of Monetary Economics* **51**, 1683–1710.
- Zeldes, S. P.: 1989, Consumption and liquidity constraints: An empirical investigation, *Journal of Political Economy* **97**(2), 305–346.



## Appendix

## A. Code for Chapter 1.

```

1 % The neoclassical growth model with irreversible investment
2 % in the setting of Christiano and Fischer (2000), model (1),
3 % solved by the method of endogenous gridpoints using a finite
4 % element method (linear interpolation is default).
5
6 % Parameters: exp(z) is the solow residual, a is the capital share
7 % of output, b is the discount factor, d is the depreciation
8 % rate and g is the coefficient of relative riskaversion.
9 % Z is the exogenous state space with associated transition
10 % matrix, Q.
11
12 %N defines the number of nodes in the endogenous state space.
13
14 N=200; p=0; z=0.23; a=0.3; b=1.03^(-1/4); d=0.02; g=1;
15 Q=[(1+p)/2, (1-p)/2; (1-p)/2, (1+p)/2]; Z=exp([z;-z]);
16
17 n=ones(size(Z')); nn=ones(N,1); d1=0.5;
18 khat=((1-b*(1-d))/(a*b))^(1/(a-1)); kmax=khat*1.9; kmin=khat*0.3;
19 kp=(linspace(kmin,kmax,N))'; kpp=(1-d)*kp*n; mp=0; mup=0*nn*n;
20 m0=(kp./(1-d)).^a*Z';
21
22 while d1>1e-8
23     up=(kp.^a)*Z'+(1-d)*kp*n-max(kpp,(1-d)*kp*n).^(-g);
24     r=a*kp.^(a-1)*Z'-d;
25     m=(b*(up.*(1+r)-max(mup,0))*Q').^(-1/g)+kp*n;
26     mu=(m0).^(-g)-b*(up.*(1+r)-max(mup,0))*Q';
27     d1=max(max(abs(mp-m)./(1+abs(m))));
28     mp=m;
29     for i=1:length(Z)
30         kpp(:,i)=interp1(m(:,i),kp,Z(i)*kp.^a+(1-d)*kp);
31         mup(:,i)=interp1(m(:,i),mu(:,i),Z(i)*kp.^a+(1-d)*kp);
32     end
33 end

```

## B. Proofs of Chapter 2

### Lemma 3.

PROOF. Equations (12) and (13) are repeated for convenience:

$$V(b_0, \theta_0) = u(c_0^*(b_0, \theta_0)) - (1 - \theta_0)v(p_0^*(b_0, \theta_0)) + \beta \int_{\Theta^1} V(\sigma^*, b^*(\theta_1), \theta_1) \lambda(b_0, \theta^1) d\theta^1 \quad (\text{B1})$$

$$b_0 = c_0^*(b_0, \theta_0) - \theta_0 w + q \int_{\Theta^1} b^*(\theta_1) \lambda(b_0, \theta^1) d\theta^1 \quad (\text{B2})$$

The proof proceeds in three steps: *First* it will be shown that for any *utility maximizing* or *resource minimizing* allocation, the Inada-conditions on  $u(c)$  implies that if  $c_t(b_0, \theta^t) = 0$ , then  $c_{t+s}(b_0, \theta^{t+s}) = 0$ , for  $s \in \mathbb{N}$ , almost surely. *Second*, focusing on the interior case, it will then be shown that  $b^*(\theta_1)$ , as given in equation (B2), is resource minimizing under the value  $V(\sigma^*, b^*(\theta_1), \theta_1)$ . *Third* it will be shown that *duality* holds; that is if  $b^*(\theta_1)$  is resource minimizing under  $V(\sigma^*, b^*(\theta_1), \theta_1)$ , then  $V(\sigma^*, b^*(\theta_1), \theta_1)$  is utility maximizing under  $b^*(\theta_1)$  - that is,  $V(\sigma^*, b^*(\theta_1), \theta_1) = V(b^*(\theta_1), \theta_1)$ .

*Step 1.* For any utility maximizing or resource minimizing allocation, define  $\delta(b_0, \theta^t)$  as

$$\delta(b_0, \theta^t) = u(c_t) - v(p_t) + \beta(p_t u(c_{t+1}^1) + (1 - p_t)u(c_{t+1}^0))$$

The dependency of  $c_t$ ,  $p_t$  and  $c_{t+1}$ , on  $(b_0, \theta^t)$  and  $(b_0, (\theta^t, \theta_{t+1}))$  is here left implicit. Assume that  $\lambda(b_0, \theta^t) > 0$ . Consider the following problem

$$\begin{aligned} & \max_{x, z, y} \{y - q(px + (1 - p)z)\} \\ \text{s.t. } & \delta(b_0, \theta^t) = u(c_t - y) - v(p_t) + \beta(p_t u(c_{t+1}^1 + x) + (1 - p_t)u(c_{t+1}^0 + z)) \\ & u(c_{t+1}^1 + x) - u(c_{t+1}^0 + z) = u(c_{t+1}^1) - u(c_{t+1}^0) \\ & c_t \geq y, \quad c_{t+1}^1 \geq -x, \quad c_{t+1}^0 \geq -z \end{aligned}$$

where the allocation  $\{c_t, p_t\}_{t=0}^\infty$  is incentive feasible. At the optimal allocation, the solution to the above problem is given by  $x = y = z = 0$ . To see why, notice that any deviation of  $x$ ,  $y$ , and  $z$  from zero, fulfilling the above restrictions, is feasible and incentive compatible. Moreover, such a perturbation frees up period  $t$  resources equal to  $y - q(px + (1 - p)z)$ . These additional resources may, if properly discounted, be allocated as period zero consumption - or, in a resource minimizing setting, as less period zero resources - without inflicting with incentive compatibility.

Assume that  $c_t = 0$ . Then the first order necessary conditions to the above problem with respect to  $x$ ,  $y$  and  $z$ , evaluated at zero, must observe

$$\frac{1}{u'(0)} \geq \frac{\beta}{q} \left( p_t \frac{1}{u'(c_{t+1}^1)} + (1 - p_t) \frac{1}{u'(c_{t+1}^0)} \right) \quad (\text{B3})$$

Since  $u'(0) = \infty$ ,  $c_{t+1}^1$  must also equal zero whenever  $p_t > 0$ . The same holds for  $c_{t+1}^0$  whenever  $(1 - p_t) > 0$ . Thus if  $c_t(b_0, \theta^t) = 0$  for any  $\theta^t$  with  $\lambda(b_0, \theta^t) > 0$ , then  $c_{t+s}(b_0, \theta^{t+s}) = 0$ ,  $\lambda(b_0, \theta^{t+s})$ -a.s.

*Step 2.* Consider the following resource minimization problem:

$$b(V, \hat{\theta}_0) = \min_{\sigma} \sum_{t=0}^{\infty} q^t \int_{\Theta^t} \{c_t(V, \hat{\theta}^t) - \hat{\theta}_t w\} \lambda(V, \hat{\theta}^t) d\hat{\theta}^t \quad (\text{B4})$$

$$\text{s.t. } V \leq \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} \{u(c_t(V, \hat{\theta}^t)) - (1 - \hat{\theta}_t)v(p_t(V, \hat{\theta}^t))\} \lambda(V, \hat{\theta}^t) d\hat{\theta}^t \quad (\text{B5})$$

and subject to the incentive compatibility constraint. A “hat” on the sequence  $\theta_t$  is used to distinguish it from the values of  $\theta_t$  in the original problem (9)-(11). If the constraint in (B5) is non-binding, then  $c_0 = 0$  and, by *Step 1.* above,  $c_t(V, \theta^t) = 0 \forall \theta^t$ . I will henceforth refer to this solution as the *zero solution*. It is important to note that a non-zero solution attains at least as high utility as the zero solution; at any non-zero solution, the agent could exert the same search effort as at the zero solution (which is zero), and attain a strictly higher level of utility. Thus, independently of  $c_0$  being interior, constraint (B5) must hold as an equality.

Assume that  $V$  in (B5) equals  $V(\sigma^*, b^*(\theta_1), \theta_1)$  in (B1). Assume further that  $\theta_1 = \hat{\theta}_0$ . Could  $b(V, \hat{\theta}_0)$  in (B4) take on a smaller value than  $b^*(\theta_1)$  in (B2)? If so,  $V(b(V, \hat{\theta}_0), \theta_1) = V(\sigma^*, b^*(\theta_1), \theta_1) \forall \theta_1 \in \Theta$ , and  $b(V, \hat{\theta}_0) < b^*(\theta_1)$  for at least one value of  $\theta_1$ . At this alternative allocation,  $p_0^*$  is still incentive compatible and

$$\begin{aligned} V(b_0, \theta_0) &= u(c_0^*(b_0, \theta^0)) - (1 - \theta_0)v(p_0^*(b_0, \theta^0)) + \beta \int_{\Theta^1} V(b(V, \hat{\theta}_0), \theta_1) \lambda(b_0, \theta^1) d\theta^1 \\ b_0 &> c_0^*(b_0, \theta^0) - \theta_0 w + q \int_{\Theta^1} b(V, \hat{\theta}_0) \lambda(b_0, \theta^1) d\theta^1 \end{aligned}$$

Where the last inequality together with monotonicity of  $V(b_0, \theta_0)$  implies thus that  $\sigma^*$  could not have attained the maximum in (9).

*Step 3.* In order to complete the proof, it must be shown that  $V(\sigma^*, b^*(\theta_1), \theta_1)$  attains the maximum value under resources  $b^*(\theta_1)$ .

Assume that  $V(b^*(\theta_1), \theta_1) > V(\sigma^*, b^*(\theta_1), \theta_1)$ . By Berge’s Maximum Theorem (Aliprantis and Border, 1999),  $V(b^*(\theta_1), \theta_1)$  is continuous in  $b$ . Since any non-zero solution renders greater utility than the zero solution,  $c_1(b^*(\theta_1), \theta_1) > 0$ , and there exist a  $b^{**}(\theta_1)$  arbitrarily close to  $b^*(\theta_1)$  such that  $b^*(\theta_1) > b^{**}(\theta_1)$  and  $V(b^{**}(\theta_1), \theta_1) > V(\sigma^*, b^*(\theta_1), \theta_1)$ . This contradicts that  $b^*(\theta_1)$  was resource minimizing for  $V(\sigma^*, b^*(\theta_1), \theta_1)$ . Thus  $V(b^*(\theta_1), \theta_1) = V(\sigma^*, b^*(\theta_1), \theta_1)$ .  $\square$

#### Proposition 4.

PROOF. The proof is direct and derives the implied marginal taxes from an infinitesimal change in assets.

By construction, the equilibrium tax functions satisfies

$$a' = p(a')(a' - T_e(a')) + (1 - p(a))(a' - T_u(a'))$$

Thus, if the tax functions are differentiable, the following must hold for the marginal tax

$$p'(a')(T_u(a') - T_e(a')) = pT_e'(a') + (1 - p)T_u'(a') \quad (\text{B6})$$

From the incentive compatibility constraint we have

$$v''(p)p'(a') = \beta(V'_e(a')(1 - T'_e(a')) - V'_u(a')(1 - T'_u(a'))) \quad (\text{B7})$$

Substituting the relationships  $b_e = a' - T_e(a')$  and  $b_u = a' - T_u(a')$  into (19) (the government's first order condition for  $p$ ) gives

$$q(T_u(a') - T_e(a')) = \frac{\mu}{\lambda} v''(p) \quad (\text{B8})$$

Where  $\lambda$  and  $\mu$  are the multipliers on the budget and incentive compatibility constraint, respectively. Substituting (B8) into (B6)

$$p'(a')v''(p)\frac{\mu}{\lambda q} = pT'_e(a') + (1 - p)T'_u(a') \quad (\text{B9})$$

Substituting (B7) into (B9)

$$\beta(V'_e(a')(1 - T'_e(a')) - V'_u(a')(1 - T'_u(a')))\frac{\mu}{\lambda q} = pT'_e(a') + (1 - p)T'_u(a') \quad (\text{B10})$$

In addition, the agent's decentralized first order condition must hold:

$$u'(c) = \frac{\beta}{q}(pu'(c_e)(1 - T'_e(a')) + (1 - p)u'(c_u)(1 - T'_u(a'))) \quad (\text{B11})$$

Using equation (18) and solving equations (B10) and (B11) yields

$$T'_e(a') = 1 - \frac{u'(c_u)}{pu'(c_u) + (1 - p)u'(c_e)}, \quad T'_u(a') = 1 - \frac{u'(c_e)}{pu'(c_u) + (1 - p)u'(c_e)} \quad \square$$

### C. Proofs of Chapter 3

#### Lemma 6.

PROOF. By the Principle of Optimality, the problem in (33)-(36) can be split up as

$$V(b_0, \theta_0) = \max_{c_0, a_1} \{u(c_0) - \theta_0 v(p_0) + X(a_1, \theta_0)\} \quad (\text{C1})$$

$$\text{s.t. } b_0 = c_0 - \theta_0 w + qa_1 \quad (\text{C2})$$

$$0 \geq \phi - a_1 \quad (\text{C3})$$

$$X(a_1, \theta_0) = \max_{\sigma(a_1)} \sum_{t=1}^{\infty} \beta^t \int_{\Theta^t} \{u(c_t(a_1, \theta^t)) - (1 - \theta_t)v(p_t(a_1, \theta^t))\} \lambda(a_1, \theta^t) d\theta^t \quad (\text{C4})$$

$$\text{s.t. } \{p_t\}_{t=0}^{\infty} = \text{argmax}\{X(a_1, \theta_0)\} \quad (\text{C5})$$

$$a_1 = \sum_{t=1}^{\infty} q^t \int_{\Theta^t} \{c_t(a_1, \theta^t) - \theta_t w\} \lambda(a_1, \theta^t) d\theta^t \quad (\text{C6})$$

$$0 \geq \phi - \sum_{s=1}^{\infty} q^{s-1} \int_{\Theta^{t+s}} \{c_{t+s}(a_1, \theta^{t+s}) - \theta_{t+s} w\} \frac{\lambda(a_1, \theta^{t+s})}{\lambda(a_1, \theta^t)} d\theta^{t+s}, \quad \text{for } t = 1, 2, \dots \quad (\text{C7})$$

Under an optimal allocation, equations (C4) and (C6) can be written as

$$X(a_1, \theta_0) = -v(p_0^*(a_1, \theta_0)) + \beta \int_{\Theta^1} V(\sigma^*, b^*(a_1, \theta_1), \theta_1) \lambda(a_1, \theta^1) d\theta^1 \quad (\text{C8})$$

$$a_1 = q \int_{\Theta^1} b^*(a_1, \theta_1) \lambda(b_0, \theta^1) d\theta^1 \quad (\text{C9})$$

The proof then proceeds in three steps. The first step shows that  $X(a_1, \theta_0)$  is strictly increasing in  $a_1$ . By exploiting this fact, the second step will then proceed by showing that  $b^*(a_1, \theta_1)$  must be resource minimizing under promised utility  $V(\sigma^*, b^*(a_1, \theta_1), \theta_1)$ . Lastly, the third step then shows that duality holds: If  $b^*(a_1, \theta_1)$  is resource minimizing under utility  $V(\sigma^*, b^*(a_1, \theta_1), \theta_1)$ , then  $V(\sigma^*, b^*(a_1, \theta_1), \theta_1)$  must be utility maximizing under resources  $b^*(a_1, \theta_1)$ .

*Step 1.* Assume that there is an inflow of resources to the left-hand side of (C6) equal to  $\varepsilon > 0$ . For notational convenience, define  $c_1$  and  $c_0$  as period one consumption in the employed and unemployed state respectively. Pick an  $\varepsilon_1 \geq 0$  and  $\varepsilon_0 \geq 0$  such that

$$u(c_1 + \varepsilon_1) - u(c_0 + \varepsilon_0) = u(c_1) - u(c_0)$$

$$\varepsilon_1 + \varepsilon_0 = \varepsilon$$

Since the relative value between employment states are unaltered,  $p_0^*(a_1, \theta_0)$  is still incentive compatible and period zero expected utility has increased by

$$\beta(p_0(u(c_1 + \varepsilon_1) - u(c_1)) + (1 - p_0)(u(c_0 + \varepsilon_0) - u(c_0))) > 0$$

Where  $p_0 = p_0^*(a_1, \theta_0)$ .

*Step 2.* Consider the following resource minimization problem:

$$b(V, \hat{\theta}_0) = \min_{\sigma} \sum_{t=0}^{\infty} q^t \int_{\Theta^t} \{c_t(V, \hat{\theta}^t) - \hat{\theta}_t w\} \lambda(V, \hat{\theta}^t) d\hat{\theta}^t \quad (\text{C10})$$

$$\text{s.t. } V \leq \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} \{u(c_t(V, \hat{\theta}^t)) - (1 - \hat{\theta}_t)v(p_t(V, \hat{\theta}^t))\} \lambda(V, \hat{\theta}^t) d\hat{\theta}^t \quad (\text{C11})$$

and subject to the incentive compatibility and liquidity constraint. A “hat” on the sequence  $\theta_t$  is used to distinguish it from the values of  $\theta_t$  in the original problem (33)-(36). It is important to note if constraint (C11) in problem (C10)-(C11) is slack, then  $c_0(V, \hat{\theta}^0)$  is interior; if it was not, since  $u(0) = -\infty$ , the right-hand side in (C11) would equal minus infinity, and  $V \geq -\infty$ . It is then straightforward to see that constraint (C11) will hold as an equality. If it did not, period zero consumption could simply be reduced without interfering with neither incentive compatibility nor the liquidity constraint, reducing the objective function.

Now, consider the scenario in which  $\hat{\theta}_0 = \theta_1$  and  $V = V(\sigma^*, b^*(a_1, \theta_1), \theta_1)$ . Could  $b(V, \hat{\theta}_0)$  in (C10) be smaller than  $b^*(a_1, \theta_1)$ , for at least one value of  $\theta_1$ ? Assume that it is. Define  $a'_1$  as  $a'_1 = p_0(a_1, \theta^0)b(V, 1) + (1 - p_0(a_1, \theta^0))b(V, 0)$ , and note that  $a_1 > a'_1$ , and that  $a'_1$  is budget feasible, incentive compatible and delivers utility  $V(b_0, \theta_0)$ .  $a'_1$  might not, however, respect the time zero liquidity constraint. Pick an  $a''_1$  such that  $a_1 > a''_1 > a'_1$ . Then, since  $X(a'_1, \theta_0)$  is strictly increasing and continuous (Aliprantis and Border, 1999),  $X(a''_1, \theta_0) > X(a_1, \theta_0)$ , which violates the optimality of  $V(b_0, \theta_0)$ . Thus,  $b^*(a_1, \theta_1)$  is resource minimizing under promised utility  $V(\sigma^*, b^*(a_1, \theta_1), \theta_1)$ .

*Step 3.* In order to complete the proof, it must be shown that  $V(\sigma^*, b^*(a_1, \theta_1), \theta_1)$  attains the maximum value under resources  $b^*(a_1, \theta_1)$ .

Assume that  $V(b^*(a_1, \theta_1), \theta_1) > V(\sigma^*, b^*(a_1, \theta_1), \theta_1)$ . By Berge’s Maximum Theorem (Aliprantis and Border, 1999),  $V(b^*(a_1, \theta_1), \theta_1)$  is continuous in  $b$ . By the same argument as above,  $c_1(b^*(a_1, \theta_1), \theta_1) > 0$  since  $u(0) = -\infty$ . Thus there exist a  $b^{**}(a_1, \theta_1)$  arbitrarily close to  $b^*(a_1, \theta_1)$  such that  $b^*(a_1, \theta_1) > b^{**}(a_1, \theta_1)$  and  $V(b^{**}(a_1, \theta_1), \theta_1) > V(\sigma^*, b^*(a_1, \theta_1), \theta_1)$ . This contradicts that  $b^*(a_1, \theta_1)$  was resource minimizing for  $V(\sigma^*, b^*(a_1, \theta_1), \theta_1)$ . Thus  $V(b^*(a_1, \theta_1), \theta_1) = V(\sigma^*, b^*(a_1, \theta_1), \theta_1)$ .  $\square$

### Proposition 8.

PROOF. The proof is direct and derives the implied marginal taxes from an infinitesimal change in assets.

By construction, the equilibrium tax functions satisfies

$$a' = p(a')(a' - T_e(a')) + (1 - p(a))(a' - T_u(a'))$$

Thus, if the tax functions are differentiable, the following must hold for the marginal tax

$$p'(a')(T_u(a') - T_e(a')) = pT'_e(a') + (1 - p)T'_u(a') \quad (\text{C12})$$

From the incentive compatibility constraint we have

$$v''(p)p'(a') = \beta(V'_e(a')(1 - T'_e(a')) - V'_u(a')(1 - T'_u(a'))) \quad (\text{C13})$$

Substituting the relationships  $b_e = a' - T_e(a')$  and  $b_u = a' - T_u(a')$  into (52) (the government's first order condition for  $p$ ) gives

$$q(T_u(a') - T_e(a')) = \zeta v''(p) \quad (\text{C14})$$

Where  $\zeta$  is the ratio of the multipliers on the budget and incentive compatibility constraint, respectively. Elementary algebra shows that  $\zeta = p(1-p)(1/u'(c_u) - 1/u'(c_e))$ . Substituting (C14) into (C12)

$$p'(a')v''(p)\zeta = pT'_e(a') + (1-p)T'_u(a') \quad (\text{C15})$$

Substituting (C13) into (C15)

$$\frac{\beta}{q}(V'_e(a')(1 - T'_e(a')) - V'_u(a')(1 - T'_u(a')))\zeta = pT'_e(a') + (1-p)T'_u(a') \quad (\text{C16})$$

In addition, the agent's decentralized first order condition must hold:

$$u'(c) - \mu = \frac{\beta}{q}(pu'(c_e)(1 - T'_e(a')) + (1-p)u'(c_u)(1 - T'_u(a'))) \quad (\text{C17})$$

Using equation (51) and solving equations (C16) and (C17) yields

$$T'_e(a') = 1 - \frac{q}{\beta} \frac{u'(c) - \mu}{u'(c_e)}, \quad T'_u(a') = 1 - \frac{q}{\beta} \frac{u'(c) - \mu}{u'(c_u)} \quad \square$$